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# Non-commutative Multiple-Valued Logic Algebras

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*Dedicated to my precious son David Edward*

# Introduction

In 1920 Łukasiewicz introduced his three valued logic ([223]), the first model of multiple-valued logic. The $n$-valued propositional logic for $n > 3$ was constructed in 1922 and the $\aleph_0$-valued Łukasiewicz-Tarski logic in 1930 ([224]). The first completeness theorem for $\aleph_0$-valued Łukasiewicz-Tarski logic was given by Wajsberg in 1935. As a direct generalization of two-valued calculus, Post introduced in 1921 an $n$-valued propositional calculus distinct from that of Łukasiewicz ([239]).

In the early 1940s Gr.C. Moisil was the first to develop the theory of $n$-valued Łukasiewicz algebras with the intention of algebraizing Łukasiewicz's logic ([226, 227]), but an example of A. Rose from 1956 established that for $n \geq 5$ the Łukasiewicz implication can no longer be defined on a Łukasiewicz algebra. Consequently, the structures introduced by Moisil are models for Łukasiewicz logic only for $n = 3$ and $n = 4$. These algebras are now called *Łukasiewicz-Moisil algebras* or *LM algebras* for short ([14]).

The loss of implication has led to another type of logic, today called *Moisil logic*, distinct from the Łukasiewicz system. The logic corresponding to $n$-valued Łukasiewicz-Moisil algebras was created by Moisil in 1964. The fundamental concept of Moisil logic is *nuancing*. During 1954–1973 Moisil introduced the $\theta$-valued LM algebras without negation, applied multiple-valued logics to switching theory and studied algebraic properties of LM algebras (representation, ideals, residuation) ([228]). Moisil's works have been continued by many mathematicians ([149, 151]). A. Iorgulescu introduced and studied $\theta$-valued LM algebras with negation ([170]), while V. Boicescu defined and investigated $n$-valued LM algebras without negation ([13]).

Today these multiple-valued logics have been developed into fuzzy logics, which connect quantum mechanics, mathematical logic, probability theory, algebra and soft computing.

In 1958 Chang defined *MV-algebras* ([38]) as the algebraic counterpart of $\aleph_0$-valued Łukasiewicz logic and he gave another completeness proof of this logic ([39]).

An *MV-algebra* is an algebra $(A, \oplus, ^-, 0)$ with a binary operation $\oplus$, a unary operation $^-$ and a constant 0 satisfying the following equations:

$(MV_1)$ $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;
$(MV_2)$ $x \oplus y = y \oplus x$;
$(MV_3)$ $x \oplus 0 = x$;
$(MV_4)$ $(x^-)^- = x$;
$(MV_5)$ $x \oplus 0^- = 0^-$;
$(MV_6)$ $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

Studies on MV-algebras have been developed in [5–8, 22, 77, 81, 87, 89, 91, 120, 139, 146, 147, 153, 213, 214, 217–219, 247].

Starting from the systems of positive implicational calculus, weak systems of positive implicational calculus and BCI and BCK systems, in 1966 Y. Imai and K. Iséki introduced the BCK-algebras ([168]).

In 1977 R. Grigolia introduced $MV_n$-algebras to model the $n$-valued Łukasiewicz logic ([157]) and it was proved that there is a connection between $n$-valued Łukasiewicz algebras and $MV_n$-algebras ([171–173, 191, 216]).

One of the most famous results in the theory of MV-algebras was Mundici's theorem from 1986 which states that the category of MV-algebras is equivalent to the category of Abelian $\ell$-groups with strong unit ([229]).

The non-commutative generalizations of MV-algebras called *pseudo-MV algebras* were introduced by G. Georgescu and A. Iorgulescu in [135] and [137] and they can be regarded as algebraic semantics for a non-commutative generalization of a multiple-valued reasoning ([215]). The pseudo-MV algebras were introduced independently by J. Rachůnek ([241]) under the name of *generalized MV-algebras*.

A. Dvurečenskij proved in [97] that any pseudo-MV algebra is isomorphic with some interval in an $\ell$-group with strong unit, that is, the category of pseudo-MV algebras is equivalent to the category of unital $\ell$-groups.

Residuation is a fundamental concept of ordered structures and categories and Ward and Dilworth were the first to introduce the concept of a *residuated lattice* as a generalization of ideal lattices of rings ([262]). The theory of residuated lattices was used to develop algebraic counterparts of fuzzy logics ([256]) and substructural logics ([234]).

A residuated lattice is defined as an algebra $\mathcal{A} = (A, \wedge, \vee, \odot, \to, \rightsquigarrow, e)$ of type $(2, 2, 2, 2, 2, 0)$ satisfying the following conditions:

$(A_1)$ $(A, \wedge, \vee)$ is a lattice;
$(A_2)$ $(A, \odot, e)$ is a monoid;
$(A_3)$ $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x, y, z \in A$ (*pseudo-residuation*).

A residuated lattice with a constant 0 (which can denote any element) is called a *pointed residuated lattice* or *full Lambek algebra* (*FL-algebra*, for short). If $x \leq e$ for all $x \in A$, then $\mathcal{A}$ is called an *integral residuated lattice*. An FL-algebra $\mathcal{A}$ which satisfies the condition $0 \leq x \leq e$ for all $x \in A$ is called *$FL_w$-algebra* or *bounded integral residuated lattice* ([129]). In this case we put $e = 1$, so that an $\mathrm{FL}_w$-algebra will be denoted $(A, \wedge, \vee, \odot, \to, \rightsquigarrow, 0, 1)$. Clearly, if $\mathcal{A}$ is an $\mathrm{FL}_w$-algebra, then $(A, \wedge, \vee, 0, 1)$ is a bounded lattice.

In order to formalize the multiple-valued logics induced by continuous t-norms on the real unit interval $[0, 1]$, P. Hájek introduced in 1998 a very general multiple-valued logic, called *Basic Logic* (or BL) ([158]). Basic Logic turns out to be a common ingredient in three important multiple-valued logics: $\aleph_0$-valued Łukasiewicz logic, Gödel logic and Product logic. The Lindenbaum-Tarski algebras for Basic Logic are called *BL-algebras* ([23, 82, 220–222, 255–257]). Apart from their logical interest, BL-algebras have important algebraic properties and they have been intensively studied from an algebraic point of view.

The well-known result that a t-norm on $[0, 1]$ has residuum if and only if the t-norm is left-continuous makes clear that BL is not the most general t-norm based logic. In fact, a weaker logic than BL, called *Monoidal t-norm based logic* (MTL, for short) was defined in [117] and proved in [197] to be the logic of left-continuous t-norms and their residua. The algebraic counterpart of this logic is MTL-algebra, also introduced in [117].

G. Georgescu and A. Iorgulescu introduced in [136] the *pseudo-BL algebras* as a natural generalization of BL-algebras in the non-commutative case. A pseudo-BL algebra is an $\mathrm{FL}_w$-algebra which satisfies the conditions:

$(A_4)$ $(x \to y) \odot x = x \odot (x \rightsquigarrow y) = x \wedge y$ (*pseudo-divisibility*);
$(A_5)$ $(x \to y) \vee (y \to x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$ (*pseudo-prelinearity*).

Properties of pseudo-BL algebras were deeply investigated by A. Di Nola, G. Georgescu and A. Iorgulescu in [85] and [86]. Some classes of pseudo-BL algebras were investigated in [143] and the corresponding propositional logic was established by Hájek in [158] and [159].

A more general structure than the pseudo-BL algebra is the *weak pseudo-BL algebra* or *pseudo-MTL algebra* introduced by P. Flondor, G. Georgescu and A. Iorgulescu in [122]. Pseudo-MTL algebras are $\mathrm{FL}_w$-algebras satisfying condition $(A_5)$ and they include as a particular case the *weak BL-algebras* which is an alternative name for MTL-algebras.

Properties of pseudo-MTL algebras are also studied in [46, 144, 181].

An $\mathrm{FL}_w$-algebra which satisfies condition $(A_4)$ is called a *divisible residuated lattice* or *bounded R$\ell$-monoid*. Properties of divisible residuated lattices were studied by A. Dvurečenskij, J. Rachůnek and J. Kühr ([105, 111, 205, 240]).

Pseudo-BCK algebras were introduced in 2001 by G. Georgescu and A. Iorgulescu ([138]) as non-commutative generalizations of BCK-algebras. Properties of pseudo-BCK algebras and their connection with other fuzzy structures were established by A. Iorgulescu in [179–182].

For a guide through the pseudo-BCK algebras realm we refer the reader to the monograph [186].

Another generalization of pseudo-BL algebras was given in [148], where *pseudo-hoops* were defined and studied. Pseudo-hoops were originally introduced by Bosbach in [15] and [16] under the name of *complementary semigroups*. It was proved that a pseudo-hoop has the pseudo-divisibility condition and it is a meet-semilattice, so a bounded R$\ell$-monoid can be viewed as a bounded pseudo-hoop together with the join-semilattice property. In other words, a bounded pseudo-hoop is a meet-semilattice ordered residuated, integral and divisible monoid.

Other topics in multiple-valued logic algebras have been studied in [34, 36, 92, 132, 141, 150, 248].

