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To Sammy and our unborn child

Lin Bai

Preface

As an effective means to improve the spectral efficiency in wireless communications, multiple-input multiple-output (MIMO) systems equipped with multiple antennas at both transmitter and receiver sides have been well studied in recent years. In MIMO systems, more careful receiver design than in single-input single-output (SISO) systems is highly desirable in order to achieve good performances due to interfering signals by multiple antennas. To this end, we may consider joint detection and decoding. However, it leads to prohibitively high computational complexity, which grows exponentially with the number of equipped antennas and thus it becomes impractical for actual systems. Therefore, it is desirable to develop suboptimal MIMO receivers to provide good performances with reasonably low complexities, especially for large systems.

In the book entitled *Low Complexity MIMO Detection* published by Springer in 2012, a number of complexity-efficient MIMO detection methods and algorithms have been reviewed and studied. However, the MIMO detection was discussed only for uncoded systems. Since the separation of signal detection and decoding may lead to a performance degradation for coded MIMO systems, the MIMO detection has to be considered with decoding, which becomes the motivation of this book.

In this book, we focus on the design of low-complexity and high-performance MIMO receivers, where two techniques, successive interference cancellation (SIC) and lattice reduction (LR), become the key ingredients in deriving such receivers. In addition, in conjunction with the receiver design, other topics including the channel estimation, multiuser, and multicell systems, are further discussed in the later part of the book. Our book is summarized as follows.

We first present point-to-point MIMO systems and various low complexity detection methods. In order to provide a background, the detection theory is reviewed in Chap. 2, the signal detection in a vector space and principles of MIMO detection are introduced in Chap. 3, different computationally efficient SIC-based detection approaches are presented in Chap. 4, and the principles of LR and corresponding detection schemes are discussed in Chap. 5.

In the second part of this book, we focus on iterative detection and decoding (IDD) schemes in MIMO-bit interleaved coded modulation (MIMOBICM) systems. A background of MIMO iterative receivers is introduced in Chap. 6. Low complexity iterative receivers using LR at bit-level are studied in Chap. 7. Randomized sampling-based IDD is presented in Chap. 8.

Other issues in conjunction with the LR-based detection schemes are presented in Chaps. 9 and 10. In particular, various channel estimation techniques are discussed in Chap. 9. Multiuser and multicell MIMO systems are considered in Chap. 10.

Our book is intended to introduce the low complexity receiver design in MIMO systems from fundamentals to practical applications. This book makes an easy-to-follow presentation from the elementary to the profound level and includes not only theories, but also updated research outcomes that could be useful for both graduate students and practicing engineers in wireless communications.

Beijing, January 2014 Lin Bai
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Acronyms

API	A priori information
APP	A posteriori probability
APRP	A priori probability
AWGN	Additive white Gaussian noise
BCJR	Bahl, Cocke, Jelinek, and Raviv
BER	Bit error rate
BICM	Bit interleaved coded modulation
Bit-LR	Bit-level LR-aided
BLAST	Bell laboratories layered space time
BPSK	Binary phase shift keying
BS	Base station
cdf	Cumulative distribution function
CLLL	Complex-valued LLL
CLR-RLGA	Complex-valued LR-based list generation algorithm
CLT	Central limit theorem
CMs	Complex multiplications
CRC	Column reordering criteria
CRIS	Column reordering index set
CSCG	Circular symmetric complex Gaussian
CSI	Channel state information
DFE	Decision feedback equalizer
DMT	Diversity-multiplexing trade-off
DRC	Dimension reduction condition
EM	Expectation-maximization
EP	Error probability
EP-CRC	Error probability based CRC
EXIT	Extrinsic information transfer
FCA	Fixed candidate algorithm
flops	Floating point operation
FSM	Finite state machine
GAL	Genie-aided list
GLR	Generalized likelihood ratio
GLRT	Generalized likelihood ratio test
ICED	Iterative channel estimation and detection

