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Gabriel N. Gatica

A Simple Introduction to the Mixed Finite Element Method

Theory and Applications

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Gabriel N. Gatica
Centro de Investigación en Ingeniería Matemática
and Departamento de Ingeniería Matemática
Universidad de Concepción
Concepción, Chile

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To the memory of Professor Dr. HERMANN ALDER WELLER,
founder of the numerical analysis *discipline at Concepción*
with great affection and enormous gratitude.

Preface

The main purpose of this monograph is to provide a simple and accessible introduction to the mixed finite element method as a fundamental tool to numerically solve a wide class of boundary value problems arising in physics and engineering sciences. The book is based on material that I have used to teach corresponding undergraduate and graduate courses at Universidad de Concepción, Concepción, Chile, during the last 10 years. As compared with several other classic books on the subject, and in addition to being of a limited scope, the main features of the present work concern, on the one hand, my attempt to present and explain most of the details in the proofs and in the various applications. In particular, several results and aspects of the corresponding analysis that are usually available only in papers or proceedings are included here. In addition, keeping in mind that the subject is growing and evolving very quickly, I concentrate the discussion mainly on those core concepts and fundamental results that need to be understood by thesis students and young researchers so that they can read more advanced textbooks and make their own contributions in this and related fields. As a consequence, one of the main emphases of the book is on most of the mathematical and numerical issues involved in the application of the mixed finite element method to simple modeling problems in continuum mechanics. This includes classical Poisson and linear elasticity problems, both under several kinds of boundary conditions for which, among other matters, complete proofs of the continuous and discrete inf-sup conditions required by the theory are provided.

The contents of the book, which assume a basic knowledge of functional analysis, partial differential equations, and Sobolev spaces (e.g., [7, 15, 50, 51, 53, 54]) are described next. Throughout the text, I employ the usual notations from those disciplines, especially the standard terminology for Sobolev spaces. For example, if $\mathscr{O}$ is an open set, its closure, a curve, or a surface, and $s \in \mathbb{R}$, then $\langle \cdot, \cdot \rangle_{s,\mathscr{O}}$, $|\cdot|_{s,\mathscr{O}}$, and $\|\cdot\|_{s,\mathscr{O}}$ denote, respectively, the inner product, seminorm, and norm of the Sobolev space $H^s(\mathscr{O})$. In particular, given Γ, a boundary or part of a boundary, $\langle \cdot, \cdot \rangle_{0,\Gamma}$ represents the inner product of $L^2(\Gamma)$, whereas $\langle \cdot, \cdot \rangle$ stands for the duality pairings of $H^{-s}(\Gamma) \times H^s(\Gamma)$, $H_{00}^{-s}(\Gamma) \times H_{00}^s(\Gamma)$, and any vector version of them, for each $s > 0$. However, when it is necessary to identify the underlying Γ, the corresponding duality expression is replaced by $\langle \cdot, \cdot \rangle_\Gamma$. Furthermore, when using the norm $\|\cdot\|_X$ of a given normed space X and when no confusion arises, the subscript $_X$ will usually be omitted. Finally, I use 0 to denote the null scalar as well as the null vector of any space and use C and c, with or without subscripts, bars, tildes, or hats, to denote generic constants independent of eventual discretization parameters, which may take different values at different places.

In Chap. 1, which is of an introductory character, I present a detailed discussion of the classical and general versions of the Lax–Milgram lemma, provide a couple of examples of mixed variational formulations, and prove the main results on traces and Green's identities in $H^1(\Omega)$ and $H(\mathrm{div};\Omega)$. The analysis of the Babuška–Brezzi theory for the aforementioned formulations is the main subject of Chap. 2. The continuous and discrete versions of the theorem, with the necessary and sufficient conditions for unique solvability and the corresponding Cea estimate of the error in the general case, are presented here. In addition, applications to several problems from continuum mechanics, whose respective analyses employ known results from functional analysis and Sobolev spaces, are also provided here. Then, in Chap. 3, I discuss the main facts about the classical Raviart–Thomas spaces. This includes the unisolvency that characterizes their definitions, and the approximation properties of the local and global interpolation operators involved. All the necessary theoretical tools, such as the Denny–Lions and Bramble–Hilbert lemmas and related arguments, are described in this part. Subsequently, after assimilating the contents of this chapter, the reader will easily understand the analysis of similar finite element subspaces available in the literature, such as Brezzi–Douglas–Marini (BDM) and Brezzi–Douglas–Fortin–Marini (BDFM) (e.g., [13, 16]). Finally, specific mixed finite element methods for the boundary value problems discussed in Chap. 2, which consider the Raviart–Thomas finite element subspaces from Chap. 3, are examined in Chap. 4. The corresponding numerical analyses include, among other aspects, the derivation of stable discrete liftings of the associated normal traces, which is particularly relevant for the treatment of Neumann or mixed boundary conditions in three dimensions. The devising of well-posed mixed finite element methods for the linear elasticity problem, which is based on the approach establishing its connection with stable finite element schemes for the usual primal formulation of the Stokes problem, is also discussed briefly in this chapter.