The notion of a *state* is an analogue of a probability measure and it has a very important role in the theory of quantum structures ([108]). The basic idea of states is an averaging of events (elements) of a given algebraic structure. Since in the case of Łukasiewicz $\infty$-valued logic the set of events has the structure of an MV-algebra, the theory of probability on this logic is based on the notion of a state defined on an MV-algebra. Besides mathematical logic, Riečan and Neubrunn studied MV-algebras as fields of events in generalized probability theory ([250]). Therefore, the study of states on MV-algebras is a very active field of research ([40, 83, 84, 119, 133, 246]) which arises from the general problem of investigating probabilities defined for logical systems.

States on an MV-algebra $(A, \oplus, ^-, 0)$ were first introduced by D. Mundici in [230] as functions $s : A \longrightarrow [0, 1]$ satisfying the conditions:

$s(1) = 1$ (*normality*);
$s(x \oplus y) = s(x) + s(y)$ if $x \odot y = 0$ (*additivity*),

where $x \odot y = (x^- \oplus y^-)^-$.

They are analogous to finitely additive probability measures on Boolean algebras and play a crucial role in MV-algebraic probability theory ([249]).

States on other commutative and non-commutative algebraic structures have been defined and investigated by many authors ([20, 21, 102, 133, 134, 140, 142, 258, 259]).

The aim of this book is to present new results regarding non-commutative multiple-valued logic algebras and some of their applications. Almost all the results are based on the author's recent papers ([42–75]).

The book consists of nine chapters.

The Chap. 1 is devoted to pseudo-BCK algebras. After presenting the basic definitions and properties, we prove new properties of pseudo-BCK algebras with pseudo-product and pseudo-BCK algebras with pseudo-double negation. Examples of proper pseudo-BCK algebras, good pseudo-BCK algebras and pseudo-BCK lattices are given, and the orthogonal elements in a pseudo-BCK algebra are characterized. Finally, we define the maximal and normal deductive systems of a pseudo-BCK algebra with pseudo-product and we study their properties.

In Chap. 2 we recall the basic properties of pseudo-hoops, we introduce the notions of join-center and cancellative-center of pseudo-hoops and we define and study algebras on subintervals of pseudo-hoops. Additionally, new properties of a pseudo-hoop are proved.

Chapter 3 is devoted to residuated lattices. We investigate the properties of the Boolean center of an $\mathrm{FL}_w$-algebra and we define and study the directly indecomposable $\mathrm{FL}_w$-algebras. One of the main results consists of proving that any linearly ordered $\mathrm{FL}_w$-algebra is directly indecomposable. Finally, we define and study $\mathrm{FL}_w$-algebras of fractions relative to a meet-closed system.

In Chap. 4 we present some specific properties of other non-commutative multiple-valued logic algebras: pseudo-MTL algebras, bounded R$\ell$-monoids, pseudo-BL algebras and pseudo-MV algebras. As main results, we extend to the case of pseudo-MTL algebras some results regarding prime filters proved for pseudo-BL algebras. The Glivenko property for a good pseudo-BCK algebra is defined and it is shown that a good pseudo-hoop has the Glivenko property.

Chapter 5 deals with special classes of non-commutative residuated structures: local, perfect and Archimedean structures. The local bounded pseudo-BCK(pP) algebras are characterized in terms of primary deductive systems, while the perfect pseudo-BCK(pP) algebras are characterized in terms of perfect deductive systems. One of the main results consists of proving that the radical of a bounded pseudo-BCK(pP) algebra is a normal deductive system. We also prove that any linearly ordered pseudo-BCK(pP) algebra and any locally finite pseudo-BCK(pP) algebra are local. Other results state that any local $\mathrm{FL}_w$-algebra and any locally finite $\mathrm{FL}_w$-algebra are directly indecomposable. The classes of Archimedean and hyperarchimedean $\mathrm{FL}_w$-algebras are introduced and it is proved that any locally finite $\mathrm{FL}_w$-algebra is hyperarchimedean and any hyperarchimedean $\mathrm{FL}_w$-algebra is Archimedean.

Chapter 6 is devoted to the presentation of states on multiple-valued logic algebras. We introduce the notion of states on pseudo-BCK algebras and we study their properties. One of the main results consists of proving that any Bosbach state on a good pseudo-BCK algebra is a Riečan state, however the converse turns out not to be true. We also prove that every Riečan state on a good pseudo-BCK algebra with pseudo-double negation is a Bosbach state. In contrast to the case of pseudo-BL algebras, we show that there exist linearly ordered pseudo-BCK algebras having no Bosbach states and that there exist pseudo-BCK algebras having normal filters which are maximal, but having no Bosbach states.

Some specific properties of states on $\mathrm{FL}_w$-algebras, pseudo-MTL algebras, bounded R$\ell$-monoids and subinterval algebras of pseudo-hoops are proved.

A special section is dedicated to the existence of states on the residuated structures, showing that every perfect $\mathrm{FL}_w$-algebra admits at least a Bosbach state and every perfect pseudo-BL algebra has a unique state-morphism.

Finally, we introduce the notion of a local state on a perfect pseudo-MTL algebra and we prove that every local state can be extended to a Riečan state.

In Chap. 7 we generalize measures on BCK algebras introduced by A. Dvurečenskij in [94] and [108] to pseudo-BCK algebras that are not necessarily bounded. In particular, we show that if $A$ is a downwards-directed pseudo-BCK algebra and $m$ a measure on it, then the quotient over the kernel of $m$ can be embedded into the negative cone of an Abelian, Archimedean $\ell$-group as its subalgebra. This result will enable us to characterize nonzero measure-morphisms on downwards-directed pseudo-BCK algebras as measures whose kernel is a maximal filter. We study state-measures on pseudo-BCK algebras with strong unit and we show how to characterize state-measure-morphisms as extremal state-measures or as state-measures whose kernel is a maximal filter. In particular, we show that for unital pseudo-BCK algebras that are downwards-directed, the quotient over the kernel can be embedded into the negative cone of an Abelian, Archimedean $\ell$-group with strong unit. We generalize to pseudo-BCK algebras the identity between de Finetti maps and Bosbach states, following the results proved by Kühr and Mundici in [211] who showed that de Finetti's coherence principle, which has its origins in Dutch bookmaking, has a strong relationship with MV-states on MV-algebras. We also generalize this for state-measures on unital pseudo-BCK algebras that are downwards-directed.

Chapter 8 is devoted to generalized states on residuated structures. The study of these generalized states is motivated by their interpretation as a new type of semantics for non-commutative fuzzy logics. Usually, the truth degree of sentences in a fuzzy logic is a number in the interval [0, 1] or, more generally, an element of an $\mathrm{FL}_w$-algebra. Similarly, for generalized states, the probability of sentences is evaluated in an arbitrary $\mathrm{FL}_w$-algebra.

We define the generalized states of type I and type II and generalized state-morphisms and we study the relationship between them. We prove that any perfect $\mathrm{FL}_w$-algebra admits strong type I and type II states. Some conditions are given for a generalized state of type I on a linearly ordered bounded $\mathrm{R}\ell$-monoid to be a state operator. The notion of a strong perfect $\mathrm{FL}_w$-algebra is introduced and it is proved that any strong perfect $\mathrm{FL}_w$-algebra admits a generalized state-morphism. The notion of a generalized Riečan state is also introduced and the main results are proved based on the Glivenko property defined for the non-commutative case. The main results consist of proving that any order-preserving type I state is a generalized Riečan state and in some particular conditions the two states coincide. We introduce the notion of a generalized local state on a perfect pseudo-MTL algebra $A$ and we prove that, if $A$ is relatively free of zero divisors, then every generalized local state can be extended to a generalized Riečan state.

Chapter 9 deals with residuated structures with internal states. We define the notions of state operator, strong state operator, state-morphism operator, weak state-morphism operator and we study their properties. We prove that every strong state pseudo-hoop is a state pseudo-hoop and any state operator on an idempotent pseudo-hoop is a weak state-morphism operator. It is proved that for an idempotent pseudo-hoop $A$ a state operator on $\mathrm{Reg}(A)$ can be extended to a state operator on $A$. One of the main results of this chapter consists of proving that every perfect pseudo-hoop admits a nontrivial state operator. Other results compare the state operators with states and generalized states on a pseudo-hoop. Some conditions are given for a state operator to be a generalized state and for a generalized state to be a state operator.

We hope that this book will be useful to graduate students and researchers in the area of algebras of multiple-valued logics.

I wish to firstly thank my adviser George Georgescu for guiding many of my steps in this field.

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On a personal note, I am very grateful to my parents for all their support and encouragement over the years.

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Iowa City, USA Lavinia Corina Ciungu
May 2013

# Contents

# Chapter 1
# Pseudo-BCK Algebras

BCK algebras were originally introduced by K. Isèki in [194] with a binary operation $*$ modeling the set-theoretical difference and with a constant element 0, that is, a least element. Another motivation is from classical and non-classical propositional calculi modeling logical implications. Such algebras contain as a special subfamily the family of MV-algebras where some important fuzzy structures can be studied. For more about BCK algebras, see [167, 174–179, 182–187, 189, 192, 193, 225].