ICI	Inter-cell interference
IDD	Iterative detection and decoding
iid	Independently and identically distributed
ISI	Inter-symbol interference
ITS	Iterative tree search
K–L	Karhunen–Loéve
LAPP	Log a posteriori probability
LAPPR	Logarithms of a posteriori probability ratios
LAPRP	Log-ratio of a priori probability
LBR	Lattice basis reduced
LLL	Lenstra–Lenstra–Lovász
LLR	Log-likelihood ratio
LR	Likelihood ratio or lattice reduction
LRG	LR-based greedy
LR-RLGA	LR-based randomized list generation algorithm
LSD	List-sphere decoding
MAP	Maximum a posteriori probability
MCMC	Monte Carlo Markov chain
MD	Max–min diagonal term
MDist	Max–min distance
ME	Max–min eigenvalue
MFB	Matched filter bound
MGF	Moment generating function
MIMO	Multiple-input multiple-output
ML	Maximum likelihood
MLE	Maximum likelihood estimate
MMI	Maximum mutual information
MMMSE	Min–max mean square error
MMSE	Minimum mean square error
MMSE-PIC	Minimum mean square error parallel interference cancellation
MSB	Most significant bit
MSE	Mean square error
OD	Orthogonal defect or orthogonality deficiency
OD-CRC	Orthogonality deficiency based CRC
ODR	Optimal decision region
OFDMA	Orthogonal frequency division multiple access
PAM	Pulse amplitude modulation
pdf	Probability density function
PDR	Probability of dimension reduction
PEP	Pairwise error probability
QAM	Quadrature amplitude modulation
Rand-SIC	Randomized SIC
RLR-RLGA	Real-valued LR-based randomized list generation algorithm
ROCs	Receiver operating characteristics
SC	Soft (interference) cancellation

SIC	Successive interference cancellation
SINR	Signal to interference plus noise ratio
SISO	Single-input single-output or soft-input soft-output
SNR	Signal-to-noise ratio
SSE	Sum of squared error
SVP	Shortest vector problem
UBLR	Updated basis LR
UBLRG	UBLR-based greedy
ZF	Zero forcing

Notations

$\mathbf{A}$, a, a	(boldface upper, boldface lower, lower italic fonts) complex-valued matrix, vector, scalar
$\mathbf{A}$, $\mathbf{a}$, $\mathbf{a}$	(boldface upper, boldface lower, lower sans-serif fonts) real-valued matrix, vector, scalar
$\mathbf{A}^{\mathrm{T}}$, $\mathbf{A}^{\mathrm{H}}$, $\mathbf{A}^{\dagger}$	Transpose, Hermitian transpose, Pseudo inverse, respectively
$[\mathbf{A}]_{p,q}$	The (p, q)th element of $\mathbf{A}$
$[\mathbf{A}]_{a:b,c:d}$	A sub-matrix of $\mathbf{A}$ with the elements obtained from rows $a,\ldots,b$ and columns $c,\ldots,d$
$[\mathbf{A}]:,n$	The nth column vector of $\mathbf{A}$
$[\mathbf{A}]n,:$	The nth row vector of $\mathbf{A}$
$\mathrm{Tr}(\mathbf{A})$	Trace operation of a square matrix $\mathbf{A}$
$\det(\mathbf{A})$	Determinant of matrix $\mathbf{A}$
$\mathrm{adj}(\mathbf{A})$	Adjoint of matrix $\mathbf{A}$
$\mathcal{D}(\mathcal{A})$	Length of the shortest nonzero vector of the lattice generated by $\mathbf{A}$
$\lambda_{\min}(\mathbf{A})$	Minimum eigenvalue of $\mathbf{A}$
$\mathcal{L}(\mathcal{A})$	Lattice generated by $\mathbf{A}$
$\Pr(X)$	Probability of random event X
$E[\bullet]$	Statistical expectation
$<\mathbf{a}, \mathbf{b}>$	Inner product of two vectors $\mathbf{a}$ and $\mathbf{b}$
$\Re(\cdot), \Im(\cdot)$	Real and imaginary parts
$\mathcal{N}(m, C)$	Gaussian probability density function with mean $\mathbf{m}$ and covariance $\mathbf{C}$
$\mathcal{CN}(m, C)$	Circularly symmetric complex Gaussian probability density function with mean $\mathbf{m}$ and covariance $\mathbf{C}$
$\log(\bullet)$	Natural (base e) logarithm
$\ln(\bullet)$	Common (base 10) logarithm
$\mathbf{0}$	Matrix with all entries of 0
$\lVert \cdot \rVert$	2-norm
$\lVert \cdot \rVert_{\mathrm{F}}$	The Frobenius norm
$\lceil \beta \rfloor$	The nearest integer to β
$\lfloor \beta \rfloor$	The closest integer which is smaller than β
$\lvert \beta \rvert$	Absolute value of scalar β

