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The Fields Institute for Research in Mathematical Sciences

Dong Eui Chang
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Geometry, Mechanics, and Dynamics

The Legacy of Jerry Marsden

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Preface to Fields Volume

Jerrold Eldon Marsden, an eminent mathematician of our time, passed away on September 21, 2010, at the age of 68. This volume contains 20 papers presented at a focus program in honor of Jerry Marsden's legacy in Geometry, Mechanics, and Dynamics held in July 2012 at the Fields Institute for Research in Mathematical Sciences. Marsden helped found the Fields Institute in 1992. It was initially directed by him, and memorable workshops in honor of his 50th and 60th birthday celebrations were held there in 1992 and 2002, respectively. Information about Jerry's contributions, influence, and personal interactions and an overview of the Marsden Legacy focus program at Fields are available at http://www.fields.utoronto.ca/programs/scientific/12-13/Marsden/index.html. Many of Jerry's research publications and more information about him may also be found at http://www.cds.caltech.edu/~marsden/.

No single volume could do justice to the legacy of Jerry Marsden's research and his personal influence on the field of Geometric Mechanics, which he helped create. Many of Jerry's papers and books have received hundreds of citations; some of them have received thousands. The legacy of his influence in mathematics, physics, and computational science, though, is not countable. We miss him dearly.

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Contents

A Global Version of the Koon-Marsden Jacobiator Formula

Paula Balseiro

Dedicated to the memory of J.E. Marsden

Abstract In this paper we study the Jacobiator (the cyclic sum that vanishes when the Jacobi identity holds) of the almost Poisson brackets describing nonholonomic systems. We revisit the local formula for the Jacobiator established by Koon and Marsden (Rep Math Phys 42:101–134, 1998) using suitable local coordinates and explain how it is related to the global formula obtained in Balseiro (Arch. Ration. Mech. Anal. 214(2):453-501, 2014), based on the choice of a complement to the constraint distribution. We use an example to illustrate the benefits of the coordinate-free viewpoint.

1 Introduction

The geometric approach to nonholonomic systems was among the many research interests of J.E. Marsden, and his contributions to this area were fundamental. A system with nonholonomic constraints can be geometrically described by an almost Poisson bracket [10, 15, 18], whose failure to satisfy the Jacobi identity, measured by the so-called Jacobiator, is precisely what encodes the nonholonomic nature of the system. There is a vast literature on the study of such nonholonomic brackets and their properties, starting with the early work of Chaplygin [6], see e.g. [2, 5, 8–12]. An explicit formula for the Jacobiator of nonholonomic brackets, expressed in suitable local coordinates, was obtained by Koon and Marsden in their 1998 paper [14]. In the present paper, we revisit the Koon-Marsden formula of [14] and explain how it can be derived from the coordinate-free Jacobiator formula for nonholonomic brackets obtained in [1].

We organize the paper as follows. In Section 2, we recall the hamiltonian viewpoint to systems with nonholonomic constraints. For a nonholonomic system

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on a configuration manifold Q, determined by a lagrangian $L : TQ \to \mathbb{R}$ and a nonintegrable distribution D on Q (the *constraint distribution*, defining the permitted velocities of the system), we consider the induced *nonholonomic bracket* $\{\cdot,\cdot\}_{\rm nh}$ defined on the submanifold $\mathcal{M} := \mathrm{Leg}(D)$ of T^*Q, where $\mathrm{Leg} : T^*Q \to TQ$ is the Legendre transform (see Section 2.2). In Section 2.3 (see Theorem 1) we recall the global formula for the Jacobiator of $\{\cdot,\cdot\}_{\rm nh}$ from [1], which depends on the choice of a complement W of the constraint distribution D such that $TQ = D \oplus W$. As shown in [1], this formula is useful to provide information about properties of reduced nonholonomic brackets in the presence of symmetries.

In Section 3 we recall the choice of coordinates, suitably adapted to the constraints, used by Koon and Marsden in [14], and in terms of which their Jacobiator formula is expressed. We then compare the global and local viewpoints in Section 4, explaining how one can derive the local Jacobiator formula in [14] from the coordinate-free formula in [1].

Since the formula in [1] is coordinate free, it can be used in examples without specific choices of coordinates. We illustrate this fact studying the *snakeboard*, following [14, 17]; here the natural coordinates in the problem are not adapted to the constraints so, in principle, the local formula from [14] cannot be directly applied.

2 Nonholonomic Systems

2.1 The Hamiltonian Viewpoint

A nonholonomic system is a mechanical system on a configuration manifold Q with constraints on the velocities which are not derived from constraints in the positions. Mathematically, it is defined by a lagrangian $L : TQ \to \mathbb{R}$ of mechanical type, i.e., $L = \kappa - U$ where κ is the kinetic energy metric and $U \in C^\infty(Q)$ is the potential energy, and a nonintegrable distribution D on Q determining the constraints, see [3, 7]. If D is an integrable distribution then the system is called *holonomic*.

In order to have an intrinsic formulation of the dynamics of nonholonomic systems, let us consider the Legendre transform $\mathrm{Leg} : TQ \to T^*Q$ associated to the lagrangian L. The Legendre transform is a diffeomorphism since $\mathrm{Leg} = \kappa^\flat$, where $\kappa^\flat : TQ \to T^*Q$ is defined by $\kappa^\flat(X)(Y) = \kappa(X,Y)$. We denote by $\mathcal{H} : T^*Q \to \mathbb{R}$ the hamiltonian function associated to the lagragian L.

We define the constraint submanifold $\mathcal{M}$ of T^*Q by $\mathcal{M} = \kappa^\flat(D)$. Note that $\mathcal{M}$ is a vector subbundle of T^*Q. We denote by $\tau : \mathcal{M} \to Q$ the restriction to $\mathcal{M}$ of the canonical projection $\tau_Q : T^*Q \to Q$.

On $\mathcal{M}$ we have a natural 2-form $\Omega_{\mathcal{M}}$ given by $\Omega_{\mathcal{M}} := \iota^*\Omega_Q$ where $\iota : \mathcal{M} \to T^*Q$ is the inclusion and Ω_Q is the canonical 2-form on T^*Q. The constraints are encoded on a (regular) distribution $\mathcal{C}$ on $\mathcal{M}$ defined, at each $m \in \mathcal{M}$, by

$$\mathcal{C}_m = \{v \in T_m\mathcal{M} \ : \ T\tau(v) \in D_{\tau(m)}\}. \tag{1}$$

It was proven in [2] that the point-wise restriction of the 2-form $\Omega_{\mathcal{M}}$ to $\mathcal{C}$, denoted by $\Omega_{\mathcal{M}}|_{\mathcal{C}}$, is nondegenerate. That is, if $X \in \Gamma(\mathcal{C})$ is such that $\mathbf{i}_X \Omega_{\mathcal{M}}|_{\mathcal{C}} \equiv 0$, then $X = 0$. Therefore, there is a unique vector field X_{nh} on $\mathcal{M}$, called the *nonholonomic vector field*, such that $X_{\text{nh}}(m) \in \mathcal{C}_m$ and

$$\mathbf{i}_{X_{\text{nh}}} \Omega_{\mathcal{M}}|_{\mathcal{C}} = d\mathcal{H}_{\mathcal{M}}|_{\mathcal{C}}, \tag{2}$$

where $\mathcal{H}_{\mathcal{M}} := \iota^*\mathcal{H} : \mathcal{M} \to \mathbb{R}$. The integral curves of X_{nh} are solutions of the nonholonomic dynamics [2].

In order to write (2) in local coordinates, suppose that the constraint distribution D is described (locally) by the annihilators of 1-forms ϵ^a for $a = 1, \ldots, k$, that is $D = \{(q, \dot{q}) \ : \ \epsilon^a(q)(\dot{q}) = 0 \text{ for all } a = 1, \ldots, k\}$. If we consider canonical coordinates (q^i, p_i) on T^*Q then the constraints are given by

$$\epsilon_i^a(q)\frac{\partial \mathcal{H}}{\partial p_i} = 0, \qquad \text{for } a = 1, \ldots, k,$$

and (2) becomes

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i} + \lambda_a \epsilon^a,$$

where λ_a are functions (called the Lagrange multipliers) which are uniquely determined by the fact that the constraints are satisfied.

