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Ronald G. Douglas
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The Corona Problem

Connections Between Operator Theory, Function Theory, and Geometry

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Preface

In June 2012, a workshop on the corona problem was held at the Fields Institute in Toronto, Ontario, Canada. The organizers were Ronald G. Douglas, Steven G. Krantz, Eric T. Sawyer, Sergei Treil, and Brett D. Wick. About forty people attended the workshop. The weeklong event was exciting, stimulating, and productive. Several new papers grew out of the interactions, and they appear in this volume.

In particular, we offer a history of the corona problem—the first article of its kind. The other articles that we present describe various directions of research, and many offer new results. All of the articles were refereed to a high standard, and each represents original and incisive scholarship.

We thank the Fields Institute, and particularly the Director Ed Bierstone, for providing a supportive and nurturing atmosphere, and financial support, for our mathematical work. We also thank the National Science Foundation for financial support.

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Contents

A History of the Corona Problem

Ronald G. Douglas, Steven G. Krantz, Eric T. Sawyer, Sergei Treil, and Brett D. Wicks

Abstract We give a history of the Corona Problem in both the one variable and the several variable setting. We also describe connections with functional analysis and operator theory. A number of open problems are described.

Keywords Domain • Pseudoconvex • Corona • Maximal ideal space • Bounded analytic functions • Multiplicative linear functional

Subject Classifications: 30H80, 30H10, 30J99, 32A65, 32A70, 32A38, 32A35

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1 Ancient History

The idea of a *Banach algebra* was conceived by I. M. Gelfand in his thesis in 1936. A Banach algebra is a complex, normed algebra which is a complete Banach space in the norm metric.

The theory of Banach algebras rapidly revealed itself to be a rich and powerful structure for attacking many different types of problems in analysis. It represented a beautiful marriage of analysis, functional analysis, algebra, and topology. It gave elegant, soft proofs of results in classical analysis (such as the Wiener Inversion Theorem, a particular case of Wiener's Tauberian Theorem) that were quite difficult to prove by classical methods.

In 1941, S. Kakutani posed the *Corona Problem*. The question concerned the maximal ideal space of the Banach algebra $H^\infty(\mathbb{D})$, where $\mathbb{D} \subseteq \mathbb{C}$ is the unit disc. The only maximal ideals of this algebra that one can actually "write down" are the point evaluation functionals at $z \in \mathbb{D}$. The question was whether the point evaluation functionals are dense in the maximal ideal space (in the weak-$*$ topology). If the point evaluation functionals are not dense in the maximal ideal space of H^∞, then a big chunk of the maximal ideal space "sticks out" off the set of point evaluations (i.e., off the unit disc $\mathbb{D}$), akin the sun's Corona.

People were fascinated by this question, but made little headway on it for twenty years or more. Some preliminary remarks on related ideas in function algebras appear in [32]. A lovely paper [57] was written in 1961 that laid the foundations for the study of the Corona Problem.[1] Among the key results of [57] are the following:

(a) The notion of *fiber* is introduced. The maximal ideals which are not point evaluations live in fibers over boundary points of the disc.
(b) The paper gives a complete and explicit description of the Sĭlov boundary of $H^\infty(\mathbb{D})$.
(c) Even though any two fibers are homeomorphic, it is shown that the maximal ideal space less the disc (the point evaluations) is *not* the product of the circle with a fiber.
(d) It is shown that each fiber contains a homeomorphic replica of the entire maximal ideal space.

It is a good exercise in functional analysis to translate (recalling the definition of the topology in the maximal ideal space) the topological statement about density given above into the following algebraic *Bezout formulation*:

[1]This paper is very entertaining because its author I. J. Schark is a fiction. "I. J. Schark" is actually an acronym for the authors Irving Kaplansky, John Wermer, Shizuo Kakutani, R. Creighton Buck, Halsey Royden, Andrew Gleason, Richard Arens and Kenneth Hoffman. The letters of "I. J. Schark" come from their first initials. The references to this paper, plus consultation with experts, show that virtually no work was done on the Corona Problem between 1941 and 1961.

Suppose that $f_1, f_2, \ldots, f_k$ are bounded, analytic functions on the disc $\mathbb{D}$ that satisfy

$$|f_1(\zeta)| + |f_2(\zeta)| + \cdots |f_k(\zeta)| > \delta > 0 \quad \forall \zeta \in \mathbb{D}$$

for some positive, real number δ. Then do there exist bounded, analytic functions $g_1, g_2, \ldots, g_k$ such that

$$f_1(\zeta)g_1(\zeta) + f_2(\zeta)g_2(\zeta) + \cdots + f_k(\zeta)g_k(\zeta) \equiv 1\,?$$

It is a pleasure to thank Nikolai Nikolski for enlightening conversations about some of the topics of this paper. We also want to thank Ted Gamelin, Nessim Sibony, Richard Rochberg, and Tavan Trent for reading an early draft of this history and providing many interesting comments that ultimately improved the paper.