$\setminus$	Set minus
$\mathbf{I}_n$	An $n \times n$ identity matrix
$\{k_{(1)}, k_{(2)}, \ldots\}$	The collection set of $k_{(1)}, k_{(2)}, \ldots$
erfc(x)	Complementary error function of x, i.e., $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty^{23}} e^{-z^2} dz$
$\{\exists x : f(x)\}$	There is at least one x such that a function of x, f(x), is true
$\mathbb{Z}$	Set of integer numbers
$\mathbb{R}^k$	Real-valued k-dimensional vector space
$\mathbb{C}^k$	Complex-valued k-dimensional vector space

Chapter 1
Introduction

In wireless communications, the maximum achievable data rate of wireless channels is usually the most critical issue to be addressed for system design, which can be characterized by the channel capacity. The channel capacity represents the maximum reliable transmission rate of information over a given channel, which was first established by Claude E. Shannon, the father of information theory, and becomes the theoretical foundation of wireless communications. The channel capacity (in bits per second per Hz) of a single-input single-output (SISO) system can be given by

$$C_{\text{SISO}} = B \log_2 (1 + \text{SNR}), \tag{1.1}$$

where B and SNR denote the bandwidth and the signal-to-noise ratio (SNR), respectively. As we can see from (1.1) that either high SNR or wide bandwidth has to be required in order to achieve a high transmission rate.

In wireless communications, since there are always limitations to increase the SNR due to the propagation loss with practical power amplifiers, the bandwidth has to be wide enough to support high data rate services. However, the scarce wireless spectrum has posed a huge challenge in designing wireless communication systems with increasing data rate demands. Fortunately, by employing multiple antennas at both transmitter and receiver, the multiple-input multiple-output (MIMO) system [1] has been developed to improve the spectral efficiency without unrealistic high SNR or extra bandwidth. In relation to the SISO capacity, the MIMO channel capacity [2] can be characterized as

$$C_{\text{MIMO}} \simeq \min(M, N) C_{\text{SISO}}, \tag{1.2}$$

where M and N are the numbers of equipped transmit and receive antennas, respectively. According to (1.2), the capacity grows linearly with the minimum of the numbers of transmit and receive antennas. Due to this linear scaling property, a higher spectral efficiency can be easily achieved by increasing the number of antennas.

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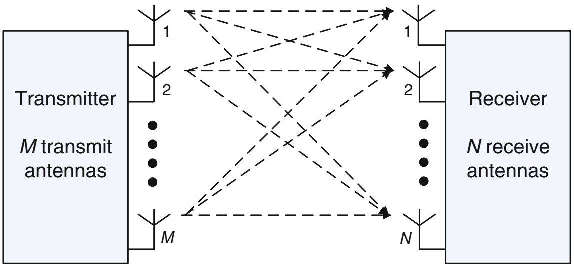

Fig. 1.1 An M by N MIMO system

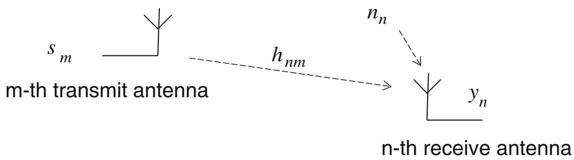

Fig. 1.2 Wireless channel from the m-th transmit antenna to the n-th receive antenna

A point-to-point MIMO system equipped with M transmit antennas and N receive antennas is illustrated in Fig. 1.1. Since each receive antenna is able to receive signals from all the transmit antennas, supposing that each transmit antenna transmits different symbols, the received signal of the nth antenna at the receiver is given by

$$y_n = h_{n1}s_1 + h_{n2}s_2 + \cdots + h_{nM}s_M + n_n, \tag{1.3}$$

where s_m, h_{nm}, and n_n are the symbols transmitted by the m-th antenna, the channel gain from the m-th transmit antenna to the n-th receive antenna, and the additive noise at the n-th receive antenna, respectively. Figure 1.2 illustrates the channel from the m-th transmit antenna to the n-th receive antenna.