It is time now for the acknowledgements. First of all, I would like to express my deep gratitude to my great collaborators and even greater friends, Salim Meddahi, Norbert Heuer, Francisco J. Sayas, and Antonio Márquez, who, beginning in the late 1990s, and the early, mid, and late following decade, respectively, up to nowadays, have strongly contributed to improving my limited original knowledge of the mixed finite element method and its diverse applications. My deep appreciation also goes to George C. Hsiao for the many fruitful discussions on this and related topics over the years. In addition, I am very thankful to all the undergraduate and graduate students from Universidad de Concepción, Chile, who have taken my regular courses on the subject or have performed their thesis work under my guidance during the last decade. Apologizing in advance for not naming them all, I would like to give special thanks to a former Ph.D. student of mine, Ricardo Oyarzúa, who took the time to read the entire manuscript and pointed out several typographical and

mathematical amendments in it. Nevertheless, I am sure that new readers will find more corrections to make, and I thank them in advance for letting me know about the errors. In addition, my gratitude is also due to Mrs. Angelina Fritz, who typeset the original version of the book (written in Spanish) in LaTeX. Finally, I would like to express my appreciation to Springer-Verlag, and especially to Donna Chernyk, Associate Editor of Mathematics, for the publication of this monograph and for the friendly and supportive collaboration along all the way.

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Concepción, Chile Gabriel N. Gatica
October 2013

Contents

Chapter 1
Introduction

In this chapter we base most of the presentation on the classical references [8, 20, 41, 51] and describe the main introductory aspects of the finite and mixed finite element methods. We first recall the particular and general versions of the Lax–Milgram lemma and then introduce two examples illustrating the use of mixed variational formulations to solve boundary value problems. Finally, we present several basic results on traces, integration by parts formulae, and Green's identities for some Sobolev spaces, and in particular for $H(\mathrm{div};\Omega)$.

1.1 The Lax–Milgram Lemma

To state and prove this result, the most classical one in the analysis of variational problems, we need some preliminary concepts.

1.1.1 Preliminaries

Definition 1.1. Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ be real Hilbert spaces. We say that $B : H_1 \times H_2 \to \mathbb{R}$ is a bilinear form if it is linear in each of its components, that is,

(i) $B(\alpha x + \beta y, z) = \alpha B(x,z) + \beta B(y,z) \quad \forall x, y \in H_1, \quad \forall z \in H_2, \quad \forall \alpha, \beta \in \mathbb{R};$
(ii) $B(x, \alpha y + \beta z) = \alpha B(x,y) + \beta B(x,z) \quad \forall x \in H_1, \quad \forall y, z \in H_2, \quad \forall \alpha, \beta \in \mathbb{R}.$

Definition 1.2. Let $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ be real Hilbert spaces with induced norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. We say that a bilinear form $B : H_1 \times H_2 \to \mathbb{R}$ is BOUNDED if there exists a constant $M > 0$ such that

$$|B(x,y)| \leq M \, \|x\|_1 \, \|y\|_2 \quad \forall (x,y) \in H_1 \times H_2.$$

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Definition 1.3. Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space with induced norm $\|\cdot\|$, and let $B : H \times H \to \mathbb{R}$ be a bilinear form. We say that B is STRONGLY COERCIVE (or H-ELLIPTIC) if there exists a constant $\alpha > 0$ such that

$$B(x,x) \ \geq \ \alpha \, \|x\|^2 \qquad \forall\, x \in H.$$

Now, given $(H_1, \langle\cdot,\cdot\rangle_1)$ and $(H_2, \langle\cdot,\cdot\rangle_2)$ real Hilbert spaces and $B : H_1 \times H_2 \to \mathbb{R}$ a bounded bilinear form, we are interested in defining the operator $\mathbb{B} : H_1 \to H_2$ induced by B and vice versa. To this end, we consider $v \in H_1$ and define the functional $F_v : H_2 \to \mathbb{R}$ by

$$F_v(w) := B(v,w) \quad \forall\, w \in H_2.$$

Since B is bilinear, it is clear that F_v is linear. In addition, the fact that B is bounded (with constant M) implies that

$$|F_v(w)| \leq \ M\|v\|_1 \, \|w\|_2 \quad \forall\, w \in H_2,$$

which shows that $F_v \in H_2'$ and

$$\|F_v\| \leq M\|v\|_1 \quad \forall\, v \in H_1. \tag{1.1}$$

The foregoing analysis induces the definition of the operator $\mathscr{B} : H_1 \to H_2'$ as

$$\mathscr{B}(v) := F_v \quad \forall\, v \in H_1,$$

which, in virtue of the linearity of B in its first component and the inequality (1.1), is linear and bounded with

$$\|\mathscr{B}\|_{\mathscr{L}(H_1,H_2')} \leq M.$$

Recall here that, given Banach spaces X and Y, $\mathscr{L}(X,Y)$ denotes the space of bounded linear operators from X to Y. Finally, if $\mathscr{R}_2 : H_2' \to H_2$ denotes the Riesz mapping, we let $\mathbb{B} : H_1 \to H_2$ be the operator induced by B, that is,

$$\mathbb{B} := \mathscr{R}_2 \circ \mathscr{B} \tag{1.2}$$

or, graphically,

$$\begin{array}{ccc} H_1 & \xrightarrow{\mathscr{B}} & H_2' \\ & \underset{\mathbb{B}}{\searrow} & \downarrow \mathscr{R}_2 \\ & & H_2 \end{array} .$$

Note that the linearity and boundedness of $\mathscr{R}_2$ and $\mathscr{B}$ yield the same properties for $\mathbb{B}$, and there holds

$$\langle \mathbb{B}(v), w\rangle_2 = \langle \mathscr{R}_2(\mathscr{B}(v)), w\rangle_2 = \mathscr{B}(v)(w) = B(v,w) \quad \forall\, (v,w) \in H_1 \times H_2\,. \tag{1.3}$$

Conversely, given $\mathbb{B} \in \mathscr{L}(H_1, H_2)$, we define the bilinear form $B : H_1 \times H_2 \to \mathbb{R}$ induced by $\mathbb{B}$ as

$$B(v,w) := \langle \mathbb{B}(v), w\rangle_2 \quad \forall\, (v,w) \in H_1 \times H_2\,. \tag{1.4}$$