2 Modern History

The Corona Problem was finally answered by Lennart Carleson in his seminal paper [19], which built on foundational work in [18]. Carleson's paper was important not only for his main theorem, but for the techniques that he introduced to solve the problem. In particular, one of the main tools used in Carleson's solution was the idea of *Carleson measure*, an idea that has become of pre-eminent importance in function theory and harmonic analysis. Carleson uses these measures to control the lengths of certain curves in the disc that wind around the zeros of a bounded analytic function. This construction was very clever and quite involved and has proved to be useful in other areas of mathematics. In particular, as pointed out by Peter Jones, [38]: *The corona construction is widely regarded as one of the most difficult arguments in modern function theory. Those who take the time to learn it are rewarded with one of the most malleable tools available. Many of the deepest arguments concerning hyperbolic manifolds are easily accessible to those who understand well the corona construction.*

In the mid-1960s, Edgar Lee Stout [61] and Norman Alling [3] proved that the Corona Theorem remains true on a finitely-connected Riemann surface. By contrast, Brian Cole [29] gave an example of an infinitely connected Riemann surface on which the Corona Theorem fails. Cole's counterexample was built by exploiting the connections between representing measures and uniform algebras. Around the same time, Kenneth Hoffman [35] showed that there is considerable analytic structure in the fibers of the maximal ideal space of H^∞. It may be mentioned that the paper [57] also constructs analytic discs in the fibers.

In the remarkable paper [36], Lars Hörmander introduced a new method for studying the Corona Problem. His approach was first to construct a preliminary non-analytic solution of the Bezout equation, and then "correct" it to get an analytic one by solving an appropriate inhomogeneous $\overline{\partial}$-equations. He used the Koszul complex to find these equations, although in the one complex variable case (he also considered some algebras of analytic functions in several variables) one can use elementary methods to deduce the equations, especially after the equations are already known.

In the unit disc he used the result that the equation

$$\bar{\partial} w = \mu \,,$$

has a bounded solution if μ is a Carleson measure (this fact can be easily proved by duality). To construct the preliminary solution, given a Carleson measure in the right hand side of the $\bar{\partial}$-equation, Hörmander used the Carleson contours from the original Carleson construction, so the main technical difficulty remained.

However, the main advantage of the new approach was that it allowed one to solve the Bezout equation with n functions (generators). The original Carleson construction gave a method of solving the equation with two generators. To move from two to n generators Carleson used a clever trick. This trick was based on the Riemann Mapping Theorem, so there was no hope to generalize it to higher dimensions.

Hörmander's paper raised hopes that the Corona Theorem can be generalized to higher dimensions, but the hopes for an easy generalization were squashed by N. Varopoulos [82], who had shown that in the unit ball in $\mathbb{C}^2$, the Carleson measure condition on the right hand side does not imply existence of a bounded solution of the $\bar{\partial}$-equation.

In 1979 T. Wolff presented a new proof of the Corona Theorem, which followed Hörmander's approach with one critical difference. Wolff used a different condition for the existence of a bounded solution of the $\bar{\partial}$-equation $\partial w = G$, based on a "second order" Green's formula. That was crucial because, for the trivial non-analytic solution $g_k = \overline{f}_k / \sum_j |f_j|^2$ of the Bezout equation $\sum_k f_k g_k \equiv 1$, the right sides of the corresponding $\bar{\partial}$-equations satisfied this condition.

In the remaining sections we discuss some of the major contributions of this history in more detail.

3 The Corona Theorem On the Disc

Theorem 1 (Carleson, [19]). *Suppose that* $f_1, \dots, f_n \in H^\infty(\mathbb{D})$ *and there exists a* $\delta > 0$ *such that*

$$1 \geq \max_{1 \leq j \leq n} \{ |f_j(z)| \} \geq \delta > 0. \tag{1}$$

Then there exists $g_1, \dots, g_n \in H^\infty(\mathbb{D})$ *such that*

$$1 = f_1(z) g_1(z) + \cdots + f_n(z) g_n(z) \quad \forall z \in \mathbb{D}$$

and

$$\|g_j\|_{H^\infty(\mathbb{D})} \le C(\delta, n) \quad \forall j = 1, \dots, n.$$

An obvious remark is that the condition on the functions f_j is clearly necessary. In [19] the assumption $\sum_{k=1}^{n} |f_k(z)| \ge \delta > 0$ was used, but if one is not after sharp estimates, it does not matter what ℓ^p norm we use.

3.1 Carleson Embedding Theorem

One of Carleson's powerful and influential ideas in this development is the notion of *Carleson measure*, which was introduced earlier in [18] in connection with the interpolation problem.