With all the received signals, we have

$$\begin{aligned} y_1 &= h_{11}s_1 + h_{12}s_2 + \cdots + h_{1M}s_M + n_1; \\ y_2 &= h_{21}s_1 + h_{22}s_2 + \cdots + h_{2M}s_M + n_2; \\ &\vdots \\ y_N &= h_{N1}s_1 + h_{N2}s_2 + \cdots + h_{NM}s_M + n_N, \end{aligned} \tag{1.4}$$

or

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1M} \\ h_{21} & h_{22} & \cdots & h_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NM} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_M \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_N \end{bmatrix}. \tag{1.5}$$

Letting the channel matrix be

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1M} \\ h_{21} & h_{22} & \cdots & h_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ h_{N1} & h_{N2} & \cdots & h_{NM} \end{bmatrix}, \tag{1.6}$$

the transmit signal vector be $\mathbf{s} = [s_1, s_2, \ldots, s_M]^{\mathrm{T}}$, the received signal vector be $\mathbf{y} = [y_1, y_2, \ldots, y_N]^{\mathrm{T}}$, and the additive noise vector be $\mathbf{n} = [n_1, n_2, \ldots, n_N]^{\mathrm{T}}$, (1.5) becomes

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}, \tag{1.7}$$

where the superscript T denotes the transpose operation. The system model in (1.7) will be widely considered throughout this book. At the receiver, in order to detect the transmit signal $\mathbf{s}$ from the received signal $\mathbf{y}$ under the knowledge of estimated channel state information (CSI), efficient detection schemes in terms of performance and complexity have to be considered. According to (1.7), the CSI can be defined as the knowledge of $\mathbf{H}$ (under the assumption that the statistical properties of $\mathbf{n}$ are available), which can be estimated via some channel estimation schemes. For uncoded systems, the spatial diversity order can be used as a performance metric for various MIMO detection methods, where a full receive diversity gain is the same as the number of receive antennas in MIMO systems. Asymptotically, optimal detection schemes should achieve this full receive diversity gain.

The problem of MIMO detection is a joint detection problem as all the symbols in the signal vector $\mathbf{s}$ are to be jointly detected. Using exhaustive search, the maximum likelihood (ML) detection can be carried out to provide an optimal performance with a full receive diversity. However, the complexity to detect the M signals jointly grows exponentially with the number of transmit antennas, M, which easily becomes impractical in many applications, especially when large MIMO systems are considered. To reduce the computational complexity, various suboptimal approaches have been proposed. Linear detectors such as the zero-forcing (ZF) and minimum mean square error (MMSE) detectors can be considered, in which the signals from the other antennas are treated as interfering signals. Although they have low complexity, they cannot achieve reasonably good performance and a full receive diversity gain, in particular, at a high SNR. In order to achieve the two desirable features, i.e., low computational complexity and near optimal performance, simultaneously, successive interference cancellation (SIC) and lattice reduction (LR)-based approaches become quite attractive. In this book, we discuss various MIMO detectors based on SIC and LR in detail for point-to-point MIMO systems. Prior to discussing those detectors, however, the detection theory is first reviewed in Chap. 2 and the signal detection in a vector space and principles of MIMO detection are introduced in Chap. 3 in order to provide a background. Then, different computationally efficient

SIC-based detection approaches are presented in Chap. 4 and the principles of LR and corresponding detection schemes are discussed in Chap. 5.

Since multiple antennas can be used either to improve the reliability of the system with respect to the spatial diversity or to increase the data rate with respect to the spatial multiplexing, it is not possible to increase both diversity and multiplexing gains simultaneously for given numbers of transmit and receive antennas. The spatial diversity gain d and the spatial multiplexing gain r can be used to characterize the performance of coded MIMO systems. Using the average error probability $P_e(\text{SNR})$ and the data rate $R(\text{SNR})$, it can be shown that

$$d = -\lim_{\text{SNR}\to\infty} \frac{P_e(\text{SNR})}{\log \text{SNR}} \tag{1.8}$$

and

$$r = \lim_{\text{SNR}\to\infty} \frac{R(\text{SNR})}{\log \text{SNR}}. \tag{1.9}$$