Pseudo-BCK algebras were introduced by G. Georgescu and A. Iorgulescu in [138] as algebras with "two differences", a left- and right-difference, instead of one $*$ and with a constant element 0 as the least element. In [112], a special subclass of pseudo-BCK algebras, called Łukasiewicz pseudo-BCK algebras, was introduced and it was shown that each such algebra is always a subalgebra of the positive cone of some $\ell$-group (not necessarily Abelian). The class of Łukasiewicz pseudo-BCK algebras is a variety whereas the class of pseudo-BCK algebras is not; it is only a quasivariety because it is not closed under homomorphic images. Nowadays pseudo-BCK algebras are used in a dual form, with two implications, $\to$ and $\rightsquigarrow$ and with one constant element 1, that is the greatest element. Thus such pseudo-BCK algebras are in the "negative cone" and are also called "left-ones". Further properties of pseudo-BCK algebras and their connection with other fuzzy structures were established by A. Iorgulescu in [179–182]. For a guide through the pseudo-BCK algebras realm, see the monograph [186]. Studies on pseudo-BCK algebras were also developed in [107, 163, 190, 206, 208–210].

In this chapter we prove new properties of pseudo-BCK algebras with pseudo-product and pseudo-BCK algebras with pseudo-double negation and we show that every pseudo-BCK algebra can be extended to a good one. Examples of proper pseudo-BCK algebras, good pseudo-BCK algebras and pseudo-BCK lattices are given and the orthogonal elements in a pseudo-BCK algebra are characterized. Finally, we define the maximal and normal deductive systems of a pseudo-BCK algebra with pseudo-product and we study their properties.

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## 1.1 Definitions and Properties

**Definition 1.1** A *pseudo-BCK algebra* (more precisely, *reversed left-pseudo-BCK algebra*) is a structure $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ where $\leq$ is a binary relation on $A$, $\rightarrow$ and $\rightsquigarrow$ are binary operations on $A$ and 1 is an element of $A$ satisfying, for all $x, y, z \in A$, the axioms:

($psBCK_1$) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$;
($psBCK_2$) $x \leq (x \rightarrow y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \rightarrow y$;
($psBCK_3$) $x \leq x$;
($psBCK_4$) $x \leq 1$;
($psBCK_5$) if $x \leq y$ and $y \leq x$, then $x = y$;
($psBCK_6$) $x \leq y$ iff $x \rightarrow y = 1$ iff $x \rightsquigarrow y = 1$.

A pseudo-BCK algebra $\mathcal{A} = (A, \leq, \rightarrow, \rightsquigarrow, 1)$ is *commutative* if $\rightarrow = \rightsquigarrow$. Any commutative pseudo-BCK algebra is a BCK-algebra.

In the sequel we will refer to the pseudo-BCK algebra $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ by its universe $A$.

**Proposition 1.1** *The structure* $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ *is a pseudo-BCK algebra iff the algebra* $(A, \rightarrow, \rightsquigarrow, 1)$ *of type* $(2, 2, 0)$ *satisfies the following identities and quasi-identity*:

($psBCK'_1$) $(x \rightarrow y) \rightsquigarrow [(y \rightarrow z) \rightsquigarrow (x \rightarrow z)] = 1$;
($psBCK'_2$) $(x \rightsquigarrow y) \rightarrow [(y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)] = 1$;
($psBCK'_3$) $1 \rightarrow x = x$;
($psBCK'_4$) $1 \rightsquigarrow x = x$;
($psBCK'_5$) $x \rightarrow 1 = 1$;
($psBCK'_6$) $(x \rightarrow y = 1$ *and* $y \rightarrow x = 1)$ *implies* $x = y$.

*Proof* Obviously, any pseudo-BCK algebra satisfies ($psBCK'_1$)–($psBCK'_6$).

Conversely, assume that an algebra $(A, \rightarrow, \rightsquigarrow, 1)$ satisfies ($psBCK'_1$)–($psBCK'_6$).

Applying ($psBCK'_3$) and ($psBCK'_1$) we get:

$$x \rightsquigarrow \big[(x \rightarrow y) \rightsquigarrow y\big] = (1 \rightarrow x) \rightsquigarrow \big[(x \rightarrow y) \rightsquigarrow (1 \rightarrow y)\big] = 1.$$

Similarly, by ($psBCK'_4$) and ($psBCK'_2$) we have:

$$x \rightarrow \big[(x \rightsquigarrow y) \rightarrow y\big] = (1 \rightsquigarrow x) \rightarrow \big[(x \rightsquigarrow y) \rightarrow (1 \rightsquigarrow y)\big] = 1.$$

Applying ($psBCK'_3$) and ($psBCK'_2$) we have:

$$x \rightarrow x = 1 \rightarrow (x \rightarrow x) = (1 \rightsquigarrow 1) \rightarrow \big[(1 \rightsquigarrow x) \rightarrow (1 \rightsquigarrow x)\big] = 1.$$

Similarly, by ($psBCK'_4$) and ($psBCK'_1$) we get:

$$x \rightsquigarrow x = 1 \rightsquigarrow (x \rightsquigarrow x) = (1 \rightarrow 1) \rightsquigarrow \big[(1 \rightarrow x) \rightsquigarrow (1 \rightarrow x)\big] = 1.$$

Moreover, if $x \rightarrow y = 1$ then $x \rightsquigarrow y = x \rightsquigarrow [(x \rightarrow y) \rightsquigarrow y] = 1$ and similarly, if $x \rightsquigarrow y = 1$ then $x \rightarrow y = x \rightarrow [(x \rightsquigarrow y) \rightarrow y] = 1$.

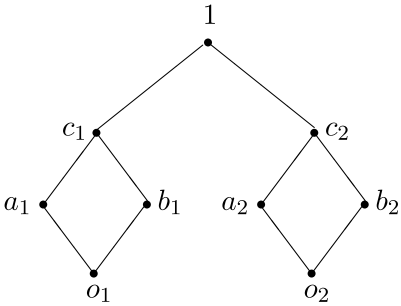


**Fig. 1.1** Example of proper pseudo-BCK algebra

It follows that $x \to y = 1$ iff $x \rightsquigarrow y = 1$.

We deduce that the relation $\leq$ defined by $x \leq y$ iff $x \to y = 1$ is a partial order on $A$ which makes $(A, \leq, \to, \rightsquigarrow, 1)$ a pseudo-BCK algebra. □

In the sequel, we shall use either $(A, \leq, \to, \rightsquigarrow, 1)$ or $(A, \to, \rightsquigarrow, 1)$ for a pseudo-BCK algebra.

*Example 1.1* Consider $A = \{o_1, a_1, b_1, c_1, o_2, a_2, b_2, c_2, 1\}$ with $o_1 < a_1, b_1 < c_1 < 1$ and $a_1$, $b_1$ incomparable, $o_2 < a_2, b_2 < c_2 < 1$ and $a_2$, $b_2$ incomparable. Assume that any element of the set $\{o_1, a_1, b_1, c_1\}$ is incomparable with any element of the set $\{o_2, a_2, b_2, c_2\}$ (see Fig. 1.1).

Consider the operations $\to$, $\rightsquigarrow$ given by the following tables:

| $\to$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $o_1$ | $1$ | $1$ | $1$ | $1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
| $a_1$ | $o_1$ | $1$ | $b_1$ | $1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
| $b_1$ | $a_1$ | $a_1$ | $1$ | $1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
| $c_1$ | $o_1$ | $a_1$ | $b_1$ | $1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
| $o_2$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_2$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $o_2$ | $1$ | $b_2$ | $1$ | $1$ |
| $b_2$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $c_2$ | $c_2$ | $1$ | $1$ | $1$ |
| $c_2$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $o_2$ | $c_2$ | $b_2$ | $1$ | $1$ |
| $1$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |

| $\rightsquigarrow$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|
| $o_1$ | $1$ | $1$ | $1$ | $1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
| $a_1$ | $b_1$ | $1$ | $b_1$ | $1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
| $b_1$ | $o_1$ | $a_1$ | $1$ | $1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
| $c_1$ | $o_1$ | $a_1$ | $b_1$ | $1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |
| $o_2$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_2$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $b_2$ | $1$ | $b_2$ | $1$ | $1$ |
| $b_2$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $b_2$ | $c_2$ | $1$ | $1$ | $1$ |
| $c_2$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $b_2$ | $c_2$ | $b_2$ | $1$ | $1$ |
| $1$ | $o_1$ | $a_1$ | $b_1$ | $c_1$ | $o_2$ | $a_2$ | $b_2$ | $c_2$ | $1$ |

Then $(A, \leq, \to, \rightsquigarrow, 1)$ is a proper pseudo-BCK algebra.

We recall the definition of an $\ell$-group. The language of lattice-ordered groups ($\ell$-groups) involves both the group operations and the binary lattice operations.

By a *lattice-ordered group* (*$\ell$-group*) we will mean an ordered group $(G, \leq)$ such that $(G, \leq)$ is a lattice. The $\ell$-group $G$ is called an *$\ell u$-group* if there exists an element $u > 0$ such that for any $x \in G$ there is an $n \in \mathbb{N}$ such that $x \leq nu$. The element $u$ is called a *strong unit*.

For details regarding $\ell$-groups we refer the reader to [2, 12, 76].

*Example 1.2* Let $(G, \vee, \wedge, +, -, 0)$ be an $\ell$-group.

On the negative cone $G^- = \{g \in G \mid g \leq 0\}$ we define:

$$g \to h := h - (g \vee h) = (h - g) \wedge 0,$$
$$g \rightsquigarrow h := -(g \vee h) + h = (-g + h) \wedge 0.$$

Then $(G^-, \leq, \to, \rightsquigarrow, 0)$ is a pseudo-BCK algebra.