A nonnegative measure μ on the unit disc $\mathbb{D}$ that satisfies

$$\mu(S) \le C \cdot \ell \tag{2}$$

for any set S of the form

$$S = \{re^{i\theta} : r \ge 1 - \ell, \theta_0 \le \theta \le \theta_0 + \ell\}$$

is called a *Carleson measure*.

Theorem 2 (Carleson Embedding Theorem). *For any* $p > 0$ *the estimate (embedding)*

$$\int_{\mathbb{D}} |f(z)|^p \, d\mu(z) \le C_1 \|f\|_{H^p} \qquad \forall f \in H^p(\mathbb{D}) \tag{3}$$

holds if and only if the measure μ *is Carleson (i.e., satisfies (2)).*

Moreover, the best constant C_1 *is the same for all* $p > 0$*, and the best* C_1 *is equivalent to the best* C *in (2) in the sense of two-sided estimates:*

$$A^{-1}C \le C_1 \le AC,$$

where A *is an absolute constant.*

The Carleson Embedding Theorem is now a staple of harmonic analysis, and many different proofs exist in the literature. The fact that the best C_1 is the same for all $p > 0$ is an easy corollary of the Nevanlinna factorization of H^p functions.

Definition. The best constant C_1 in (3) is called the *Carleson norm* of the measure μ. Note that sometimes the best constant C in (2) is used for the Carleson norm: since C_1 and C are equivalent. If one does not look for exact constants, it does not matter what definition is used.

3.2 Carleson's Original Proof and Interpolation

As was acknowledged in [19], Carleson's original strategy for attacking the Corona Problem comes from D.J. Newman [47]. Newman had shown that the Corona Theorem follows from a certain interpolation result, which Carleson then proved. According to L. Ehrenpreis (see his recollection in the paper [86] dedicated to D.J. Newman's memory) Newman's contribution to the Corona Theorem was more significant than the credit he received.

The original Carleson proof worked as follows. First, using a standard normal family argument one can assume without loss of generality that all the generators f_k are holomorphic in a slightly bigger disc. Then a pretty straightforward (although not completely trivial) argument allows one to assume that one of the f_ks, say f_n, is a finite Blaschke product with simple zeroes. The next step reduced solution of the Bezout equation with two generators to a solution of an interpolation problem.

Namely, if one wants to find g_1 and g_2 such that $f_1 g_1 + f_2 g_2 \equiv 1$, and one of the generators, say f_2, is a Blaschke product with simple zeroes then finding a bounded solution of the interpolation problem

$$g_1(\lambda) = 1/f_1(\lambda), \qquad \forall \lambda \in Z(f_2) \tag{4}$$

($Z(f)$ denotes the zero set of f) solves the Bezout equation: one just needs to define $g_2 := (1 - f_1 g_1)/f_2$. Existence of a bounded solution of the above interpolation problem follows from the following theorem, which is the technical crux of the proof.

Theorem 3. *Let A be finite Blaschke product with simple zeroes. Assume that $\delta < 1/2$ and that F is a holomorphic function in $\{z \in \mathbb{D} : |A(z)| < \delta\}$ bounded by 1. Then the interpolation problem*

$$g(\lambda) = F(\lambda)\ , \qquad \forall \lambda \in Z(A)\,, \tag{5}$$

has a solution $g \in H^\infty$ with $\|f\|_\infty \leq C(\delta) < \infty$.

To see how the solution of the interpolation problem follows from the above theorem, note that the condition (1) implies that $|f_2(z)| \geq \frac{\delta}{2}$ whenever $|f_1(z)| < \frac{\delta}{2}$. Therefore, applying Theorem 3 with $1/f_2$ on $\{z \in \mathbb{D} : |f_1(z)| < \frac{\delta}{2}\}$ being the (rescaled) function F, we get that (4) has a solution $g_1 \in H^\infty$, $\|g_1\|_\infty \leq \frac{1}{\delta} C(\delta)$

To prove the case of n generators, the following clever induction trick was used. Assuming that the theorem holds for $n-1$ generators, one can show that there exist bounded analytic functions $p_1, \ldots, p_{n-1}$ defined on $\Omega := \{z \in \mathbb{D} : |f_n(z)| < \frac{\delta}{2}\}$ such that

$$\sum_{k=1}^{n-1} f_k(z) p_k(z) \equiv 1, \qquad \forall z \in \Omega. \tag{6}$$

Indeed,

$$\max_{k=1,\dots,n-1} |f_k(z)| \geq \frac{\delta}{2} \qquad \forall z \in \Omega.$$

By the maximum principle, connected components of Ω are simply connected (and so conformally equivalent to $\mathbb{D}$), therefore by the induction hypothesis the Bezout equation (6) has a bounded solution in each connected component of Ω.