If a system achieves a higher r, d becomes lower, vice versa. However, it is not possible to increase both r and d simultaneously for given numbers of transmit and receive antennas. This is called diversity-multiplexing trade-off (DMT) [3], which can be used as a performance measure for coded MIMO systems. Although the system design based on DMT is of paramount importance, we do not discuss this issue in this book. We rather focus on practical systems where MIMO-bit interleaved coded modulation (MIMO-BICM) is adopted over MIMO channels to obtain good performance at a data rate near to the channel capacity with a relatively simple structure for the transmitter. With a soft-input soft-output channel decoder in MIMO-BICM systems, iterative detection and decoding (IDD) can be employed to improve the performance with a reasonable complexity for the receiver based on the turbo principle for MIMO systems. These are the main topics in the second part of this book, where we focus on IDD schemes in MIMO-BICM systems. A background of MIMO iterative receivers is introduced in Chap. 6. Low complexity iterative receivers using LR at bit-level are studied in Chap. 7. Randomized sampling-based IDD is presented in Chap. 8.

In this book, aiming at providing a comprehensive view to low complexity MIMO receiver design, while we mainly focus on various MIMO detection and IDD approaches, where the two techniques, SIC and LR, play a key role in developing good-performance and low-complexity MIMO receivers, we also discuss other topics (e.g., the channel estimation, multiuser and multicell systems) that are equally important in designing MIMO receivers in the later part of the book. In particular, various channel estimation techniques are discussed in Chap. 9. Multiuser and multicell MIMO systems are considered in Chap. 10.

Chapter 2
Signal Processing at Receivers: Detection Theory

As an application of the statistical hypothesis testing, signal detection plays a key role in signal processing at receivers of wireless communication systems. To accept or reject a hypothesis based on observations, the hypotheses are possible statistical descriptions of observations using statistical hypothesis testing tools. As realizations of a certain random variable, observations can be characterized by a set of candidate probability distributions of the random variable.

In this chapter, based on the statistical hypothesis testing, we introduce the theory of signal detection and key techniques for performance analysis. We focus on the fundamentals of signal detection in this chapter, while the signal detection over multiple-antenna systems will be considered in the following parts of the book.

2.1 Principles of Hypothesis Testing

Three key elements are carried out in the statistical hypothesis testing, including

(1) Observations.
(2) Set of hypotheses.
(3) Prior information.

The decision process or hypothesis testing is illustrated in Fig. 2.1. In Fig. 2.1, is shown that observations and prior information are taken into account to obtain the final decision. However, considering the cases that no prior information is available or prior information could be useless, the hypothesis test can also be developed with observations only.

Under the assumption that there exist $M (\geq 2)$ hypotheses, we can have an M-ary hypothesis testing in which we need to choose one of the M hypotheses that explains observations and prior information best. In order to choose a hypothesis, different criteria can be considered. According to these criteria, different hypothesis tests are

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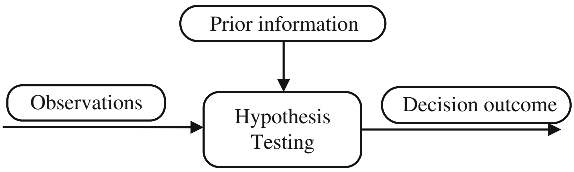

Fig. 2.1 Block diagram for hypothesis testing

available. Based on the likelihood ratio (LR)[1] hypothesis test; three well-known hypothesis tests are given as follows:

(1) Maximum a posteriori probability (MAP) hypothesis test.
(2) Baysian hypothesis test.
(3) Maximum likelihood (ML) hypothesis test.

In the following section, the hypothesis tests in the above are illustrated respectively.

2.2 Maximum a Posteriori Probability Hypothesis Test

Let us first introduce the MAP hypothesis test or MAP decision rule. Consider that there are different balls contained in two boxes (A and B), where a certain number is marked on each ball. Under the assumption that the distribution of the numbers on balls is different for each box, as a ball is drawn from one of the boxes, we want to determine the box where the ball is drawn from based on the number of the ball. Accordingly, the following two hypotheses can be founded:

$$\begin{cases} \mathcal{H}_0 : \text{the ball is drawn from box A;} \\ \mathcal{H}_1 : \text{the ball is drawn from box B.} \end{cases}$$

For example, suppose that 10 balls are drawn from each box as shown in Fig. 2.2. Based on the empirical distribution results in Fig. 2.2, conditional distributions of the number on balls are given by

$$\Pr(1|\mathcal{H}_0) = \frac{4}{10};$$
$$\Pr(2|\mathcal{H}_0) = \frac{3}{10};$$
$$\Pr(3|\mathcal{H}_0) = \frac{3}{10};$$

and

[1] Note that in Chaps. 2 and 3, we use LR to denote the term “likelihood ratio,” while in the later chapters of the book, the LR is used to represent “lattice reduction.”