*Remark 1.1* (Definition of union) Let $(A_i, \leq, \to_i, \rightsquigarrow_i, 1_i)_{i \in I}$ be a collection of pseudo-BCK algebras such that:

(i) $1_i = 1$ for all $i \in I$,
(ii) $A_i \cap A_j = \{1\}$ for all $i, j \in I, i \neq j$.

Let $A = \bigcup_{i \in I} A_i$ and define:

$$x \to y := \begin{cases} x \to_i y & \text{if } x, y \in A_i, i \in I \\ y & \text{otherwise,} \end{cases}$$
$$x \rightsquigarrow y := \begin{cases} x \rightsquigarrow_i y & \text{if } x, y \in A_i, i \in I \\ y & \text{otherwise.} \end{cases}$$

Then $(A, \leq, \to, \rightsquigarrow, 1)$ is a pseudo-BCK algebra called the *union* of the pseudo-BCK algebras $(A_i, \leq, \to_i, \rightsquigarrow_i, 1_i)_{i \in I}$.

Note that the notion of union defined above is not related to the notion of ordinal sum defined in Chap. 2.

**Proposition 1.2** *In any pseudo-BCK algebra $A$ the following properties hold*:

($psbck$-$c_1$) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$;
($psbck$-$c_2$) $x \leq y$, $y \leq z$ *implies* $x \leq z$;
($psbck$-$c_3$) $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$, $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$;
($psbck$-$c_4$) $z \leq y \to x$ *iff* $y \leq z \rightsquigarrow x$;
($psbck$-$c_5$) $z \to x \leq (y \to z) \to (y \to x)$, $z \rightsquigarrow x \leq (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)$;
($psbck$-$c_6$) $x \leq y \to x$, $x \leq y \rightsquigarrow x$;

($psbck$-$c_7$) $1 \to x = x = 1 \rightsquigarrow x$;
($psbck$-$c_8$) $x \to x = x \rightsquigarrow x = 1$;
($psbck$-$c_9$) $x \to 1 = x \rightsquigarrow 1 = 1$;
($psbck$-$c_{10}$) *$x \le y$ implies $z \to x \le z \to y$ and $z \rightsquigarrow x \le z \rightsquigarrow y$;*
($psbck$-$c_{11}$) $[(y \to x) \rightsquigarrow x] \to x = y \to x$, $[(y \rightsquigarrow x) \to x] \rightsquigarrow x = y \rightsquigarrow x$.

*Proof*

($psbck$-$c_1$) Since $x \le y$, applying ($psBCK_6$), ($psBCK_1$) and ($psBCK_4$) we get $1 = x \to y \le (y \to z) \rightsquigarrow (x \to z)$, so $(y \to z) \rightsquigarrow (x \to z) = 1$ for all $z \in A$.
Applying ($psBCK_6$) again we get $y \to z \le x \to z$.
Similarly, $y \rightsquigarrow z \le x \rightsquigarrow z$.

($psbck$-$c_2$) By ($psbck$-$c_1$), $x \le y$ implies $y \to z \le x \to z$. Since $y \le z$ we have $y \to z = 1$, so $x \to z = 1$. Applying ($psBCK_6$) we get $x \le z$.

($psbck$-$c_3$) Applying ($psBCK_1$) we have $y \to x \le (x \to z) \rightsquigarrow (y \to z)$ and by ($psbck$-$c_1$) we get $[(x \to z) \rightsquigarrow (y \to z)] \rightsquigarrow u \le (y \to x) \rightsquigarrow u$ for any $u \in A$.
From this inequality, replacing $z$ with $u \rightsquigarrow z$, $x$ with $x \rightsquigarrow z$ and $u$ with $(u \rightsquigarrow x) \rightsquigarrow [y \to (u \rightsquigarrow z)]$ we get

$$\Big[\big[(x \rightsquigarrow z) \to (u \rightsquigarrow z)\big] \rightsquigarrow \big[y \to (u \rightsquigarrow z)\big]\Big] \rightsquigarrow \Big[(u \rightsquigarrow x) \rightsquigarrow \big[y \to (u \rightsquigarrow z)\big]\Big]$$
$$\le \big[y \to (x \rightsquigarrow z)\big] \rightsquigarrow \Big[(u \rightsquigarrow x) \rightsquigarrow \big[y \to (u \rightsquigarrow z)\big]\Big].$$

By ($psBCK_1$) we have $u \rightsquigarrow x \le (x \rightsquigarrow z) \to (u \rightsquigarrow z)$ and applying ($psbck$-$c_1$) it follows that the left-hand side of the above inequality is equal to 1.
Thus the right-hand side is also equal to 1, so $y \to (x \rightsquigarrow z) \le (u \rightsquigarrow x) \rightsquigarrow [y \to (u \rightsquigarrow z)]$.
Replacing $x$ with $y \to z$ and $u$ with $x$ we get

$$y \to \big[(y \to z) \rightsquigarrow z\big] \le \big[x \rightsquigarrow (y \to z)\big] \rightsquigarrow \big[y \to (x \rightsquigarrow z)\big].$$

But, by ($psBCK_2$) we have $y \le (y \to z) \rightsquigarrow z$, so $y \to [(y \to z) \rightsquigarrow z] = 1$.
It follows that $[x \rightsquigarrow (y \to z)] \rightsquigarrow [y \to (x \rightsquigarrow z)] = 1$.
Therefore $x \rightsquigarrow (y \to z) \le y \to (x \rightsquigarrow z)$.
On the other hand, by ($psBCK_2$) we have $x \le (x \rightsquigarrow z) \to z$ and applying ($psbck$-$c_1$) we get $[(x \rightsquigarrow z) \to z] \rightsquigarrow (y \to z) \le x \rightsquigarrow (y \to z)$.
By ($psBCK_1$) we have $y \to x \le (x \to z) \rightsquigarrow (y \to z)$ and replacing $x$ with $x \rightsquigarrow z$ we get $y \to (x \rightsquigarrow z) \le [(x \rightsquigarrow z) \to z] \rightsquigarrow (y \to z) \le x \rightsquigarrow (y \to z)$.
We conclude that $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$.
Similarly, $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$.

($psbck$-$c_4$) From $z \le y \to x$, by ($psBCK_2$) and ($psbck$-$c_1$) we have

$$y \le (y \to x) \rightsquigarrow x \le z \rightsquigarrow x.$$

Similarly, from $y \le z \rightsquigarrow x$ we get $z \le (z \rightsquigarrow x) \to x \le y \to x$.

(*psbck-c*$_5$) Applying (*psBCK*$_1$) we have $y \to z \le (z \to x) \rightsquigarrow (y \to x)$ and according to (*psbck-c*$_1$) we get

$$\big[(z \to x) \rightsquigarrow (y \to x)\big] \to (y \to x) \le (y \to z) \to (y \to x).$$

By (*psBCK*$_2$) it follows that $z \to x \le [(z \to x) \rightsquigarrow (y \to x)] \to (y \to x)$, and applying (*psbck-c*$_2$) we conclude that $z \to x \le (y \to z) \to (y \to x)$.
Similarly, from $y \rightsquigarrow z \le (z \rightsquigarrow x) \to (y \rightsquigarrow x)$ we get $z \rightsquigarrow x \le (y \rightsquigarrow z) \rightsquigarrow (y \rightsquigarrow x)$.

(*psbck-c*$_6$) Since $y \le 1 = x \to x$, it follows by (*psbck-c*$_4$) that $x \le y \rightsquigarrow x$.
Similarly, from $y \le 1 = x \rightsquigarrow x$ we get $x \le y \to x$.

(*psbck-c*$_7$) By (*psbck-c*$_6$) we have $x \le 1 \to x$ and $x \le 1 \rightsquigarrow x$.
By (*psBCK*$_2$) we get $1 \le (1 \to x) \rightsquigarrow x$ and $1 \le (1 \rightsquigarrow x) \to x$.
It follows that $(1 \to x) \rightsquigarrow x = 1$ and $(1 \rightsquigarrow x) \to x = 1$, so $1 \to x \le x$ and $1 \rightsquigarrow x \le x$. Thus $1 \to x = x = 1 \rightsquigarrow x$.

(*psbck-c*$_8$) and (*psbck-c*$_9$) are consequences of the axiom (*psBCK*$_6$).

(*psbck-c*$_{10}$) Applying (*psbck-c*$_7$), (*psBCK*$_6$) and (*psBCK*$_1$) we have:

$$z \to y = 1 \rightsquigarrow (z \to y) = (x \to y) \rightsquigarrow (z \to y) \ge z \to x \quad \text{and}$$
$$z \rightsquigarrow y = 1 \to (z \rightsquigarrow y) = (x \rightsquigarrow y) \to (z \rightsquigarrow y) \ge z \rightsquigarrow x.$$

(*psbck-c*$_{11}$) By (*psBCK*$_2$) we have $y \le (y \to x) \rightsquigarrow x$ and $y \le (y \rightsquigarrow x) \to x$.
Applying (*psbck-c*$_1$) we get

$$\big[(y \to x) \rightsquigarrow x\big] \to x \le y \to x \quad \text{and} \quad \big[(y \rightsquigarrow x) \to x\big] \rightsquigarrow x \le y \rightsquigarrow x.$$

On the other hand, by (*psBCK*$_2$) we have:

$$y \to x \le \big[(y \to x) \rightsquigarrow x\big] \to x \quad \text{and} \quad y \rightsquigarrow x \le \big[(y \rightsquigarrow x) \to x\big] \rightsquigarrow x.$$

We conclude that

$$\big[(y \to x) \rightsquigarrow x\big] \to x = y \to x \quad \text{and} \quad \big[(y \rightsquigarrow x) \to x\big] \rightsquigarrow x = y \rightsquigarrow x. \qquad \square$$

**Proposition 1.3** *Let* $(A, \le, \to, \rightsquigarrow, 1)$ *be a pseudo-BCK algebra.*
*If* $\bigvee_{i \in I} x_i$ *exists, then so does* $\bigwedge_{i \in I}(x_i \to y)$ *and* $\bigwedge_{i \in I}(x_i \rightsquigarrow y)$ *and we have*:

(*psbck-c*$_{12}$) $(\bigvee_{i \in I} x_i) \to y = \bigwedge_{i \in I}(x_i \to y)$, $(\bigvee_{i \in I} x_i) \rightsquigarrow y = \bigwedge_{i \in I}(x_i \rightsquigarrow y)$.

*Proof* If we let $x = \bigvee_{i \in I} x_i$, it follows that $x_i \le x$ and applying (*psbck-c*$_1$) we have $x \to y \le x_i \to y$ for all $i \in I$. Let $z$ be a lower bound of $\{x_i \to y \mid i \in I\}$. Then, by (*psbck-c*$_4$), $z \le x_i \to y$ implies $x_i \le z \rightsquigarrow y$ for all $i \in I$, so $x \le z \rightsquigarrow y$. Applying (*psbck-c*$_4$) again, we get $z \le x \to y$.
Thus $x \to y$ is the g.l.b. of $\{x_i \to y \mid i \in I\}$.
We conclude that $\bigwedge_{i \in I}(x_i \to y)$ exists and $(\bigvee_{i \in I} x_i) \to y = \bigwedge_{i \in I}(x_i \to y)$.
Similarly, $\bigwedge_{i \in I}(x_i \rightsquigarrow y)$ exists and $(\bigvee_{i \in I} x_i) \rightsquigarrow y = \bigwedge_{i \in I}(x_i \rightsquigarrow y)$. $\square$

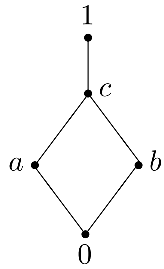


**Fig. 1.2** Example of bounded pseudo-BCK algebra

**Definition 1.2** If there is an element 0 of a pseudo-BCK algebra $(A, \leq, \rightarrow, \rightsquigarrow, 1)$, such that $0 \leq x$ (i.e. $0 \rightarrow x = 0 \rightsquigarrow x = 1$), for all $x \in A$, then 0 is called the *zero* of $A$. A pseudo-BCK algebra with zero is called a *bounded pseudo-BCK algebra* and it is denoted by $(A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$.

*Example 1.3* Consider $A = \{0, a, b, c, 1\}$ with $0 < a, b < c < 1$ and $a$, $b$ incomparable (see Fig. 1.2).

Consider the operations $\rightarrow$, $\rightsquigarrow$ given by the following tables:

| $\rightarrow$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $0$ | $1$ | $b$ | $1$ | $1$ |
| $b$ | $a$ | $a$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $a$ | $b$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $1$ |

| $\rightsquigarrow$ | $0$ | $a$ | $b$ | $c$ | $1$ |
|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $b$ | $1$ | $b$ | $1$ | $1$ |
| $b$ | $0$ | $a$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $a$ | $b$ | $1$ | $1$ |
| $1$ | $0$ | $a$ | $b$ | $c$ | $1$ |

Then $(A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ is a bounded pseudo-BCK algebra. (As we will see later, $A$ is even a pseudo-BCK lattice.)

Let $(A, \leq, \rightarrow, \rightsquigarrow, 0, 1)$ be a bounded pseudo-BCK algebra. We define two negations $^-$ and $^\sim$: for all $x \in A$,

$$x^- := x \rightarrow 0, \qquad x^\sim := x \rightsquigarrow 0.$$

In the sequel we will use the following notation:

$$x^{--} = \left(x^-\right)^-; \qquad x^{\sim\sim} = \left(x^\sim\right)^\sim; \qquad x^{-\sim} = \left(x^-\right)^\sim; \qquad x^{\sim-} = \left(x^\sim\right)^-.$$

*Example 1.4* Let $(G, \vee, \wedge, +, -, 0)$ be an $\ell$-group with a strong unit $u \geq 0$. On the interval $[-u, 0]$ we define:

$$x \rightarrow y := (y - x) \wedge 0, \qquad x \rightsquigarrow y := (-x + y) \wedge 0.$$

Then $([-u, 0], \leq, \rightarrow, \rightsquigarrow, -u, 0)$ is a bounded pseudo-BCK algebra with $x^- = -u - x$ and $x^\sim = -x - u$. In a similar way, $((-u, 0], \leq, \rightarrow, \rightsquigarrow, 0)$ is a pseudo-BCK algebra that is not bounded.

*Example 1.5* Let $(G, \vee, \wedge, +, -, 0)$ be an $\ell$-group with a strong unit $u \geq 0$. On the interval $[0, u]$ we define:

$$x \to y := (u - x + y) \wedge u, \qquad x \rightsquigarrow y := (y - x + u) \wedge u.$$

Then $([0, u], \leq, \to, \rightsquigarrow, 0, u)$ is a bounded pseudo-BCK algebra with $x^- = u - x$ and $x^\sim = -x + u$. If on $[0, u]$ we set $\to_1 = \rightsquigarrow$ and $\rightsquigarrow_1 = \to$, then $([0, u], \leq, \to_1, \rightsquigarrow_1, 0, u)$ is isomorphic with $([-u, 0], \leq, \to, \rightsquigarrow, -u, 0)$ under the isomorphism $x \mapsto x - u$, $x \in [0, u]$.

**Proposition 1.4** *In a bounded pseudo-BCK algebra the following hold*:

($psbck$-$c_{13}$) $1^- = 0 = 1^\sim$, $0^- = 1 = 0^\sim$;
($psbck$-$c_{14}$) $x \leq x^{-\sim}$, $x \leq x^{\sim-}$;
($psbck$-$c_{15}$) $x \to y \leq y^- \rightsquigarrow x^-$, $x \rightsquigarrow y \leq y^\sim \to x^\sim$;
($psbck$-$c_{16}$) $x \leq y$ *implies* $y^- \leq x^-$ *and* $y^\sim \leq x^\sim$;
($psbck$-$c_{17}$) $x \to y^\sim = y \rightsquigarrow x^-$ *and* $x \rightsquigarrow y^- = y \to x^\sim$;
($psbck$-$c_{18}$) $x^{-\sim-} = x^-$, $x^{\sim-\sim} = x^\sim$;
($psbck$-$c_{19}$) $x \to y^{-\sim} = y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim}$ *and* $x \rightsquigarrow y^{\sim-} = y^\sim \to x^\sim = x^{\sim-} \rightsquigarrow y^{\sim-}$;
($psbck$-$c_{20}$) $x \to y^\sim = y^{\sim-} \rightsquigarrow x^- = x^{-\sim} \to y^\sim$ *and* $x \rightsquigarrow y^- = y^{-\sim} \to x^\sim = x^{\sim-} \rightsquigarrow y^-$;
($psbck$-$c_{21}$) $(x \to y^{\sim-})^{\sim-} = x \to y^{\sim-}$ *and* $(x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}$.

*Proof*

($psbck$-$c_{13}$) Since $0 \leq 0$, by ($psBCK_6$) we get $0 \to 0 = 1$ and $0 \rightsquigarrow 0 = 1$, that is, $0^- = 1$ and $0^\sim = 1$.
Taking $x = 1$ and $y = 0$ in ($psBCK_2$) we have $1 \leq (1 \to 0) \rightsquigarrow 0$, hence $(1 \to 0) \rightsquigarrow 0 = 1$. Thus by ($psBCK_6$) we get $1 \to 0 \leq 0$, so $1 \to 0 = 0$, i.e. $1^- = 0$. Similarly, $1^\sim = 0$.
($psbck$-$c_{14}$) This follows by taking $y = 0$ in ($psBCK_2$).
($psbck$-$c_{15}$) Applying ($psBCK_1$) for $z = 0$ we get:

$$x \to y \leq (y \to 0) \rightsquigarrow (x \to 0) = y^- \rightsquigarrow x^- \quad \text{and}$$
$$x \rightsquigarrow y \leq (y \rightsquigarrow 0) \to (x \rightsquigarrow 0) = y^\sim \to x^\sim.$$

($psbck$-$c_{16}$) From $x \leq y$, applying ($psbck$-$c_1$) we get $y \to 0 \leq x \to 0$, so $y^- \leq x^-$. Similarly, $y^\sim \leq x^\sim$.
($psbck$-$c_{17}$) By ($psbck$-$c_{15}$), ($psbck$-$c_{14}$) and ($psbck$-$c_1$) we get:

$$x \to y^\sim \leq y^{\sim-} \rightsquigarrow x^- \leq y \rightsquigarrow x^- \quad \text{and} \quad x \rightsquigarrow y^- \leq y^{-\sim} \to x^\sim \leq y \to x^\sim.$$

In the above inequalities we change $x$ and $y$ obtaining:

$$y \to x^\sim \leq x \rightsquigarrow y^- \quad \text{and} \quad y \rightsquigarrow x^- \leq x \to y^\sim.$$

Thus $x \to y^\sim = y \rightsquigarrow x^-$ and $x \rightsquigarrow y^- = y \to x^\sim$.

($psbck$-$c_{18}$) By ($psbck$-$c_{14}$) and ($psbck$-$c_{16}$) we get $x^{\sim-\sim} \le x^{\sim}$ and $x^{-\sim-} \le x^{-}$.
By ($psbck$-$c_{14}$), replacing $x$ with $x^{\sim}$ and $x^{-}$ we get $x^{\sim} \le x^{\sim-\sim}$ and $x^{-} \le x^{-\sim-}$, respectively. Thus $x^{\sim-\sim} = x^{\sim}$ and $x^{-\sim-} = x^{-}$.

($psbck$-$c_{19}$) By ($psbck$-$c_{17}$) we have: $y \rightsquigarrow x^{-} = x \to y^{\sim}$.
Replacing $y$ with $y^{-}$ we get: $y^{-} \rightsquigarrow x^{-} = x \to y^{-\sim}$.
Replacing $x$ by $x^{-\sim}$ in the last equality we get: $y^{-} \rightsquigarrow x^{-\sim-} = x^{-\sim} \to y^{-\sim}$.
Hence applying ($psbck$-$c_{18}$) it follows that: $y^{-} \rightsquigarrow x^{-} = x^{-\sim} \to y^{-\sim}$.
Thus $x \to y^{-\sim} = y^{-} \rightsquigarrow x^{-} = x^{-\sim} \to y^{-\sim}$.
Similarly, $x \rightsquigarrow y^{\sim-} = y^{\sim} \to x^{\sim} = x^{\sim-} \rightsquigarrow y^{\sim-}$.

($psbck$-$c_{20}$) The assertions follow by replacing in ($psbck$-$c_{19}$) $y$ with $y^{\sim}$ and $y$ with $y^{-}$, respectively and applying ($psbck$-$c_{18}$).

($psbck$-$c_{21}$) Applying ($psbck$-$c_{3}$) and ($psbck$-$c_{19}$) we have:

$$\begin{aligned}1 &= \left(x \to y^{\sim-}\right) \rightsquigarrow \left(x \to y^{\sim-}\right) = x \to \left(\left(x \to y^{\sim-}\right) \rightsquigarrow y^{\sim-}\right)\\ &= x \to \left(\left(x \to y^{\sim-}\right)^{\sim-} \rightsquigarrow y^{\sim-}\right) = \left(x \to y^{\sim-}\right)^{\sim-} \rightsquigarrow \left(x \to y^{\sim-}\right).\end{aligned}$$

Hence $(x \to y^{\sim-})^{\sim-} \le x \to y^{\sim-}$.
On the other hand, by ($psbck$-$c_{14}$) we have $x \to y^{\sim-} \le (x \to y^{\sim-})^{\sim-}$, thus $(x \to y^{\sim-})^{\sim-} = x \to y^{\sim-}$. Similarly, $(x \rightsquigarrow y^{-\sim})^{-\sim} = x \rightsquigarrow y^{-\sim}$. □

We recall some notions and results regarding pseudo-BCK semilattices (see [209]).

**Definition 1.3** A *pseudo-BCK join-semilattice* is an algebra $(A, \vee, \to, \rightsquigarrow, 1)$ such that $(A, \vee)$ is a join-semilattice, $(A, \to, \rightsquigarrow, 1)$ is a pseudo-BCK algebra and $x \to y = 1$ iff $x \vee y = y$.

*Remark 1.2* It is easy to show that an algebra $(A, \vee, \to, \rightsquigarrow, 1)$ of type $(2, 2, 2, 0)$ is a pseudo-BCK join-semilattice if and only if $(A, \vee)$ is a join-semilattice and $(A, \to, \rightsquigarrow, 1)$ satisfies ($psBCK'_1$)–($psBCK'_5$) and the following identities:

($psBCK'_7$) $x \vee [(x \to y) \rightsquigarrow y] = (x \to y) \rightsquigarrow y$;
($psBCK'_8$) $x \to (x \vee y) = 1$.

**Definition 1.4** A *pseudo-BCK meet-semilattice* is an algebra $(A, \wedge, \to, \rightsquigarrow, 1)$ such that $(A, \wedge)$ is a meet-semilattice, $(A, \to, \rightsquigarrow, 1)$ is a pseudo-BCK algebra and $x \to y = 1$ iff $x \wedge y = x$.

*Remark 1.3* It is easy to show that an algebra $(A, \wedge, \to, \rightsquigarrow, 1)$ of type $(2, 2, 2, 0)$ is a pseudo-BCK meet-semilattice if and only if $(A, \wedge)$ is a meet-semilattice and $(A, \to, \rightsquigarrow, 1)$ satisfies the identities ($psBCK'_1$)–($psBCK'_5$) and the identities:

($psBCK_7''$) $x \wedge [(x \to y) \rightsquigarrow y] = x$;
($psBCK_8''$) $(x \wedge y) \to y = 1$.

*Example 1.6* Given a pseudo-hoop $(A, \odot, \to, \rightsquigarrow, 1)$ (see Chap. 2), then $(A, \wedge, \to, \rightsquigarrow, 1)$ is a pseudo-BCK meet-semilattice, where $x \wedge y = x \odot (x \rightsquigarrow y) = (x \to y) \odot x$.

In the sequel by a *pseudo-BCK semilattice* we mean a pseudo-BCK join-semilattice.

**Definition 1.5** Let $(A, \leq, \to, \rightsquigarrow, 1)$ be a pseudo-BCK algebra. If the poset $(A, \leq)$ is a lattice, then we say that $A$ is a *pseudo-BCK lattice*.

A pseudo-BCK lattice is denoted by $(A, \wedge, \vee, \to, \rightsquigarrow, 1)$.

*Example 1.7* Consider the bounded pseudo-BCK algebra $(A, \leq, \to, \rightsquigarrow, 0, 1)$ from Example 1.3. Since $(A, \leq)$ is a lattice, it follows that $A$ is a pseudo-BCK lattice.

Let $A$ be a pseudo-BCK algebra. For all $x, y \in A$, define:

$$x \vee_1 y = (x \to y) \rightsquigarrow y, \qquad x \vee_2 y = (x \rightsquigarrow y) \to y.$$

**Proposition 1.5** *In any bounded pseudo-BCK algebra $A$ the following hold for all $x, y \in A$:*

(1) $0 \vee_1 x = x = 0 \vee_2 x$;
(2) $x \vee_1 0 = x^{-\sim}$, $x \vee_2 0 = x^{\sim-}$;
(3) $1 \vee_1 x = x \vee_1 1 = 1 = 1 \vee_2 x = x \vee_2 1$;
(4) $x \leq y$ *implies* $x \vee_1 y = y$ *and* $x \vee_2 y = y$;
(5) $x \vee_1 x = x \vee_2 x = x$.

*Proof*

(1) $0 \vee_1 x = (0 \to x) \rightsquigarrow x = 1 \rightsquigarrow x = x$ and similarly $0 \vee_2 x = x$.
(2) $x \vee_1 0 = (x \to 0) \rightsquigarrow 0 = x^{-\sim}$ and similarly $x \vee_2 0 = x^{\sim-}$.
(3) We have: $1 \vee_1 x = (1 \to x) \rightsquigarrow x = 1$ and $x \vee_1 1 = (x \to 1) \rightsquigarrow 1 = 1$, so $1 \vee_1 x = x \vee_1 1 = 1$. Similarly, $1 \vee_2 x = x \vee_2 1 = 1$.
(4) $x \vee_1 y = (x \to y) \rightsquigarrow y = 1 \rightsquigarrow y = y$. Similarly, $x \vee_2 y = y$.
(5) This follows from the definitions of $\vee_1$ and $\vee_2$. □

**Proposition 1.6** *In any bounded pseudo-BCK algebra $A$ the following hold for all $x, y \in A$:*

(1) $x \vee_1 y^{-\sim} = x^{-\sim} \vee_1 y^{-\sim}$ *and* $x \vee_2 y^{\sim-} = x^{\sim-} \vee_2 y^{\sim-}$;
(2) $x \vee_1 y^{\sim} = x^{-\sim} \vee_1 y^{\sim}$ *and* $x \vee_2 y^{-} = x^{\sim-} \vee_2 y^{-}$;
(3) $(x^{-\sim} \vee_1 y^{-\sim})^{-\sim} = x^{-\sim} \vee_1 y^{-\sim}$ *and* $(x^{\sim-} \vee_2 y^{\sim-})^{\sim-} = x^{\sim-} \vee_2 y^{\sim-}$.

*Proof*

(1) Applying ($psbck$-$c_{19}$) we have:

$$x \vee_1 y^{-\sim} = \big(x \to y^{-\sim}\big) \rightsquigarrow y^{-\sim} = \big(x^{-\sim} \to y^{-\sim}\big) \rightsquigarrow y^{-\sim} = x^{-\sim} \vee_1 y^{-\sim};$$
$$x \vee_2 y^{\sim-} = \big(x \rightsquigarrow y^{\sim-}\big) \to y^{\sim-} = \big(x^{\sim-} \rightsquigarrow y^{\sim-}\big) \to y^{\sim-} = x^{\sim-} \vee_2 y^{\sim-}.$$

(2) Applying ($psbck$-$c_{20}$) we have:

$$x \vee_1 y^{\sim} = \big(x \to y^{\sim}\big) \rightsquigarrow y^{\sim} = \big(x^{-\sim} \to y^{\sim}\big) \rightsquigarrow y^{\sim} = x^{-\sim} \vee_1 y^{\sim};$$
$$x \vee_2 y^{-} = \big(x \rightsquigarrow y^{-}\big) \to y^{-} = \big(x^{\sim-} \rightsquigarrow y^{-}\big) \to y^{-} = x^{\sim-} \vee_2 y^{-}.$$

(3) Applying ($psbck$-$c_{21}$) we have:

$$\big(x^{-\sim} \vee_1 y^{-\sim}\big)^{-\sim} = \big[\big(x^{-\sim} \to y^{-\sim}\big) \rightsquigarrow y^{-\sim}\big]^{-\sim} = \big(x^{-\sim} \to y^{-\sim}\big) \rightsquigarrow y^{-\sim}$$
$$= x^{-\sim} \vee_1 y^{-\sim};$$
$$\big(x^{\sim-} \vee_2 y^{\sim-}\big)^{\sim-} = \big[\big(x^{\sim-} \rightsquigarrow y^{\sim-}\big) \to y^{\sim-}\big]^{\sim-} = \big(x^{\sim-} \rightsquigarrow y^{\sim-}\big) \to y^{\sim-}$$
$$= x^{\sim-} \vee_2 y^{\sim-}.$$

□

**Proposition 1.7** *In any pseudo-BCK algebra the following hold for all* $x, y \in A$:

($psbck$-$c_{22}$) $(x \vee_1 y) \to y = x \to y$ *and* $(x \vee_2 y) \rightsquigarrow y = x \rightsquigarrow y$.

*Proof* This is a consequence of the property ($psbck$-$c_{11}$). □

**Lemma 1.1** *Let* $A$ *be a pseudo-BCK algebra. Then*:

(1) $x \vee_1 y$ $(y \vee_1 x)$ *is an upper bound of* $\{x, y\}$;
(2) $x \vee_2 y$ $(y \vee_2 x)$ *is an upper bound of* $\{x, y\}$

*for all* $x, y \in A$.

*Proof*

(1) By ($psBCK_2$) we have $x \leq (x \to y) \rightsquigarrow y$.
Since by ($psbck$-$c_6$), $y \leq (x \to y) \rightsquigarrow y$, we conclude that $x, y \leq x \vee_1 y$.
Similarly we get $x, y \leq y \vee_1 x$.
(2) Similar to (1). □

**Definition 1.6** Let $A$ be a pseudo-BCK algebra.

(1) If $x \vee_1 y = y \vee_1 x$ for all $x, y \in A$, then $A$ is called $\vee_1$-*commutative*;
(2) If $x \vee_2 y = y \vee_2 x$ for all $x, y \in A$, then $A$ is called $\vee_2$-*commutative*.

**Lemma 1.2** *Let* $A$ *be a pseudo-BCK algebra.*

(1) *If for all* $x, y \in A$, $x \vee_1 y$ $(y \vee_1 x)$ *is the l.u.b. of* $\{x, y\}$, *then* $A$ *is* $\vee_1$-*commutative*;
(2) *If for all* $x, y \in A$, $x \vee_2 y$ $(y \vee_2 x)$ *is the l.u.b. of* $\{x, y\}$, *then* $A$ *is* $\vee_2$-*commutative*.

*Proof*

(1) Suppose that for all $x, y \in A$, $x \vee_1 y$ $(y \vee_1 x)$ is the l.u.b. of $\{x, y\}$. Then by Lemma 1.1, for all $x, y \in A$ we have $y \vee_1 x \le x \vee_1 y$ and $x \vee_1 y \le y \vee_1 x$. Applying $(psBCK_5)$ we get $x \vee_1 y = y \vee_1 x$. Thus $A$ is $\vee_1$-commutative.
(2) Similar to (1). □

**Proposition 1.8** *Let* $A$ *be a pseudo-BCK algebra.*

(1) *If* $A$ *is* $\vee_1$-*commutative*, *then* $x \vee_1 y$ *is the l.u.b. of* $\{x, y\}$, *for all* $x, y \in A$;
(2) *If* $A$ *is* $\vee_2$-*commutative*, *then* $x \vee_2 y$ *is the l.u.b. of* $\{x, y\}$, *for all* $x, y \in A$.

*Proof*

(1) Let $x, y \in A$. According to Lemma 1.1, $x \vee_1 y$ is an upper bound of $\{x, y\}$. Let $z$ be another upper bound of $\{x, y\}$, i.e. $x \le z$ and $y \le z$. We will prove that $x \vee_1 y \le z$. Indeed, applying Proposition 1.5(4) and taking into consideration that $A$ is $\vee_1$-commutative we have:

$$\begin{aligned} x \vee_1 y \to z &= x \vee_1 y \to y \vee_1 z = x \vee_1 y \to z \vee_1 y \\ &= \big((x \to y) \rightsquigarrow y\big) \to \big((z \to y) \rightsquigarrow y\big). \end{aligned}$$

According to $(psBCK_1)$ we have $(b \to c) \rightsquigarrow (a \to c) \ge a \to b$ and replacing $a$ with $z \to y$, $b$ with $x \to y$ and $c$ with $y$ we get:

$$\begin{aligned} \big((x \to y) \rightsquigarrow y\big) \to \big((z \to y) \rightsquigarrow y\big) &\ge (z \to y) \rightsquigarrow (x \to y) \\ &\ge x \to z \quad \big(\text{by } (psBCK_1)\big). \end{aligned}$$

Hence $x \vee_1 y \to z \ge x \to z = 1$ (since $x \le z$). It follows that $x \vee_1 y \to z = 1$, thus $x \vee_1 y \le z$. We conclude that $x \vee_1 y$ is the l.u.b. of $\{x, y\}$.
(2) Similar to (1). □

**Theorem 1.1** *If* $A$ *is a pseudo-BCK algebra*, *then*:

(1) $A$ *is* $\vee_1$-*commutative iff it is a join-semilattice with respect to* $\vee_1$ (*under* $\le$);
(2) $A$ *is* $\vee_2$-*commutative iff it is a join-semilattice with respect to* $\vee_2$ (*under* $\le$).

*Proof* This is a consequence of Lemma 1.2 and Proposition 1.8. □

**Corollary 1.1** *Let* $A$ *be a pseudo-BCK algebra. Then*:

(1) *If* $A$ *is* $\vee_1$-*commutative*, *then* $x \vee_1 y \le x \vee_2 y, y \vee_2 x$ *for all* $x, y \in A$;

(2) *If A is* $\vee_2$*-commutative, then* $x \vee_2 y \leq x \vee_1 y, y \vee_1 x$ *for all* $x, y \in A$.

*Proof*

(1) According to Lemma 1.1, $x \vee_2 y$, $y \vee_2 x$ are upper bounds of $\{x, y\}$. By Proposition 1.8, $x \vee_1 y$ is the l.u.b. of $\{x, y\}$, thus $x \vee_1 y \leq x \vee_2 y, y \vee_2 x$.
(2) Similar to (1). □

**Definition 1.7** A pseudo-BCK algebra is called *sup-commutative* if it is both $\vee_1$-commutative and $\vee_2$-commutative.

**Theorem 1.2** *A pseudo-BCK algebra is sup-commutative iff it is a join-semilattice with respect to both* $\vee_1$ *and* $\vee_2$.

*Proof* This follows from Theorem 1.1. □

**Corollary 1.2** *If A is a sup-commutative pseudo-BCK algebra, then* $x \vee_1 y = x \vee_2 y$ *for all* $x, y \in A$.

*Proof* By Corollary 1.1, $x \vee_1 y \leq x \vee_2 y$ and $x \vee_2 y \leq x \vee_1 y$, hence $x \vee_1 y = x \vee_2 y$. □

**Lemma 1.3** *In a* $\vee_1$*-commutative* ($\vee_2$*-commutative*) *bounded pseudo-BCK algebra* $A$, *we have* $x^{-\sim} = x$ ($x^{\sim-} = x$, *respectively*), *for all* $x \in A$.

*Proof* Replacing $y$ with 0 in the identity $x \vee_1 y = y \vee_1 x$, we get $(x \to 0) \rightsquigarrow 0 = (0 \to x) \rightsquigarrow x$, i.e. $x^{-\sim} = x$.

Similarly, replacing $y$ with 0 in $x \vee_2 y = y \vee_2 x$, we get $x^{\sim-} = x$. □

**Corollary 1.3** *Let* $A$ *be a sup-commutative, bounded pseudo-BCK algebra. Then* $x^{-\sim} = x^{\sim-} = x$, *for all* $x \in A$.

*Proof* This follows by replacing $y$ with 0 in the equality $x \vee_1 y = x \vee_2 y$ and applying Lemma 1.3. □

In a bounded pseudo-BCK algebra $A$, define, for all $x, y \in A$:

$$x \wedge_1 y := \left(x^- \vee_1 y^-\right)^\sim,$$
$$x \wedge_2 y := \left(x^- \vee_2 y^-\right)^\sim.$$

**Lemma 1.4** *Let* $A$ *be a pseudo-BCK algebra. Then for all* $x, y \in A$:

(1) $x \wedge_1 y$ ($y \wedge_1 x$) *is a lower bound of* $\{x^{-\sim}, y^{-\sim}\}$;
(2) $x \wedge_2 y$ ($y \wedge_2 x$) *is a lower bound of* $\{x^{\sim-}, y^{\sim-}\}$.