2.2 The Nonholonomic Bracket

Recall that an *almost Poisson bracket* on $\mathcal{M}$ is an $\mathbb{R}$-bilinear bracket $\{\cdot, \cdot\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ that is skew-symmetric and satisfies the Leibniz condition:

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \qquad \text{for } f, g, h \in C^\infty(\mathcal{M}).$$

If $\{\cdot, \cdot\}$ satisfies the Jacobi identity, then the bracket is called *Poisson*. The *hamiltonian vector field* X_f on $\mathcal{M}$ associated to a $f \in C^\infty(\mathcal{M})$ is defined by

$$X_f = \{\cdot, f\} \tag{3}$$

and the *characteristic distribution* of $\{\cdot, \cdot\}$ is the distribution on the manifold $\mathcal{M}$ whose fibers are spanned by the hamiltonian vector fields. If the bracket is Poisson, then its characteristic distribution is integrable. However, the converse is not always true.

From the Leibniz identity it follows that there is a one-to-one correspondence between almost Poisson brackets $\{\cdot,\cdot\}$ and bivector fields $\pi\in\bigwedge^2(T\mathcal{M})$ given by

$$\{f,g\}=\pi(df,dg),\qquad f,g,\in C^\infty(\mathcal{M}). \tag{4}$$

Let us denote by $\pi^\sharp : T^*\mathcal{M}\to T\mathcal{M}$ the map defined by $\beta(\pi^\sharp(\alpha))=\pi(\alpha,\beta)$. Then, using (3), the hamiltonian vector field X_f is also given by $X_f=-\pi^\sharp(df)$ and the characteristic distribution of π is the image of $\pi^\sharp$. The Schouten bracket $[\pi,\pi]$ (see [16]) measures the failure of the Jacobi identity of $\{\cdot,\cdot\}$ through the relation

$$\frac{1}{2}[\pi,\pi](df,dg,dh)=\{f,\{g,h\}\}+\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\} \tag{5}$$

for $f,g,h\in C^\infty(\mathcal{M})$. So we refer to the trivector $\frac{1}{2}[\pi,\pi]$ as the *Jacobiator* of π, which is zero when π is a Poisson bivector.

Coming back to our context, consider a nonholonomic system on a manifold Q defined by a lagrangian L and a constraint distribution D. Due to the nondegeneracy of $\Omega_{\mathcal{M}}|_{\mathcal{C}}$, there is an induced bivector field $\pi_{\text{nh}}\in\bigwedge^2(T\mathcal{M})$ defined at each $\alpha\in T^*\mathcal{M}$ by

$$\pi^\sharp_{\text{nh}}(\alpha)=X\quad\text{if and only if}\quad \mathbf{i}_X\Omega_{\mathcal{M}}|_{\mathcal{C}}=-\alpha|_{\mathcal{C}}. \tag{6}$$

The characteristic distribution of π_{nh} is the distribution $\mathcal{C}$ defined in (1). Since $\mathcal{C}$ is not integrable, π_{nh} is not Poisson.

The bivector field π_{nh} is called the *nonholonomic bivector field* [10, 15, 18] and it describes the dynamics in the sense that

$$\pi^\sharp_{\text{nh}}(d\mathcal{H}_{\mathcal{M}})=-X_{\text{nh}}. \tag{7}$$

By (4), the nonholonomic bivector π_{nh} defines uniquely an almost Poisson bracket $\{\cdot,\cdot\}_{\text{nh}}$ on $\mathcal{M}$, called the *nonholonomic bracket*. From (6) we observe that

$$\{f,g\}_{\text{nh}}=\Omega_{\mathcal{M}}(X_f,X_g)\qquad\text{for } f,g\in C^\infty(\mathcal{M}),$$

where $X_f=-\pi^\sharp_{\text{nh}}(df)$ and $X_g=-\pi^\sharp_{\text{nh}}(dg)$. The nonholonomic vector field (7) is equivalently defined through the equation $X_{\text{nh}}=\{\cdot,\mathcal{H}_{\mathcal{M}}\}_{\text{nh}}$.

2.3 The Jacobiator Formula

Recall that $\mathcal{C}$ is a smooth distribution on $\mathcal{M}$. Choose a complement $\mathcal{W}$ of $\mathcal{C}$ on $T\mathcal{M}$ such that, for each $m\in\mathcal{M}$,

$$T_m\mathcal{M}=\mathcal{C}_m\oplus\mathcal{W}_m. \tag{8}$$

Let $P_{\mathcal{C}} : T\mathcal{M} \to \mathcal{C}$ and $P_{\mathcal{W}} : T\mathcal{M} \to \mathcal{W}$ be the projections associated to the decomposition (8). Since $P_{\mathcal{W}} : T\mathcal{M} \to \mathcal{W}$ can be seen as a $\mathcal{W}$-valued 1-form, following [1], we define the $\mathcal{W}$-valued 2-form $\mathbf{K}_{\mathcal{W}}$ given by

$$\mathbf{K}_{\mathcal{W}}(X,Y) = -P_{\mathcal{W}}([P_{\mathcal{C}}(X), P_{\mathcal{C}}(Y)]) \qquad \text{for } X, Y \in \mathfrak{X}(\mathcal{M}). \tag{9}$$

Once a complement $\mathcal{W}$ of $\mathcal{C}$ is chosen, we obtain a coordinate-free formula for the Jacobiator of the nonholonomic bracket.

Theorem 1 ([1]). *The following holds:*

$$\begin{aligned} &\frac{1}{2}[\pi_{\text{nh}}, \pi_{\text{nh}}](\alpha, \beta, \gamma) \\ &= \Omega_{\mathcal{M}}(\mathbf{K}_{\mathcal{W}}(\pi_{\text{nh}}^\sharp(\alpha), \pi_{\text{nh}}^\sharp(\beta)), \pi_{\text{nh}}^\sharp(\gamma)) - \gamma\left(\mathbf{K}_{\mathcal{W}}(\pi_{\text{nh}}^\sharp(\alpha), \pi_{\text{nh}}^\sharp(\beta))\right) + \text{cyclic.} \end{aligned} \tag{10}$$

for $\alpha, \beta, \gamma \in T^*\mathcal{M}$.

In fact, a more general formula appeared in [1], valid for any bivector field π_B gauge related to π_{nh}. In that context, this formula was used to understand under which circumstances the reduction of π_B by symmetries had an integrable characteristic distribution (even if it was not Poisson).

We will now show how this formula recovers the coordinate Jacobiator formula obtained in [14].

3 The Koon-Marsden Adapted Coordinates

In this section we will recall the Koon-Marsden approach to writing the Jacobiator of a nonholonomic bracket, based on a suitable choice of coordinates of the manifold Q. After this, we will write the objects presented in Section 2 (such as the 2-forms $\Omega_{\mathcal{M}}$ and $\mathbf{K}_{\mathcal{W}}$, and the bivector π_{nh}) in such local coordinates in order to see the equivalence between the local and global viewpoints.

We start by recalling the coordinates chosen in [14]. Consider a nonholonomic system given by a lagrangian L and a nonintegrable distribution D. Let ϵ^a for $a = 1, \dots, k$ be 1-forms that span the annihilator of D, i.e., $D^\circ = \text{span}\{\epsilon^a\}$. The authors in [14] introduce local coordinates $(q^i) = (r^\alpha, s^a)$ on Q for which each 1-form ϵ^a has the form

$$\epsilon^a = ds^a + A^a_\alpha(r,s)dr^\alpha, \tag{11}$$

where A^a_α are functions on Q for $\alpha = 1, \dots, n-k$ and $a = 1, \dots, k$. During the present paper, we refer to the coordinates (r^α, s^a) such that (11) is satisfied as *coordinates adapted to the constraints*.

These coordinates induce a (local) basis of D given by $\left\{X_\alpha := \frac{\partial}{\partial r^\alpha} - A_\alpha^a \frac{\partial}{\partial s^a}\right\}$. We complete the basis $\{X_\alpha\}$ and $\{\epsilon^a\}$ in order to obtain dual basis on TQ and T^*Q, that is

$$TQ = \text{span}\left\{X_\alpha, \frac{\partial}{\partial s^a}\right\} \quad \text{and} \quad T^*Q = \text{span}\{dr^\alpha, \epsilon^a\}.$$

Let $(\tilde{p}_\alpha, \tilde{p}_a)$ be the coordinates on T^*Q associated to the basis $\{dr^\alpha, \epsilon^a\}$. Since $\mathcal{M} = \text{span}\{\kappa^\flat(X_\alpha)\} \subset T^*Q$ then

$$\mathcal{M} = \{(q^i, \tilde{p}_a, \tilde{p}_\alpha) \ : \ \tilde{p}_a = [\kappa_{a\alpha}][\kappa_{\alpha\beta}]^{-1}\tilde{p}_\beta = J_a^\beta \tilde{p}_\beta\}, \tag{12}$$

where $[\kappa_{a\alpha}]$ denotes the $(k \times (n-k))$-matrix with entries given by $\kappa_{a\alpha} = \kappa(\frac{\partial}{\partial s^a}, X_\alpha)$, $[\kappa_{\alpha\beta}]^{-1}$ is the inverse matrix associated to the invertible $((n-k) \times (n-k))$-matrix with entries given by $\kappa_{\alpha\beta} = \kappa(X_\alpha, X_\beta)$ and J_a^β are the functions on Q representing the entries of the matrix $[\kappa_{a\alpha}][\kappa_{\alpha\beta}]^{-1}$. Therefore, each element $(r^\alpha, s^a; \tilde{p}_\alpha)$ represents a point on the manifold $\mathcal{M}$.

In [14] the Jacobiator formula is written in terms of the curvature of an Ehresmann connection. The local coordinates (r^α, s^a) induce a fiber bundle with projection given by $\upsilon(r^\alpha, s^a) = r^\alpha$. Let us call W the vertical distribution defined by this projection.

The Ehresmann connection A on $\upsilon : Q = \{r^\alpha, s^a\} \to R = \{r^\alpha\}$ is chosen in such a way that its horizontal space agrees with the distribution D. The connection A is represented by a vector-valued differential form given, at each $X \in TQ$, by

$$A(X) = \epsilon^a(X)\frac{\partial}{\partial s^a}. \tag{13}$$

The *curvature* associated to this connection is a vector-valued 2-form $\mathbf{K}_W$ defined on $X, Y \in \mathfrak{X}(Q)$ by

$$\mathbf{K}_W(X, Y) = -A([P_D(X), P_D(Y)]), \tag{14}$$

where $P_D : TQ \to TQ$ is the projection to D given by $P_D(X) = dr^\alpha(X)X_\alpha$.

In coordinates, the curvature $\mathbf{K}_W$ is given by the following formula [14, Sec. 2.1]:

$$\mathbf{K}_W(X, Y) = d\epsilon^a(P_D(X), P_D(Y))\frac{\partial}{\partial s^a},$$

hence, locally,

$$d\epsilon^a|_D = C_{\alpha\beta}^a dr^\alpha \wedge dr^\beta|_D, \tag{15}$$

where $C^a_{\alpha\beta}(r,s) = \dfrac{\partial A^a_\beta}{\partial r^\alpha} - A^b_\alpha \dfrac{\partial A^a_\beta}{\partial s^b}$. Let us define

$$K^a_{\alpha\beta} = C^a_{\alpha\beta} - C^a_{\beta\alpha}. \tag{16}$$

For each $a = 1, \ldots, k$ the coefficients $K^a_{\alpha\beta}$ are skew-symmetric and $d\epsilon^a|_D = K^a_{\alpha\beta} dr^\alpha \wedge dr^\beta|_D$, for $\alpha < \beta$. Therefore, if $X, \bar{X} \in D$ then $d\epsilon^a(X, \bar{X}) = K^a_{\alpha\beta} v^\alpha \bar{v}^\beta$ where $X = v^\alpha X_\alpha$ and $\bar{X} = \bar{v}^\beta X_\beta$.

Remark 1. Observe that in [14], the 1-forms ϵ^a where denoted by ω^a while $\mathbf{K}_W$ was denoted by B and the coefficients $K^a_{\alpha\beta}$ were $-B^a_{\alpha\beta}$. In this case, for $\dot{q} \in D$ then $d\omega^b(\dot{q}, \cdot)|_D = -B^b_{\alpha\beta}\dot{r}^\alpha dr^\beta|_D$ (observe the correction in the sign with respect to the equation in [14, Sec. 2.1]).

Finally, in [14, Theorem 2.1] the almost Poisson bracket $\{\cdot,\cdot\}_{\mathcal{M}}$ describing the dynamics of a nonholonomic system was written following [18] but in local coordinates on Q adapted to the constraints (11). That is, $\{\cdot,\cdot\}_{\mathcal{M}}$ was computed from the canonical Poisson bracket on T^*Q but written in terms of the adapted coordinates $(r^\alpha, s^a, \tilde{p}_\alpha, \tilde{p}_a)$. As a result, the almost Poisson bracket $\{\cdot,\cdot\}_{\mathcal{M}}$ on $\mathcal{M}$, written in local coordinates $(r^\alpha, s^a, \tilde{p}_\alpha)$, has the following form [14]

$$\{q^i, q^j\}_{\mathcal{M}} = 0, \;\; \{r^\alpha, \tilde{p}_\beta\}_{\mathcal{M}} = \delta^\beta_\alpha, \;\; \{s^a, \tilde{p}_\alpha\}_{\mathcal{M}} = -A^a_\alpha, \;\; \{\tilde{p}_\alpha, \tilde{p}_\beta\}_{\mathcal{M}} = K^b_{\alpha\beta} J^\gamma_b \tilde{p}_\gamma \tag{17}$$

4 The Coordinate Version of the Jacobiator Formula

4.1 Interpretation of the Adapted Coordinates

In this section, we will relate the choice of the coordinates proposed in [14] with the choice of a complement $\mathcal{W}$ done in [1] (see (11) and (8), respectively). We will also connect the *curvature* (14) with the 2-form (9), and the nonholonomic bivector π_{nh} with the bracket $\{\cdot,\cdot\}_{\mathcal{M}}$ given in (6) and (17), respectively.

Consider a nonholonomic system on a manifold Q given by a lagrangian L and a nonintegrable distribution D. Let us consider local coordinates (r^α, s^a) adapted to the constraints as in (11).

Lemma 1. *The choice of coordinates (r^α, s^a) adapted to the constraints* (11)*, induce a complement W of D on TQ such that*

$$TQ = D \oplus W, \qquad \textit{where} \quad W = \operatorname{span}\left\{\frac{\partial}{\partial s^a}\right\}. \tag{18}$$

The projection $P_W : TQ \to W$ associated to the decomposition (18) is interpreted in [14] as the Ehresmann connection A (13). In this context we compare the curvature $\mathbf{K}_W$ defined in (14) (see [14]) with the $\mathcal{W}$-valued 2-form $\mathbf{K}_{\mathcal{W}}$ defined in (9).

Recall that the submanifold $\mathcal{M} = \kappa^\flat(D) \subset T^*Q$ is described by local coordinates $(r^\alpha, s^a; \tilde{p}_\alpha)$ (see (12)). Locally $T^*\mathcal{M}$ is generated by the basis $\mathfrak{B}_{T^*\mathcal{M}} = \{dr^\alpha, \epsilon^a, d\tilde{p}_\alpha\}$. During the rest of the paper, when there is no risk of confusion, we will use the same notation for 1-forms on Q and their pull back to $\mathcal{M}$ and T^*Q, (i.e., $\tau^*dr^\alpha = dr^\alpha$ and $\tau^*\epsilon^a = \epsilon^a$ where $\tau : \mathcal{M} \to Q$ is the canonical projection).

Since τ-projectable vector fields generate $T\mathcal{M}$ at each point, we can consider a complement $\mathcal{W}$ of $\mathcal{C}$ generated by τ-projectable vector fields Z_a such that $T\tau(Z_a) \in W$. That is,

$$\mathcal{C} = \operatorname{span}\left\{X_\alpha, \frac{\partial}{\partial \tilde{p}_\alpha}\right\} \quad \text{and} \quad \mathcal{W} = \operatorname{span}\left\{Z_a \, : \, T\tau(Z_a) = \frac{\partial}{\partial s^a}\right\}. \tag{19}$$

Lemma 2. *Let $\mathcal{W}$ be a complement of $\mathcal{C}$ as in (19) where W is the complement of D induced by the coordinates (r^α, s^a) as in Lemma 1.*

(i) The $\mathcal{W}$-valued 2-form $\mathbf{K}_{\mathcal{W}}$ and the curvature $\mathbf{K}_W$, defined in (9) and (14) respectively, are related, at each $X, Y \in T\mathcal{M}$, by $\mathbf{K}_W(T\tau(X), T\tau(Y)) = T\tau(\mathbf{K}_{\mathcal{W}}(X, Y))$. In local coordinates (r^α, s^a) adapted to the constraints (11), the following holds:

$$\mathbf{K}_{\mathcal{W}}|_{\mathcal{C}} = (C^a_{\alpha\beta} dr^\alpha \wedge dr^\beta|_{\mathcal{C}}) \otimes Z_a.$$

(ii) Let $\bar{\mathcal{W}}$ be a different complement of $\mathcal{C}$ such that $T\tau(\mathcal{W}) = T\tau(\bar{\mathcal{W}}) = W$. For $X, Y \in \Gamma(\mathcal{C})$ we have

$$\mathbf{K}_{\mathcal{W}}(X, Y) - \mathbf{K}_{\bar{\mathcal{W}}}(X, Y) \in \Gamma(\mathcal{C}).$$

Proof.

(i) During this proof and to avoid confusion, we will work with the basis $\{\tau^*dr^\alpha, \tau^*\epsilon^a, d\tilde{p}_\alpha\}$ of $T^*\mathcal{M}$, keeping dr^α and ds^a to denote 1-forms on Q. Let us consider the basis $\mathfrak{B} = \{X_\alpha, \frac{\partial}{\partial \tilde{p}_\alpha}, Z_a\}$ of $T\mathcal{M}$ adapted to $\mathcal{C} \oplus \mathcal{W}$ and its dual $\mathfrak{B}^* = \{\tau^*dr^a, \Psi_\alpha, \tau^*\epsilon^a\}$ where $\Psi_\beta(X_\alpha) = \Psi_\beta(Z_a) = 0$ and $\Psi_\beta(\frac{\partial}{\partial \tilde{p}_\alpha}) = \delta_{\alpha\beta}$. Then, for $X, Y \in \Gamma(\mathcal{C})$,

$$\begin{aligned}\mathbf{K}_{\mathcal{W}}(X, Y) &= -P_{\mathcal{W}}([X, Y]) = -\tau^*\epsilon^a([X, Y])Z_a \\ &= d\tau^*\epsilon^a(X, Y)Z_a = d\epsilon^a(T\tau(X), T\tau(Y))Z_a.\end{aligned}$$

Therefore, $T\tau(\mathbf{K}_{\mathcal{W}}(X, Y)) = d\epsilon^a(T\tau(X), T\tau(Y)) \otimes \frac{\partial}{\partial s^a} = \mathbf{K}_W(T\tau(X), T\tau(Y))$.

Finally, since $T\tau(X), T\tau(Y) \in \Gamma(D)$ (see (1)) and using (15) we obtain

$$\mathbf{K}_{\mathcal{W}}|_{\mathcal{C}} = (C^a_{\alpha\beta}\tau^* dr^\alpha \wedge \tau^* dr^\beta|_{\mathcal{C}}) \otimes Z_a.$$

Using our simplified notation ($\tau^* dr^\alpha = dr^\alpha$) we obtain the desired formula.

(ii) Let $\mathfrak{B}$ and $\mathfrak{B}^*$ be the basis as in item (i). Consider also $\bar{\mathfrak{B}} = \{X_\alpha, \frac{\partial}{\partial \tilde{p}_\alpha}, \bar{Z}_a\}$ a basis of $T\mathcal{M}$ adapted to $T\mathcal{M} = \mathcal{C} \oplus \bar{\mathcal{W}}$ such that $T\tau(\bar{Z}_a) = \frac{\partial}{\partial s^a}$ and its dual $\bar{\mathfrak{B}}^* = \{dr^\alpha, \bar{\Psi}_\alpha, \epsilon^a\}$, such that $\bar{\Psi}_\beta(X_\alpha) = \bar{\Psi}_\beta(\bar{Z}_a) = 0$ and $\bar{\Psi}_\beta(\frac{\partial}{\partial \tilde{p}_\alpha}) = \delta_{\alpha\beta}$. Then we have that, for $X, Y \in \mathcal{C}$,

$$\begin{aligned}\mathbf{K}_{\bar{\mathcal{W}}}(X, Y) &= -P_{\bar{\mathcal{W}}}([X, Y]) \\ &= \epsilon^a([X, Y])\bar{Z}_a = \mathbf{K}_{\mathcal{W}}(X, Y) + \epsilon^a([X, Y]) \otimes (\bar{Z}_a - Z_a).\end{aligned}$$

Since $\bar{Z}_a - Z_a \in \operatorname{Ker} T\tau \subset \mathcal{C}$ then $\mathbf{K}_{\mathcal{W}}(X, Y) - \mathbf{K}_{\bar{\mathcal{W}}}(X, Y) \in \mathcal{C}$. □

Remark 2. Note that the coordinates description of $\mathbf{K}_{\mathcal{W}}$ shows that it is semi-basic with respect to the bundle projection $\tau : \mathcal{M} \to Q$, i.e., $\mathbf{i}_X \mathbf{K}_{\mathcal{W}} = 0$ if $T\tau(X) = 0$. This is in agreement with [1, Prop. 3.1]

In order to write the nonholonomic bivector π_{nh} using (6) but in local coordinates $(r^\alpha, s^a; \tilde{p}_\alpha)$ on $\mathcal{M}$ we study the local description of the 2-section $\Omega_{\mathcal{M}}|_{\mathcal{C}}$.

The canonical 1-form Θ_Q on T^*Q is given, in local coordinates $(r^\alpha, s^a; \tilde{p}_\alpha, \tilde{p}_a)$, by $\Theta_Q = \tilde{p}_\alpha dr^\alpha + \tilde{p}_a \epsilon^\alpha$. Then, it is straightforward to see that the canonical 2-form Ω_Q is written locally as

$$\Omega_Q = dr^\alpha \wedge d\tilde{p}_\alpha + \epsilon^a \wedge d\tilde{p}_a - \tilde{p}_a d\epsilon^a.$$

Recall that $\iota : \mathcal{M} \to T^*Q$ is the natural inclusion, so the pull back of Ω_Q to $\mathcal{M}$ is given by

$$\Omega_{\mathcal{M}} = \iota^*\Omega_Q = dr^\alpha \wedge d\tilde{p}_\alpha + \iota^*\epsilon^a \wedge d\iota^*(\tilde{p}_a) - \iota^*(\tilde{p}_a)d(\iota^*\epsilon^a), \tag{20}$$

where dr^α and $d\tilde{p}_\alpha$ are considered as 1-forms on $\mathcal{M}$.

Therefore,

$$\begin{aligned}\Omega_{\mathcal{M}}|_{\mathcal{C}} &= dr^\alpha \wedge d\tilde{p}_\alpha - \iota^*(\tilde{p}_a)\iota^*(d\epsilon^a)|_{\mathcal{C}} \\ &= dr^\alpha \wedge d\tilde{p}_\alpha - J^\delta_a \tilde{p}_\delta C^a_{\alpha\beta} dr^\alpha \wedge dr^\beta|_{\mathcal{C}},\end{aligned} \tag{21}$$

where in the last equation we use (12) and the coordinate version of $d\epsilon|_D$ given in (15). Applying (6) to the 2-form $\Omega_{\mathcal{M}}$ and $\mathcal{C}$, given in (20) and (19) respectively, we compute the nonholonomic bivector field π_{nh} on $\mathcal{M}$:

$$\pi^\sharp_{\text{nh}}(dr^\alpha) = \frac{\partial}{\partial \tilde{p}_\alpha}, \quad \pi^\sharp_{\text{nh}}(ds^a) = -A^a_\alpha \frac{\partial}{\partial \tilde{p}_\alpha}, \quad \pi^\sharp_{\text{nh}}(d\tilde{p}_\alpha) = -X_\alpha + J^\delta_a \tilde{p}_\delta K^a_{\alpha\beta} \frac{\partial}{\partial \tilde{p}_\beta}. \tag{22}$$

Lemma 3. *The almost Poisson bracket $\{\cdot,\cdot\}_{\mathcal{M}}$ given in (17) (see [14, Theorem 2.1]) is the coordinate version of the nonholonomic bracket $\{\cdot,\cdot\}_{\rm nh}$ associated to the bivector field $\pi_{\rm nh}$ obtained from (6).*

4.2 The Jacobiator in Adapted Coordinates

Consider a nonholonomic system on a manifold Q given by a lagrangian L and a constraint distribution D such that ϵ^a, for $a = 1,\ldots,k$, are 1-forms generating D°. Consider local coordinates (r^α, s^a) on Q adapted to the constraints as in (11). Let $(r^\alpha, s^a; \tilde{p}_\alpha)$ be the coordinates on the manifold $\mathcal{M} = \kappa^\flat(D)$. By Lemma 3, the almost Poisson bracket $\{\cdot,\cdot\}_{\mathcal{M}}$ (17) is the coordinate version of the bivector field $\pi_{\rm nh}$ given in (6), and thus Koon-Marsden formula for the Jacobiator can be written directly with respect to $\{\cdot,\cdot\}_{\rm nh}$.

Theorem 2 ([14, Sec. 2.5]). *The Jacobiator of the nonholonomic bracket $\{\cdot,\cdot\}_{\rm nh}$, in coordinates $(r^a, s^a; \tilde{p}_\alpha)$ on $\mathcal{M}$, is given by the following formula*

$$
\begin{aligned}
\{\tilde{p}_\gamma, \{r^\alpha, \tilde{p}_\beta\}_{\rm nh}\}_{\rm nh} + \text{cyclic} &= J^\alpha_b K^b_{\beta\gamma},\\
\{\tilde{p}_\beta, \{s^a, \tilde{p}_\alpha\}_{\rm nh}\}_{\rm nh} + \text{cyclic} &= -K^a_{\alpha\beta} - A^a_\gamma J^\gamma_b K^b_{\alpha\beta},\\
\{\tilde{p}_\gamma, \{\tilde{p}_\alpha, \tilde{p}_\beta\}_{\rm nh}\}_{\rm nh} + \text{cyclic} &= \tilde{p}_\tau J^\tau_a \frac{\partial A^a_\gamma}{\partial s^b} K^b_{\alpha\beta} + \tilde{p}_\tau J^\tau_a K^a_{\delta\gamma} J^\delta_b K^b_{\alpha\beta}\\
&\quad - \tilde{p}_\tau K^b_{\alpha\beta}\left(\frac{\partial J^\tau_b}{\partial r^\gamma} - A^a_\gamma \frac{\partial J^\tau_b}{\partial s^a}\right) + \text{cyclic},
\end{aligned}
\tag{23}
$$

with all other combinations equal to zero and where J^α_b, $K^a_{\alpha\beta}$ and A^a_α are the functions on Q defined in (12), (16) and (11), respectively.

The next result relates the coordinate formula (23) of the Jacobiator with the coordinate-free formula given in Theorem 1.

Theorem 3. *Let (r^α, s^a) be coordinates on Q adapted to the constraints as in (11) and let W be the complement of D induced by the coordinates (Lemma 1). The Koon-Marsden Jacobiator formula (23) for the nonholonomic bracket $\{\cdot,\cdot\}_{\rm nh}$ is the coordinate version of the Jacobiator formula given in Theorem 1 for $\mathcal{W}$ any complement of $\mathcal{C}$ as in (19).*

Proof. In order to prove the equivalence we write the Schouten bracket $[\pi_{\rm nh}, \pi_{\rm nh}]$ using Theorem 1 evaluated on the elements $\{dr^\alpha, ds^a, d\tilde{p}_\alpha\}$.

First, observe that by Remark 2, the 2-form $\mathbf{K}_{\mathcal{W}}$ defined in (9) is annihilated by any of the elements $\pi^\sharp_{\rm nh}(dr^\alpha)$ or $\pi^\sharp_{\rm nh}(ds^\alpha)$ (see (22)). On the other hand, by Lemma 2(ii), we have that $\mathbf{K}_{\mathcal{W}}(\pi^\sharp_{\rm nh}(d\tilde{p}_\alpha), \pi^\sharp_{\rm nh}(d\tilde{p}_\beta)) = K^a_{\alpha\beta} Z_a$, where $Z_a \in T\mathcal{M}$ such that $T\tau(Z_a) = \frac{\partial}{\partial s^a}$. Moreover, observe that $\epsilon^a(Z_b) = \delta_{\alpha\beta}$ and $dr^\alpha(Z_a) = 0$.

Therefore, using the coordinate version of $\Omega_{\mathcal{M}}$ (20) in Theorem 1 we obtain

$$\begin{aligned}\frac{1}{2}[\pi_{\mathrm{nh}},\pi_{\mathrm{nh}}](dr^{\alpha},d\tilde{p}_{\beta},d\tilde{p}_{\gamma}) &= \Omega_{\mathcal{M}}(\mathbf{K}_{\mathcal{W}}(\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\beta}),\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\gamma})),\pi^{\sharp}_{\mathrm{nh}}(dr^{\alpha}))\\ &\quad - dr^{\alpha}(\mathbf{K}_{\mathcal{W}}(\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\beta}),\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\gamma})))\\ &= J^{\alpha}_{a}K^{a}_{\beta\gamma},\\ \frac{1}{2}[\pi_{\mathrm{nh}},\pi_{\mathrm{nh}}](ds^{a},d\tilde{p}_{\alpha},d\tilde{p}_{\beta}) &= \Omega_{\mathcal{M}}(\mathbf{K}_{\mathcal{W}}(\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\alpha}),\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\beta})),\pi^{\sharp}_{\mathrm{nh}}(ds^{a}))\\ &\quad - ds^{a}(\mathbf{K}_{\mathcal{W}}(\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\alpha}),\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\beta})))\\ &= -K^{b}_{\alpha\beta}A^{a}_{\gamma}J^{\gamma}_{b} - K^{a}_{\alpha\beta}.\end{aligned}$$

Finally, let $Y_a := Z_a - \frac{\partial}{\partial s^a} \in \operatorname{Ker} T\tau \subset \mathcal{C}$. Then, we have that

$$\begin{aligned}\frac{1}{2}[\pi_{\mathrm{nh}},\pi_{\mathrm{nh}}](d\tilde{p}_{\alpha},d\tilde{p}_{\beta},d\tilde{p}_{\gamma}) &= \Omega_{\mathcal{M}}(K^{a}_{\alpha\beta}Z_a,\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\gamma})) - d\tilde{p}_{\gamma}(K^{a}_{\alpha\beta}Z_a) + \text{cyclic}\\ &= \Omega_{\mathcal{M}}(K^{a}_{\alpha\beta}\frac{\partial}{\partial s^a},\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\gamma})) + \Omega_{\mathcal{M}}(K^{a}_{\alpha\beta}Y_a,\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\gamma}))\\ &\quad - d\tilde{p}_{\gamma}(K^{a}_{\alpha\beta}Y_a) + \text{cyclic}\\ &= \Omega_{\mathcal{M}}(K^{a}_{\alpha\beta}\frac{\partial}{\partial s^a},\pi^{\sharp}_{\mathrm{nh}}(d\tilde{p}_{\gamma})) + \text{cyclic}\\ &= \tilde{p}_{\tau}J^{\tau}_{a}\frac{\partial A^{a}_{\gamma}}{\partial s^b}K^{b}_{\alpha\beta} + \tilde{p}_{\tau}J^{\tau}_{a}K^{a}_{\delta\gamma}J^{\delta}_{b}K^{b}_{\alpha\beta}\\ &\quad - \tilde{p}_{\tau}K^{b}_{\alpha\beta}\left(\frac{\partial J^{\tau}_{b}}{\partial r^{\gamma}} - A^{a}_{\gamma}\frac{\partial J^{\tau}_{b}}{\partial s^{a}}\right) + \text{cyclic}.\end{aligned}$$

The Jacobiator on the other combinations of elements of the basis $\{dr^{\alpha}, ds^{a}, d\tilde{p}_{\alpha}\}$ is zero. Thus, the relation (5) implies that the Jacobiator formula in Theorem 1 evaluated in coordinates (11) gives the Koon-Marsden formula (23).

Observe that in this proof we are implicitly using Lemma 2 for $\mathcal{W}$ and $\bar{\mathcal{W}} = \operatorname{span}\left\{\frac{\partial}{\partial s^a}\right\}$.

Remark 3. From (10) it is straightforward to see that if the 2-form $\mathbf{K}_{\mathcal{W}}$ is zero then the bivector π_{nh} is Poisson. On the other hand, it was observed in [14] that if the curvature $\mathbf{K}_{W}$ is zero then the Jacobi identity of $\{\cdot,\cdot\}_{\mathrm{nh}}$ is satisfied. Using the equivalence between $\mathbf{K}_{\mathcal{W}}$ and $\mathbf{K}_{W}$ (Lemma 2(i)) we see that both 2-forms are zero when D is involutive, i.e., the system is holonomic.

Remark 4 (Symmetries). If the nonholonomic system admits a group of symmetries G then π_{nh} is G-invariant with respect to the induced (lifted) action on $\mathcal{M}$. As a consequence, the orbit projection $\mathcal{M} \to \mathcal{M}/G$ induces a reduced bivector field $\pi^{\mathrm{nh}}_{\mathrm{red}}$ on $\mathcal{M}/G$ describing the reduced dynamics. Let V (respectively $\mathcal{V}$) be the tangent space to the orbit of the G-action on Q (respect. on $\mathcal{M}$). If $W \subset V$ then there is a unique choice of the complement $\mathcal{W}$ contained in $\mathcal{V}$:

$$\mathcal{W} := (T\tau|_{\mathcal{V}})^{-1}(W).$$

With this choice of $\mathcal{W}$, Theorem 1 induces a formula for the Jacobiator of the reduced bivector $\pi_{\text{red}}^{\text{nh}}$ (see [1, Sec. 4]).