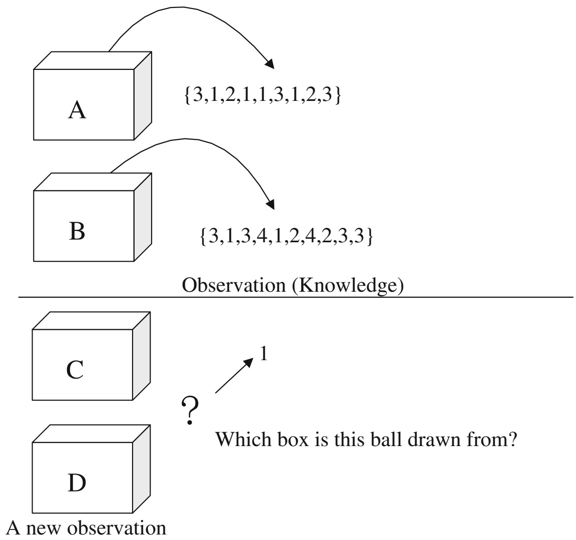

Fig. 2.2 Balls drawn from two boxes

$$\Pr(1|\mathcal{H}_1) = \frac{2}{10};$$
$$\Pr(2|\mathcal{H}_1) = \frac{2}{10};$$
$$\Pr(3|\mathcal{H}_1) = \frac{4}{10};$$
$$\Pr(4|\mathcal{H}_1) = \frac{2}{10}.$$

In addition, the probability that A ($\mathcal{H}_0$) or B ($\mathcal{H}_1$) box is chosen is assumed to be the same, i.e.,

$$\Pr(\mathcal{H}_0) = \Pr(\mathcal{H}_1) = \frac{1}{2}. \tag{2.1}$$

Then, we can easily have

$$\Pr(1) = \Pr(\mathcal{H}_0)\Pr(1|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\Pr(1|\mathcal{H}_1) = \frac{6}{20};$$
$$\Pr(2) = \Pr(\mathcal{H}_0)\Pr(2|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\Pr(2|\mathcal{H}_1) = \frac{5}{20};$$
$$\Pr(3) = \Pr(\mathcal{H}_0)\Pr(3|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\Pr(3|\mathcal{H}_1) = \frac{7}{20};$$
$$\Pr(4) = \Pr(\mathcal{H}_0)\Pr(4|\mathcal{H}_0) + \Pr(\mathcal{H}_1)\Pr(4|\mathcal{H}_1) = \frac{2}{20},$$

where $\Pr(n)$ denotes the probability that the ball with number n is drawn. Taking $\Pr(\mathcal{H}_k)$ as the a priori probability (APRP) of $\mathcal{H}_k$, the a posteriori probability (APP) of $\mathcal{H}_k$ is shown as follows:

$$\Pr(\mathcal{H}_0|1) = \frac{2}{3};$$
$$\Pr(\mathcal{H}_0|2) = \frac{3}{5};$$
$$\Pr(\mathcal{H}_0|3) = \frac{3}{7};$$
$$\Pr(\mathcal{H}_0|4) = 0,$$

and

$$\Pr(\mathcal{H}_1|1) = \frac{1}{3};$$
$$\Pr(\mathcal{H}_1|2) = \frac{2}{5};$$
$$\Pr(\mathcal{H}_1|3) = \frac{4}{7};$$
$$\Pr(\mathcal{H}_1|4) = 1.$$

Here, $\Pr(\mathcal{H}_k|n)$ is formed as the conditional probability that the hypothesis $\mathcal{H}_k$ is true under the condition that the number on the drawn ball is n. For example, if the number of the ball is $n = 1$, since $\Pr(\mathcal{H}_0|1) = \frac{2}{3}$ is greater than $\Pr(\mathcal{H}_1|1) = \frac{1}{3}$, we can decide that the ball is drawn from box A, where the hypothesis $\mathcal{H}_0$ is accepted. The corresponding decision rule is named as the MAP hypothesis testing, since we choose the hypothesis that maximizes the APP.