*Proof*

(1) By Lemma 1.1 we have $x^-, y^- \le x^- \vee_1 y^-$, hence $x \wedge_1 y = (x^- \vee_1 y^-)^\sim \le x^{-\sim}, y^{-\sim}$. Thus $x \wedge_1 y$ is a lower bound of $\{x^{-\sim}, y^{-\sim}\}$.
(2) Similar to (1). □

**Proposition 1.9** *Let $A$ be a bounded pseudo-BCK algebra.*

(1) *If $A$ is $\vee_1$-commutative, then $x \wedge_1 y$ $(y \wedge_1 x)$ is the g.l.b. of $\{x, y\}$ and $x \wedge_1 y = y \wedge_1 x$, for all $x, y \in A$;*
(2) *If $A$ is $\vee_2$-commutative, then $x \wedge_2 y$ $(y \wedge_2 x)$ is the g.l.b. of $\{x, y\}$ and $x \wedge_2 y = y \wedge_2 x$, for all $x, y \in A$.*

*Proof*

(1) By Lemma 1.3, $x^{-\sim} = x$ and $y^{-\sim} = y$. Hence by Lemma 1.4, $x \wedge_1 y$ is a lower bound of $\{x, y\}$. Now let $z$ be another lower bound of $\{x, y\}$, i.e. $z \le x, y$. It follows that $x^-, y^- \le z^-$, thus $z^-$ is an upper bound of $\{x^-, y^-\}$. Since $A$ is $\vee_1$-commutative, by Proposition 1.8, $x^- \vee_1 y^-$ is the l.u.b. of $\{x^-, y^-\}$, hence $x^- \vee_1 y^- \le z^-$. Thus $z = z^{-\sim} \le (x^- \vee_1 y^-)^\sim = x \wedge_1 y$, i.e. $x \wedge_1 y$ is the g.l.b. of $\{x, y\}$. Since $A$ is $\vee_1$-commutative, we have $x^- \vee_1 y^- = y^- \vee_1 x^-$, hence by definition it follows that $x \wedge_1 y = y \wedge_1 x$, for all $x, y \in A$.
(2) Similar to (1). □

**Corollary 1.4** *Let $A$ be a bounded pseudo-BCK algebra.*

(1) *If $A$ is $\vee_1$-commutative, then $A$ is a lattice with respect to $\wedge_1, \vee_1$;*
(2) *If $A$ is $\vee_2$-commutative, then $A$ is a lattice with respect to $\wedge_2, \vee_2$.*

*Proof* This follows by Propositions 1.8 and 1.9. □

**Theorem 1.3** *A bounded sup-commutative pseudo-BCK algebra $A$ is a lattice with respect to both $\vee_1, \wedge_1$ and $\vee_2, \wedge_2$ (under $\le$) and for all $x, y$ we have:*

$$x \vee_1 y = x \vee_2 y, \qquad x \wedge_1 y = x \wedge_2 y.$$

*Proof* By Corollary 1.4, $A$ is a lattice with respect to both $\wedge_1, \vee_1$ and $\wedge_2, \vee_2$. By Corollary 1.2, $x \vee_1 y = x \vee_2 y$ for all $x, y \in A$. By Proposition 1.9 we get: $x \wedge_2 y \le x \wedge_1 y$ and $x \wedge_1 y \le x \wedge_2 y$, hence $x \wedge_1 y = x \wedge_2 y$ for all $x, y \in A$. □

We recall that a *downwards-directed set* (or a *filtered set*) is a partially ordered set $(A, \le)$ such that whenever $a, b \in A$, there exists an $x \in A$ such that $x \le a$ and $x \le b$.

Dually, an *upwards-directed set* is a partially ordered set $(A, \le)$ such that whenever $a, b \in A$, there exists an $x \in A$ such that $a \le x$ and $b \le x$.

If $X$ is a set, then a *net* in $X$ will be a set $\{x_i \mid i \in I\}$, where $(I, \le)$ is an upwards-directed set.

We say that a pseudo-BCK algebra $A$ satisfies the *relative cancellation property*, (RCP) for short, if for every $a, b, c \in A$,

$$a, b \leq c \quad \text{and} \quad c \to a = c \to b, c \rightsquigarrow a = c \rightsquigarrow b \quad \text{imply} \quad a = b.$$

We note that a pseudo-BCK algebra $A$ that is sup-commutative and satisfies the (RCP) condition is said to be a *Łukasiewicz pseudo-BCK algebra* (see [112]).

*Example 1.8* The pseudo-BCK algebra $A$ from Example 1.3 is downwards-directed with (RCP).

**Proposition 1.10** *Any downwards-directed sup-commutative pseudo-BCK algebra has* (RCP).

*Proof* Consider $a, b, c \in A$ such that $a, b \leq c$ and $c \to a = c \to b$, $c \rightsquigarrow a = c \rightsquigarrow b$. There exists an $x \in A$ such that $x \leq a, b$.

By ($psbck$-$c_1$), from $a \leq c$ it follows that $c \rightsquigarrow x \leq a \rightsquigarrow x$.

According to Proposition 1.5(4) and ($psbck$-$c_3$) we have:

$$\begin{aligned}
a \rightsquigarrow x &= (c \rightsquigarrow x) \vee_1 (a \rightsquigarrow x) = (a \rightsquigarrow x) \vee_1 (c \rightsquigarrow x) \\
&= \big[(a \rightsquigarrow x) \to (c \rightsquigarrow x)\big] \rightsquigarrow (c \rightsquigarrow x) = \big[c \rightsquigarrow \big[(a \rightsquigarrow x) \to x\big]\big] \rightsquigarrow (c \rightsquigarrow x) \\
&= \big[c \rightsquigarrow (a \vee_2 x)\big] \rightsquigarrow (c \rightsquigarrow x) = \big[c \rightsquigarrow (x \vee_2 a)\big] \rightsquigarrow (c \rightsquigarrow x) \\
&= (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow x).
\end{aligned}$$

Similarly, $b \rightsquigarrow x = (c \rightsquigarrow b) \rightsquigarrow (c \rightsquigarrow x) = (c \rightsquigarrow a) \rightsquigarrow (c \rightsquigarrow x) = a \rightsquigarrow x$.

We have: $a = x \vee_2 a = a \vee_2 x = (a \rightsquigarrow x) \to x = (b \rightsquigarrow x) \to x = b \vee_2 x = x \vee_2 b = b$.

Thus $A$ has (RCP). □

## 1.2 Pseudo-BCK Algebras with Pseudo-product

**Definition 1.8** A pseudo-BCK algebra with the *(pP) condition* (i.e. with the *pseudo-product* condition) or a *pseudo-BCK(pP) algebra* for short, is a pseudo-BCK algebra $(A, \leq, \to, \rightsquigarrow, 1)$ satisfying the (pP) condition:

(pP) For all $x, y \in A$, $x \odot y$ exists where

$$x \odot y = \min\{z \mid x \leq y \to z\} = \min\{z \mid y \leq x \rightsquigarrow z\}.$$

*Example 1.9* Take $A = \{0, a_1, a_2, s, a, b, n, c, d, m, 1\}$ with $0 < a_1 < a_2 < s < a, b < n < c, d < m < 1$ (see Fig. 1.3).

Consider the operations $\to$, $\rightsquigarrow$ given by the following tables:

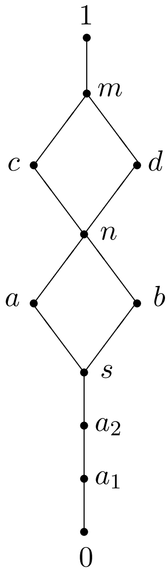


**Fig. 1.3** Example of bounded pseudo-BCK(pP) algebra

| $\to$ | $0$ | $a_1$ | $a_2$ | $s$ | $a$ | $b$ | $n$ | $c$ | $d$ | $m$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_1$ | $a_1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_2$ | $a_1$ | $a_1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $s$ | $0$ | $a_1$ | $a_2$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $0$ | $a_1$ | $a_2$ | $m$ | $1$ | $m$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $b$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $n$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $m$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $m$ | $m$ | $1$ | $m$ | $1$ | $1$ |
| $d$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $m$ | $m$ | $m$ | $1$ | $1$ | $1$ |
| $m$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | $1$ | $1$ |
| $1$ | $0$ | $a_1$ | $a_2$ | $s$ | $a$ | $b$ | $n$ | $c$ | $d$ | $m$ | $1$ |

| $\rightsquigarrow$ | $0$ | $a_1$ | $a_2$ | $s$ | $a$ | $b$ | $n$ | $c$ | $d$ | $m$ | $1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_1$ | $a_2$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a_2$ | $0$ | $a_1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $s$ | $0$ | $a_1$ | $a_2$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $a$ | $0$ | $a_1$ | $a_2$ | $m$ | $1$ | $m$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $b$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $n$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $m$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $c$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $m$ | $m$ | $1$ | $m$ | $1$ | $1$ |
| $d$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $m$ | $m$ | $m$ | $1$ | $1$ | $1$ |
| $m$ | $0$ | $a_1$ | $a_2$ | $m$ | $m$ | $m$ | $m$ | $m$ | $m$ | $1$ | $1$ |
| $1$ | $0$ | $a_1$ | $a_2$ | $s$ | $a$ | $b$ | $n$ | $c$ | $d$ | $m$ | $1$ |