There are a number of examples of systems verifying that the complement W induced by the coordinates adapted to the constraints (11) (as in Lemma 1) is vertical with respect to a G-symmetry, including the vertical rolling disk, the nonholonomic particle and the Chaplygin sphere, see [1, Sec. 7].

On the other hand, it may happen that a given example is described in coordinates that are not adapted to the constraints. Then, it is better to use the coordinate free formula of Theorem 1.

5 Example: The Snakeboard

The snakeboard describes the dynamics of a board with two sets of actuated wheels, one on each end of the board. A human rider generates forward motion by twisting his body back and forth, and thus producing a movement on the wheels. This effect is modeled as a momentum wheel which sits in the middle of the board and is allowed to spin about the vertical axis. The configuration of the snakeboard is given by the position and orientation of the board in the plane, the angle of the momentum wheel and the angles of the back and front wheels. Therefore, the configuration manifold Q is given by $Q = SE(2) \times (-\pi/2, \pi/2) \times S^1$ with local coordinates $q = (x, y, \theta, \psi, \phi)$, where (x, y, θ) represents the position and orientation of the center of the board, ψ is the angle of the momentum wheel relative to the board and ϕ is the angle of the front and back wheel as in [17] (for details see [4] and [14]).

The Lagrangian is given by

$$L(q, \dot{q}) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{mr^2}{2}\dot{\theta}^2 + \frac{J_0}{2}\dot{\psi}^2 + J_0\dot{\psi}\dot{\theta} + J_1\dot{\phi}^2,$$

where m the total mass of the board, r is the distance between the center of the board and the wheels, J_0 is the inertia of the rotor and J_1 is the inertia of each wheel.

The (nonintegrable) constraint distribution D is given by the annihilator of the following 1-forms:

$$\begin{aligned} \epsilon^1 &= -\sin(\theta + \phi)dx + \cos(\theta + \phi)dy - r\cos\phi d\theta \\ \epsilon^2 &= -\sin(\theta - \phi)dx + \cos(\theta - \phi)dy + r\cos\phi d\theta. \end{aligned} \tag{24}$$

Remark 5. The coordinates $(x, y, \theta, \psi, \phi)$ on Q are not adapted to the 1-forms of constraints ϵ^1, ϵ^2. In [13] a simplified version is considered where, taking $\phi \neq 0$, it is possible to write the 1-forms of constraints in such a way that $(x, y, \theta, \psi, \phi)$ are adapted coordinates as in (11). In this paper, we will work with the 1-forms given in (24), so, our coordinates in Q are not adapted to the constraints, even though these are the coordinates chosen in [14] to study the reduction by the group of symmetries $SE(2)$.

The distribution D on Q is given by

$$D = \operatorname{span}\left\{ X_\psi := \frac{\partial}{\partial\psi},\ X_\phi := \frac{\partial}{\partial\phi},\ X_s := -2r\cos^2\phi\cos\theta\frac{\partial}{\partial x} \right.$$
$$\left. -2r\cos^2\phi\sin\theta\frac{\partial}{\partial y} + \sin(2\phi)\frac{\partial}{\partial\theta} \right\}.$$

We choose the complement W of D generated by $\{X_1, X_2\}$ so that $\epsilon^a(X_b) = \delta^a_b$ for $a, b = 1, 2$, that is

$$W = \operatorname{span}\left\{ X_1 := -\frac{1}{2}\sin\theta\sec\phi\frac{\partial}{\partial x} + \frac{1}{2}\cos\theta\sec\phi\frac{\partial}{\partial y} - \frac{1}{2r}\sec\phi\frac{\partial}{\partial\theta}, \right.$$
$$\left. X_2 := -\frac{1}{2}\sin\theta\sec\phi\frac{\partial}{\partial x} + \frac{1}{2}\cos\theta\sec\phi\frac{\partial}{\partial y} + \frac{1}{2r}\sec\phi\frac{\partial}{\partial\theta} \right\}.$$

Consider the dual basis $\mathfrak{B}_{TQ} = \{X_\psi, X_\phi, X_s, X_1, X_2\}$ and $\mathfrak{B}_{T^*Q} = \{d\psi, d\phi, \alpha_s, \epsilon^1, \epsilon^2\}$ where

$$\alpha_s = -\frac{1}{2r}\cos\theta\sec^2\phi dx - \frac{1}{2r}\sin\theta\sec^2\phi dy.$$

Let us denote by $(q; v_\psi, v_\phi, v_s, v_1, v_2)$ the coordinates on TQ associated with the basis $\mathfrak{B}_{TQ}$ while $(q; \tilde{p}_\psi, \tilde{p}_\phi, \tilde{p}_s, \tilde{p}_1, \tilde{p}_2)$ denote the coordinates on T^*Q associated to $\mathfrak{B}_{T^*Q}$.

The submanifold $\mathcal{M} = \kappa^\flat(D) = \operatorname{span}\{\kappa^\flat(X_\psi), \kappa^\flat(X_\phi), \kappa^\flat(X_s)\}$ is defined in coordinates by

$$\mathcal{M} = \{(q; \tilde{p}_\psi, \tilde{p}_\phi, \tilde{p}_s, \tilde{p}_1, \tilde{p}_2) \ :\ \tilde{p}_1 = -\tilde{p}_2 = J_1(\phi)\tilde{p}_s + J_2(\phi)\tilde{p}_\psi\}, \tag{25}$$

where

$$J_1(\phi) = \frac{mr}{4(r^2m - J_0\sin^2\phi)}\sin\phi\sec^2\phi \qquad \text{and} \qquad J_2(\phi) = -J_1(\phi)\sin(2\phi).$$

In order to compute the nonholonomic bivector π_{nh} describing the dynamics, we write the 2-form $\Omega_{\mathcal{M}}$ and the 2-section $\Omega_{\mathcal{M}}|_{\mathcal{C}}$ in our local coordinates. The canonical 1-form Θ_Q on T^*Q is given by $\Theta_Q = \tilde{p}_\psi d\psi + \tilde{p}_\phi d\phi + \tilde{p}_s\alpha_s + \tilde{p}_a\epsilon^a$. Then,

$$\begin{aligned}\Omega_Q &= d\psi \wedge d\tilde{p}_\psi + d\phi \wedge d\tilde{p}_\phi + \alpha_s \wedge d\tilde{p}_s - \tilde{p}_s d\alpha_s + \epsilon^1 \wedge d\tilde{p}_1 \\ &\quad + \epsilon^2 \wedge d\tilde{p}_2 - \tilde{p}_1 d\epsilon^1 - \tilde{p}_2 d\epsilon^2,\end{aligned}$$

Let us consider the basis $\mathfrak{B}_{T^*\mathcal{M}} = \{d\phi, d\psi, \alpha_s, \epsilon^1, \epsilon^2, d\tilde{p}_\phi, d\tilde{p}_\psi, d\tilde{p}_s\}$ of $T^*\mathcal{M}$ (here we are using the same notation for the pullbacks of the forms to $\mathcal{M}$). Recall that, on $\mathcal{M}$, $\tilde{p}_1$ and $\tilde{p}_2$ are given by (25) and denoting $J_i = J_i(\phi)$ for $i = 1, 2$ we obtain

$$\begin{aligned}\Omega_{\mathcal{M}} &= d\psi \wedge d\tilde{p}_\psi + d\phi \wedge d\tilde{p}_\phi + \alpha_s \wedge d\tilde{p}_s - \tilde{p}_s d\alpha_s + J_1\epsilon^1 \wedge d\tilde{p}_s + J_2\epsilon^1 \wedge d\tilde{p}_\psi \\ &\quad + \tilde{p}_s J_1'\epsilon^1 \wedge d\phi + \tilde{p}_\psi J_2'\epsilon^1 \wedge d\phi - J_1\epsilon^2 \wedge d\tilde{p}_s - J_2\epsilon^2 \wedge d\tilde{p}_\psi \\ &\quad - \tilde{p}_s J_1'\epsilon^2 \wedge d\phi - \tilde{p}_\psi J_2'\epsilon^2 \wedge d\phi - (J_1\tilde{p}_s + J_2\tilde{p}_\psi)(d\epsilon^1 - d\epsilon^2).\end{aligned} \tag{26}$$

On $T\mathcal{M}$ consider the dual basis $\mathfrak{B}_{T\mathcal{M}} = \left\{X_\psi, X_\phi, X_s, X_1, X_2, \frac{\partial}{\partial\tilde{p}_\psi}, \frac{\partial}{\partial\tilde{p}_\phi}, \frac{\partial}{\partial p_s}\right\}$ associated to $\mathfrak{B}_{T^*\mathcal{M}}$. Therefore, we can decompose $T\mathcal{M} = \mathcal{C} \oplus \mathcal{W}$ such that

$$\mathcal{C} = \text{span}\left\{X_\psi\,,\, X_\phi\,,\, X_s\,,\, \frac{\partial}{\partial\tilde{p}_\psi}\,,\, \frac{\partial}{\partial\tilde{p}_\phi}\,,\, \frac{\partial}{\partial p_s}\right\} \qquad \mathcal{W} = \text{span}\{X_1\,,\, X_2\}\,. \tag{27}$$

Therefore, using that $d\epsilon^a|_{\mathcal{C}} = (-1)^a 2r\cos\phi\alpha_s \wedge d\phi|_{\mathcal{C}}$ for $a = 1, 2$ and that $d\alpha_s|_{\mathcal{C}} = 2\tan\phi d\phi \wedge \alpha_s|_{\mathcal{C}}$, the 2-section $\Omega_{\mathcal{M}}|_{\mathcal{C}}$ is given by

$$\begin{aligned}\Omega_{\mathcal{M}}|_{\mathcal{C}} &= d\psi \wedge d\tilde{p}_\psi + d\phi \wedge d\tilde{p}_\phi + \alpha_s \wedge d\tilde{p}_s \\ &\quad - \tilde{p}_s 2\tan\phi d\phi \wedge \alpha_s + (J_1\tilde{p}_s + J_2\tilde{p}_\psi)4r\cos\phi\alpha_s \wedge d\phi\,|_{\mathcal{C}}.\end{aligned}$$

Now, we compute the nonholonomic bracket π_{nh} using (6)

$$\begin{aligned}\pi_{\text{nh}} &= \frac{\partial}{\partial\psi} \wedge \frac{\partial}{\partial\tilde{p}_\psi} + \frac{\partial}{\partial\phi} \wedge \frac{\partial}{\partial\tilde{p}_\phi} + X_s \wedge \frac{\partial}{\partial\tilde{p}_s} \\ &\quad - (\tilde{p}_s 2\tan\phi + 4r(J_1\tilde{p}_s + J_2\tilde{p}_\psi)\cos\phi)\frac{\partial}{\partial\tilde{p}_s} \wedge \frac{\partial}{\partial\tilde{p}_\phi}.\end{aligned} \tag{28}$$

Therefore, the hamiltonian vector fields are

$$\begin{aligned}&\pi_{\text{nh}}^\sharp(d\psi) = \frac{\partial}{\partial\tilde{p}_\psi}, \qquad \pi_{\text{nh}}^\sharp(d\phi) = \frac{\partial}{\partial\tilde{p}_\phi}, \\ &\pi_{\text{nh}}^\sharp(\alpha_s) = \frac{\partial}{\partial\tilde{p}_s}, \qquad \pi_{\text{nh}}^\sharp(\epsilon^i) = 0, \qquad \pi_{\text{nh}}^\sharp(d\tilde{p}_\psi) = -\frac{\partial}{\partial\psi},\end{aligned} \tag{29}$$

$$\pi^\sharp_{\text{nh}}(d\tilde{p}_\phi) = -\frac{\partial}{\partial\phi} + (2\tan\phi\,\tilde{p}_{\mathrm{s}} + 4r\cos\phi(J_1\tilde{p}_{\mathrm{s}} + J_2\tilde{p}_\psi))\frac{\partial}{\partial\tilde{p}_{\mathrm{s}}},$$
$$\pi^\sharp_{\text{nh}}(d\tilde{p}_{\mathrm{s}}) = -X_{\mathrm{s}} - (2\tan\phi\,\tilde{p}_{\mathrm{s}} + 4r\cos\phi(J_1\tilde{p}_{\mathrm{s}} + J_2\tilde{p}_\psi))\frac{\partial}{\partial\tilde{p}_\phi}.$$

In order to apply Theorem 1 to compute the Jacobiator of π_{nh} we study the $\mathcal{W}$-valued 2-form $\mathbf{K}_\mathcal{W}$ defined in (9) for $\mathcal{W}$ in (27). For $X, Y \in \mathcal{C}$ and using the dual basis $\mathfrak{B}_{T\mathcal{M}}$ and $\mathfrak{B}_{T^*\mathcal{M}}$ we have that

$$\begin{aligned}\mathbf{K}_\mathcal{W}(X,Y) &= -P_\mathcal{W}([X,Y]) = -\epsilon^1([X,Y])X_1 - \epsilon^2([X,Y])X_2\\ &= d\epsilon^1(X,Y)X_1 + d\epsilon^2(X,Y)X_2.\end{aligned}$$

Therefore,

$$\mathbf{K}_\mathcal{W}|_\mathcal{C} = -2r\cos(\phi)(\alpha_{\mathrm{s}} \wedge d\phi\,|_\mathcal{C}) \otimes (X_1 - X_2). \tag{30}$$

Finally, we consider the 2-forms $\Omega_\mathcal{M}$ and $\mathbf{K}_\mathcal{W}$, described in (26) and (30) and the vector fields (29), to obtain, by (10), that

$$\begin{aligned}&[\pi_{\text{nh}}, \pi_{\text{nh}}](d\tilde{p}_\phi, d\tilde{p}_{\mathrm{s}}, d\psi) = 4r\cos(\phi)J_2(\phi),\\ &[\pi_{\text{nh}}, \pi_{\text{nh}}](d\tilde{p}_\phi, d\tilde{p}_{\mathrm{s}}, \alpha) = 4r\cos(\phi)J_1(\phi)\\ &[\pi_{\text{nh}}, \pi_{\text{nh}}](d\tilde{p}_\phi, d\tilde{p}_{\mathrm{s}}, \epsilon^i) = (-1)^i 2r\cos(\phi), \quad i = 1,2,\end{aligned} \tag{31}$$

while on other combination of elements the Jacobiator is zero.

This example admits a symmetry given by the Lie group $SE(2)$, see [14]. The reduced manifold $\mathcal{M}/G$ is $S^1 \times S(-\pi/2, -\pi/2) \times \mathbb{R}^3$ and the nonholonomic bivector field π_{nh} is invariant by the orbit projection $\rho : \mathcal{M} \to \mathcal{M}/G$. Thus, on $\mathcal{M}/G$ we have the reduced nonholonomic bivector defined at each $\alpha \in T^*(\mathcal{M}/G)$ by

$$(\pi^{\text{nh}}_{\text{red}})^\sharp(\alpha) = T\rho\,\pi^\sharp_{\text{nh}}(\rho^*\alpha).$$

The Jacobiator of the reduced nonholonomic bivector field $\pi^{\text{nh}}_{\text{red}}$ satisfies

$$[\pi^{\text{nh}}_{\text{red}}, \pi^{\text{nh}}_{\text{red}}](\alpha, \beta, \gamma) = T\rho\left([\pi_{\text{nh}}, \pi_{\text{nh}}](\rho^*\alpha, \rho^*\beta, \rho^*\gamma)\right)$$

for $\alpha, \beta, \gamma \in T^*(\mathcal{M}/G)$. So, in our example it is simple to compute the Jacobiator of $\pi^{\text{nh}}_{\text{red}}$. Taking into account that, in local coordinates, the orbit projection $\rho : \mathcal{M} \to \mathcal{M}/G$ is given by $\rho(\psi, \phi, \theta, x, y, \tilde{p}_\psi, \tilde{p}_\phi, \tilde{p}_{\mathrm{s}}) = (\psi, \phi, \tilde{p}_\psi, \tilde{p}_\phi, \tilde{p}_{\mathrm{s}})$, the Jacobiator of the reduced bivector $\pi^{\text{nh}}_{\text{red}}$ describing the dynamics is given by

$$[\pi^{\text{nh}}_{\text{red}}, \pi^{\text{nh}}_{\text{red}}](d\tilde{p}_\phi, d\tilde{p}_{\mathrm{s}}, d\psi) = 4r\cos(\phi)J_2(\phi)$$

while on other elements of $T^*(\mathcal{M}/G)$ is zero.

Just to complete the example we can write, in our coordinates, the reduced bivector field $\pi_{\text{red}}^{\text{nh}}$ on $\mathcal{M}/G$:

$$\pi_{\text{nh}}^{\text{red}} = \frac{\partial}{\partial\psi}\wedge\frac{\partial}{\partial\tilde{p}_\psi}+\frac{\partial}{\partial\phi}\wedge\frac{\partial}{\partial\tilde{p}_\phi}-(\tilde{p}_{\mathrm{S}}2\tan\phi+4r(J_1\tilde{p}_{\mathrm{S}}+J_2\tilde{p}_\psi)\cos(\phi))\frac{\partial}{\partial\tilde{p}_{\mathrm{S}}}\wedge\frac{\partial}{\partial\tilde{p}_\phi}.$$

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Geometry of Image Registration: The Diffeomorphism Group and Momentum Maps

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Abstract These lecture notes explain the geometry and discuss some of the analytical questions underlying image registration within the framework of *large deformation diffeomorphic metric mapping* (LDDMM) used in computational anatomy.

1 Introduction

The goal of computational anatomy is to model and study the variability of anatomical shape. The ideas of computational anatomy originate in the seminal book "Growth and Form" by D'Arcy Thompson [66].

> In a very large part of morphology, our essential task lies in the comparison of related forms rather than in the precise definition of each; and the *deformation* of a complicated figure may be a phenomenon easy of comprehension, though the figure itself have to be left unanalysed and undefined. This process of comparison [...] finds its solution in the elementary use of a certain method of the mathematician. This method is the Method of Coordinates, on which is based the Theory of Transformations. [66, p. 1032]

More recently Grenander [25, 26, 28] generalized these ideas to encompass a diverse collection of real-world situations and formulated the principles of pattern theory. The following formulation is adapted from [51]:

1. A wide variety of signals result from observing the world, all of which show patterns of many kinds. These patterns are caused by laws present in the world, but at least partially hidden from direct observation.
2. Observations are affected by many variables that are not conveniently modelled deterministically because they are too complex or too difficult to observe.

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3. Patterns can be described as precise *pure patterns* distorted and transformed by a limited family of *deformations*.

To have a specific example in mind, we will consider computational neuroanatomy; i.e., the study of the form and shape of the brain [27]. The observations in this case are the diagnostic tools accessible to the clinician; of particular interest to us are noninvasive imaging techniques like computed tomography (CT), magnetic resonance imaging (MRI), functional MRI and diffusion tensor imaging (DTI). The hidden laws behind the observations are all the processes taking place at cellular, organ and environmental level, which together influence and form the anatomical shape of the brain.

To avoid having to model the brain from first principles, we observe instead that topologically all brains are very similar. If we take the MRI scans of two patients—volumetric grey-scale images of two brains—then we will be able to deform the contour surfaces of one image to approximately match the other. The study of shape and variability of brains within the framework of pattern theory, thus reduces to estimating the transformations that deform one brain image into an other. Given two images, the problem of finding this transformation is called the problem of *image registration*. One then compares these transformations in order to infer information about shape and variability.

1.1 Outline of the Notes

The purpose of these lecture notes is to explain the geometry that underlies image registration within the LDDMM framework and to show how it is used in computational anatomy. Section 1 introduces the main objectives of computational anatomy. Section 2 explains the Lie group concepts and Riemannian geometry underlying the image registration problem. Finally, Sect. 3 sketches some of the analytical problems that arise in image registration within the LDDMM framework. The references are not exhaustive. Throughout, we rely on the fundamental texts [51, 75].

1.2 Image Registration with LDDMM

Mathematically we model a volumetric grey-scale image I as a function $I : \mathbb{R}^3 \to \mathbb{R}$ and we denote by $\mathscr{F}(\mathbb{R}^3)$ the collection of all such functions, subject to certain smoothness assumptions. We model transformations φ as smooth, invertible maps $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ with smooth inverses. Such maps are called *diffeomorphisms*. Invertibility ensures that tissue is not torn apart or collapsed to single points. The set of all transformations is denoted by $\mathrm{Diff}(\mathbb{R}^3)$ and since it is closed under composition and taking the inverse, it forms a group, called the *diffeomorphism group*. Deforming an image I by the transformation φ corresponds to the change of coordinates $I \circ \varphi^{-1}$. In the transformed image $I \circ \varphi^{-1}$ the voxel $\varphi(x)$ has the same grey-value as the voxel x of the original image.

Given two images $I_0, I_1 \in \mathscr{F}(\mathbb{R}^3)$, the first approach to the image registration problem would be to search for $\varphi \in \mathrm{Diff}(\mathbb{R}^3)$, such that $I_0 \circ \varphi^{-1} = I_1$. Two things can go wrong. First, such a φ may not exist and second, if it exists, it may not be unique. To address these problems, we can introduce a distance $d_1(\varphi, \psi)$ on the set of transformations and a distance $d_2(I, J)$ on the set of images and search for the minimizer of

$$\operatorname*{argmin}_{\varphi} d_1(\mathrm{Id}, \varphi)^2 + \tfrac{1}{\sigma^2} d_2(I_0 \circ \varphi, I_1)^2. \tag{1}$$

The first term addresses the problem of uniqueness by ensuring that among all the transformations that deform I_0 into I_1 we pick the simplest one, by which we mean the one closest to the identity transformation. The second term allows us to compare images for which an exact solution to the registration problem does not exist, by requiring that the transformed image is close but not necessarily equal to I_1. Taken together (1) represents a balance between finding a simple transformation and one that reproduces the given image. The parameter σ^2 controls this balance between simplicity or regularity of the transformation and the registration accuracy.

There are many possible definitions of a distance $d_1(\mathrm{Id}, \varphi)$ on the space of smooth invertible maps. We shall concentrate on the definition used in the *large deformation diffeomorphic metric mapping* (LDDMM) approach [10, 47, 49, 67], which generates the transformation $\varphi = \varphi_1$ as the flow of a time-dependent vector field. That is, $t \mapsto \varphi_t$ is a solution to the flow equation

$$\partial_t \varphi_t(x) = u_t(\varphi_t(x)), \qquad \varphi_0(x) = x.$$

The distance $d_1(\mathrm{Id}, \varphi)$ is measured using a norm on the vector field u_t,

$$d_1(\mathrm{Id}, \varphi)^2 = \inf_{\{u_t \,:\, \varphi = \varphi_1\}} \int_0^1 |u_t|^2 \, \mathrm{d}t.$$

Regarding the distance $d_2(I, J)$ on images, the simplest choice is the L^2-norm, i.e. $d_2(I, J) = |I - J|_{L^2(\mathbb{R}^3)}$, which will be used throughout these notes. The problem of image registration via LDDMM will form the basis of the following discussion.

Definition 1 (Image Registration via LDDMM). Given two images $I_0, I_1 \in V$, find a time-dependent vector field $t \mapsto u_t \in \mathfrak{X}(\mathbb{R}^3)$ that minimizes the energy

$$E(u) = \frac{1}{2} \int_0^1 |u_t|^2 \, \mathrm{d}t + \frac{1}{2\sigma^2} \left| I_0 \circ \varphi_1^{-1} - I_1 \right|^2_{L^2(\mathbb{R}^3)} , \tag{2}$$

where $\varphi_t \in \mathrm{Diff}(\mathbb{R}^3)$ is the flow of u_t, i.e.

$$\partial_t \varphi_t(x) = u_t(\varphi_t(x)), \qquad \varphi_0(x) = x \ ,$$

The vector field $t \mapsto u_t$ and the transformation φ_1 are the solutions of the image registration problem.

This is not the only possible approach to image registration. In fact, a large literature about image registration exists. An overview of the available methods can be found, e.g. in [29, 36]. The LDDMM method, whilst being computationally more expensive than others, is among the most accurate [6] registration methods. Here we will concentrate on the geometric structure of the LDDMM solutions. In particular, we will sketch some applications in which the geometry behind LDDMM helps illuminate relationships between anatomical shape and neurological functions.

Data structures other than images can be registered within the LDDMM framework. These include landmarks [23, 35], curves [17, 24], surfaces [68], tensor fields [2, 14] or functional data on a manifold [46, 54]. In fact the abstract formulation of LDDMM in Sect. 2 encompasses all these examples. Instead of the L^2-norm one can use other similarity metrics to measure the distance between images, e.g. mutual information [39]. The biggest departure from LDDMM would be to change the way diffeomorphisms are generated. Possible approaches are stationary vector fields [5], free-form deformations [62], only affine transformations [34] or demons [65, 69]. Common to all these methods however is the loss of geometric structure.

1.3 Anatomical Shape and Function

Alzheimer's disease (AD) is a neurodegenerative disease and is the most frequent type of dementia in the elderly [19]. Related to AD is mild cognitive impairment (MCI), an intermediate cognitive state between healthy ageing and dementia. Although most patients who develop AD are first diagnosed with MCI, not all of those with MCI will develop AD. There is considerable variability among the prognoses of patients with MCI: some develop into AD, while others remain stable, revert back to normal cognitive status or develop other forms of dementia. It is therefore of interest to find methods of predicting the prognosis of patients with MCI. One approach is to look for manifestations of AD and MCI in the anatomical shape and to find connections between anatomical shape and clinical measures of cognitive status that are used to diagnose and distinguish between AD, MCI and normal cognitive state (NCS) [73].