Generally, in the binary hypothesis testing, $\mathcal{H}_0$ and $\mathcal{H}_1$ are referred to as the null hypothesis and the alternative hypothesis, respectively. Under the assumption that the APRPs $\Pr(\mathcal{H}_0)$ and $\Pr(\mathcal{H}_1)$ are known and the conditional probability, $\Pr(Y|\mathcal{H}_k)$, is given, where Y denotes the random variable for an observation, the MAP decision rule for binary hypothesis testing is given by

$$\begin{cases} \mathcal{H}_0 : \Pr(\mathcal{H}_0|Y = y) > \Pr(\mathcal{H}_1|Y = y); \\ \mathcal{H}_1 : \Pr(\mathcal{H}_0|Y = y) < \Pr(\mathcal{H}_1|Y = y), \end{cases} \tag{2.2}$$

where y denotes the realization of Y. Note that $\mathcal{H}_0$ is chosen if $\Pr(\mathcal{H}_0|Y = y) > \Pr(\mathcal{H}_1|Y = y)$ and vice versa. Here, we do not consider the case of $\Pr(\mathcal{H}_0|Y = y) = \Pr(\mathcal{H}_1|Y = y)$ in (2.2), where a decision can be made arbitrarily. Thus, the decision outcome in (2.3) can be considered as a function of y. Using Bayes rule, we can also show that

$$\begin{cases} \mathcal{H}_0 : \dfrac{\Pr(Y = y|\mathcal{H}_0)}{\Pr(Y = y|\mathcal{H}_1)} > \dfrac{\Pr(\mathcal{H}_1)}{\Pr(\mathcal{H}_0)}; \\ \mathcal{H}_1 : \dfrac{\Pr(Y = y|\mathcal{H}_0)}{\Pr(Y = y|\mathcal{H}_1)} < \dfrac{\Pr(\mathcal{H}_1)}{\Pr(\mathcal{H}_0)}. \end{cases} \tag{2.3}$$

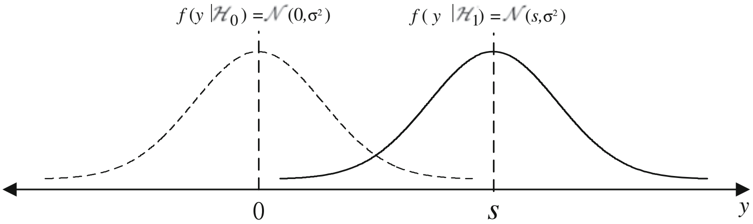

Fig. 2.3 The pdf of the hypothesis pair

Notice that as Y is a continuous random variable, $\Pr(Y = y|\mathcal{H}_k)$ is replaced by $f(Y = y|\mathcal{H}_k)$, where $f(Y = y|\mathcal{H}_k)$ represents the conditional probability density function (pdf) of Y given $\mathcal{H}_k$.

Example 2.1. Define by $\mathcal{N}\left(\mu, \sigma^2\right)$ the pdf of a Gaussian random variable (i.e., x) with mean μ and variance σ^2, where

$$\mathcal{N}\left(\mu, \sigma^2\right) = \frac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2}}. \tag{2.4}$$

Let the noise n be a Gaussian random variable with mean zero and variance σ, while s be a positive constant. Consider the case that a constant signal, s, is transmitted, while a received signal, y, may be corrupted by the noise, n, as shown in Fig. 2.3. Then, we can have the following hypothesis pair to decide whether or not s is present when y is corrupted by n:

$$\begin{cases} \mathcal{H}_0 : y = n; \\ \mathcal{H}_1 : y = s + n. \end{cases} \tag{2.5}$$

Then, as shown in Fig. 2.3, we have

$$\begin{cases} f(y|\mathcal{H}_0) = \mathcal{N}\left(0, \sigma^2\right); \\ f(y|\mathcal{H}_1) = \mathcal{N}\left(s, \sigma^2\right), \end{cases} \tag{2.6}$$

and

$$\frac{f(Y = y|\mathcal{H}_0)}{f(Y = y|\mathcal{H}_1)} = \exp\left(-\frac{s(2y - s)}{2\sigma^2}\right), \tag{2.7}$$

when $s > 0$. Letting

$$\rho = \frac{\Pr(\mathcal{H}_0)}{\Pr(\mathcal{H}_1)},$$

the MAP decision rule is simplified as follows: