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Introduction to Vortex Filaments in Equilibrium

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Introduction to Vortex Filaments in Equilibrium

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Preface

This book is an introduction to vortex filaments statistics in equilibrium. It is intended to be a survey of applications, methods, mathematical models, and computational methods for vortex filaments in ensembles wherever they appear focusing on hydrodynamics, plasmas (magnetohydrodynamics), and quantized vortices in superfluids and superconductors. The goal is to bring these distinct cases under the single umbrella of vortex statistics. We pay considerable attention to applications in this book, more than is typical in many mathematical treatments. These applications serve as motivation for the models we examine.

This book is intended to be for a graduate seminar or survey course. It is also intended to be a broad reference on a variety of topics in vortex filaments. Typically, in these courses one or two large projects are ideal for assignments. The book emphasizes understanding over computation, but, occasionally, we will ask questions or suggest problems to solve. The student should be familiar with calculus, vector calculus, differential equations, and partial differential equations, as well as know at least some programming. We do not assume prior knowledge of statistical mechanics.

This book differs from Lim and Nebus [90] in that it focuses on vortex filaments in several application areas rather than only hydrodynamics, including plasmas and quantum filaments. Also, while the previous book was directed at undergraduates and beginning graduate students, this book is at a more advanced level, focusing on graduate students and researchers.

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Notation

Ordinary vectors are usually given in bold, $\mathbf{v}$, while unit vectors have a caret $\hat{\cdot}$. Often we will use the unbolded letter to stand for the magnitude, e.g. $B = |\mathbf{B}|$. Integrals use the typical physicist's notation $\int d^d x f(\mathbf{x})$ where $\mathbf{x}$ is the d-vector variable of integration, d is the dimension, and $f(\mathbf{x})$ is the integrand. Line integrals are given by $\oint \mathbf{F} \cdot d\mathbf{l}$. Derivatives of the form $\partial f / \partial x$, f_x, $\partial_x f$, and f' are all equivalent. For simplicity we will also sometimes write $f(x)$ as f with the understanding that these are equivalent. If $\mathbf{F}$ is a vector valued function, then F_x is the x component rather than the derivative. We will use divergence, $\nabla \cdot \mathbf{F} = \partial_x F_x + \partial_y F_y + \partial_z F_z$, often.

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Chapter 1
Introduction

Vortex filaments and vortex dynamics have been an important subfield of fluid mechanics since Helmholtz's 1858 paper (translated to English by P. G. Tait) "On the Integrals of the Hydrodynamical equations, which Express Vortex Motion" and the work of William Thomson (Lord Kelvin) [143] and has since become a field of study in its own right. A vortex filament is a thin curve or line of vorticity embedded within a fluid and arises mainly because of rotational motion such as convection currents. In order for a vortex to be a filament it needs to obey the asymptotic relation: $\delta \ll R$ where δ is the core size and R is the curvature. A common further constraint can be placed, $R \ll 1$, in which case the filament is nearly straight as well.

In this book we will cover vorticity in several domains, including magnetohydrodynamics and quantum fluids, but here we will give a derivation for hydrodynamics. Vortex filaments come from the Navier–Stokes equations. Let $\omega = \nabla \times \mathbf{u}$ be the vorticity where $\mathbf{u}(\mathbf{x},t)$ is the fluid velocity localized in some domain D. The vorticity is given by

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega) + \nu \nabla^2 \omega,$$

where ν is viscosity. The first term on the right-hand side is the convection while the second term is the dissipation. We are primarily interested in fluid evolution where convection dominates (if dissipation dominates, as it does inside animal cells, then vorticity is not a concern). If fluid behavior is dominated by large-scale nonlinear interactions and dissipation is negligible, we can let $\nu \approx 0$, and use the Euler equations for ideal fluids,

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega).$$

Vorticity is said to be localized if its magnitude decreases super-exponentially outside a bounded region, i.e., $\omega(\mathbf{x},t)$ is localized in D at time t if there exists an $d > 0$ such that

$$\lim_{|\mathbf{x}| \to \infty} \left| e^{\mathbf{x}/d} \cdot \omega(\mathbf{x},t) \right| = 0.$$

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Also, if vorticity is localized at a time t_0 it is localized at any instant $t > t_0$. Let vorticity be localized.

Given that $\omega = \nabla \times \mathbf{u}$ and the fluid is divergence free, $\nabla \cdot \mathbf{u} = 0$, we can define a vector potential $\psi(\mathbf{x},t)$ such that $\mathbf{u} = \nabla \times \psi$ and $\nabla \cdot \psi = 0$, where ψ limits to a constant at infinity. Since,

$$\omega = \nabla \times (\nabla \times \psi) = \nabla(\nabla \cdot \psi) - \nabla^2 \psi,$$

the vorticity is given by Poisson's equation (in vector form),

$$-\omega = \nabla^2 \psi.$$

Poisson's equation has a solution,

$$\psi(\mathbf{x},t) = \frac{1}{4\pi} \int_\Omega \mathrm{d}^3 x' \frac{\omega(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|},$$

where Ω is a closed domain over which the vorticity is defined. Taking the curl, we get the fluid velocity,

$$\mathbf{u}(\mathbf{x},t) = \nabla \times \psi = \frac{1}{4\pi} \int_\Omega \mathrm{d}^3 x' \frac{\omega(\mathbf{x}',t) \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}. \tag{1.1}$$

Equation (1.1) is the Biot–Savart law of vortex dynamics in hydrodynamics [131] and can be generalized to magnetohydrodynamics (where the vorticity is a combination of vorticity and magnetic field discussed in Chap. 7). This law is the starting point for all vortex–vortex interactions and is critical to all studies of vortices in equilibrium.

Helmholtz gives three laws of vortex motion for an ideal, barotropic fluid being acted on by conservative external forces [131],

1. Fluid particles that are irrotational remain irrotational.
2. Fluid particles on a vortex line at any time remain on that vortex line for all past and future times.
3. The strength of a vortex tube is constant with respect to the fluid motion.

The first law is analogous to the conservation of particle number because it says that vortices cannot be created out of nothing. In other words, just as particles cannot appear in the universe out of nothing, because irrotational particles of the fluid remain irrotational, a vortex cannot simply appear in a fluid spontaneously. The second law means that vortex lines move with the fluid, i.e., if a particle moves, the vortex line moves with it so the particle always stays on the line. This law is analogous to the law of inertia or conservation of momentum. We know from general relativity that particles move with space and time, i.e., space and time act, in some respects, like a fluid medium. The conservation of momentum says that particles move with this medium. In the same way, a vortex line cannot change direction or move in a direction contrary to its medium. The third law is analogous to the conservation of charge/mass since vortex strength is similar to charge/mass in that it governs the strength of vortex interactions.

Once strong equivalences between vortex lines and ordinary particles were created, it made sense to study vortex gases, liquids, and solids as phenomena in their own right. Statistical equilibrium ensembles of vortex lines or points, however, were not considered until the 1940s when Lars Onsager calculated negative temperature states for fixed energy ensembles of vortex points in a circular container [117]. While, prior to his discovery, negative temperature states were believed to be unphysical, Onsager proved that above a certain maximal energy the temperature becomes negative and vortices of like sign, which at lower energies repelled one another, now attracted one another. This discovery encouraged continued study, and, in the 1970s, Edwards and Taylor [46] and Joyce and Montgomery [70] showed that Onsager's results also applied to magnetically confined plasmas where columns of electrons interacted in the same way as vortex lines, including having negative temperature states. Because these plasmas provide a potential route to hot magnetic nuclear fusion, the study of vortex filaments has since intensified.

Statistical mechanics forms the basis for the study of large numbers of vortices. Although macroscopic, systems or ensembles of vortex filaments have state variables such as temperature, entropy, pressure, specific heat, etc. which are the result of their combined interaction. These state variables, although statistical, become, in the limit as the number of vortices becomes large, exactly related to one another through equations of state (e.g., the ideal gas law for non-interacting particles, $PV = nRT$, also applies to non-interacting vortex lines, albeit in two dimensions).

In this book we will deal exclusively with equilibrium states, although we will occasionally touch on what happens when equilibrium states are not stable. What counts as a stable state and how long is enough time depend on the system and are often open problems. In the case of gases in a box, the quintessential subject of statistical mechanics, stability can be indefinite provided the box is either completely insulated from the outside world or in a stable heat bath. Gas molecules are extremely stable, of course, particularly in isolation. With more exotic types of particles that tend to decay or change over time, outer stability is no guarantee of inner stability. In many systems, the so-called stable states can be very short before outside influences bring the system to a different state. Vortices, in particular, tend to dissipate. They can, however, form configurations within a short time period that can be described by equilibrium statistics.

One of the major early discoveries in the study of interacting vortices is that they behave a lot like charged particles. When vortex lines are assumed to be perfectly straight, parallel, and infinitesimally thin, equations of state may be derived from Poisson's two-dimensional equation,

$$\nabla^2 \phi = -4\pi\rho, \tag{1.2}$$

relating the interaction potential to the density of vorticity, ρ, and the energy functional,

$$E = \int \mathrm{d}^2x \int \mathrm{d}^2x' \rho(\mathbf{x})\rho(\mathbf{x}') \log|\mathbf{x}-\mathbf{x}'|. \tag{1.3}$$