Finding bounded solutions (in all of $\mathbb{D}$) of the interpolation problems

$$g_k(\lambda) = p_k(\lambda), \qquad \forall \lambda \in Z(f_n)$$

and defining $g_n := (1 - \sum_{k=1}^{n-1} f_k g_k)/f_n$ (recall that f_n is a finite Blaschke product with simple zeroes), we get a bounded solution of the Bezout equation.

The main technical part of the Carleson proof is the proof of Theorem 3. To prove this theorem, Carleson constructed what was later named the *Carleson contour*. Namely, he proved that, for $\delta < 1/2$, there exists $\varepsilon(\delta)$, $0 < \varepsilon(\delta) \leq \delta$ such that, for any $A \in H^\infty$ and for any δ, $0 < \delta < 1/2$, there exists a domain $\Omega = \Omega_{A,\delta}$ such that

1. $\{z \in \mathbb{D} : |A(z)| < \varepsilon(\delta)\} \subset \Omega \subset \{z \in \mathbb{D} : |A(z)| < \delta\}$;
2. the arclength on $\partial\Omega$ is a Carleson measure (with the norm depending only on δ).

The boundary $\partial\Omega$ is now known as the *Carleson contour*.

Remark. In Carleson's paper $\varepsilon(\delta) = \delta^\kappa$ and the Carleson norm of the contour was estimated by $C\delta^{-\kappa_1}$. In [16] J. Bourgain constructed a Carleson contour (for an inner function) with the Carleson norm not depending on δ.

The construction of the Carleson contour was rather technical and was based on the stopping moment technique—which was "stolen" from probability. The stopping moment technique is now a commonplace in harmonic analysis, and now people often refer to a decomposition obtained using stopping moment technique (similar to that used by Carleson, but often significantly more involved) as the "Corona decomposition."

After the Carleson contour is constructed the proof of Theorem 3 is fairly straightforward. Namely, any solution g of the interpolation problem (5) can be represented as

$$g = g_0 + Ah, \qquad h \in H^\infty,$$

where g_0 is one of the solutions (the Lagrange interpolating polynomial, for example). By the Hahn–Banach Theorem, the smallest norm $\|g\|_\infty = \|g/A\|_\infty$ is the norm of the linear functional on H^∞

$$f \mapsto \int_{\mathbb{T}} \frac{fg}{B} \frac{dz}{2\pi} = \int_{\mathbb{T}} \frac{fg_0}{B} \frac{dz}{2\pi} = \int_{\partial\Omega} \frac{fg_0}{B} \frac{dz}{2\pi} = \int_{\partial\Omega} \frac{fF}{B} \frac{dz}{2\pi}$$

(all the equalities above can be verified by the Residue Theorem). Since the arclength on $\partial\Omega$ is a Carleson measure, the last integral can be estimated using the Carleson Embedding Theorem for H^1.

3.3 Hörmander's Construction

Suppose we constructed some functions φ_k, that are bounded in the unit disc $\mathbb{D}$ and also solve the Bezout equation

$$\sum_k \varphi_k f_k \equiv 1.$$

These functions are in general not analytic, and so to make them analytic we must "correct" them. To do this we define

$$g_j(z) = \varphi_j(z) + \sum_{k=1}^{n} a_{j,k}(z) f_k(z)$$

where the functions $a_{j,k}(z) = -a_{k,j}(z)$ are to be determined. The above alternating condition implies that

$$\sum_{j=1}^{n} f_j(z) g_j(z) = \sum_{j=1}^{n} f_j(z)\varphi_j(z) + \sum_{j=1}^{n}\sum_{k=1}^{n} a_{j,k}(z) f_j(z) f_k(z) = 1,$$

so the functions g_k defined as above always solve the Bezout equation.

To have the alternating condition $a_{j,k} = -a_{k,j}$ we set $a_{j,k} = b_{j,k}(z) - b_{k,j}(z)$ for some yet to be determined functions. If we chose the functions $b_{j,k}$, to be solutions to the following $\overline{\partial}$ problem:

$$\overline{\partial} b_{j,k} = \varphi_j \overline{\partial} \varphi_k := G_{j,k}, \tag{7}$$

then

$$\begin{aligned} \overline{\partial} g_j &= \overline{\partial}\varphi_j + \sum_{k=1}^{n} f_k \overline{\partial} a_{j,k} \\ &= \overline{\partial}\varphi_j + \sum_{k=1}^{n} f_k \left(\overline{\partial} b_{j,k} - \overline{\partial} b_{k,j}\right) \\ &= \overline{\partial}\varphi_j + \sum_{k=1}^{n} f_k \left(\varphi_j \overline{\partial}\varphi_k - \varphi_k \overline{\partial}\varphi_j\right) \end{aligned}$$

$$= \overline{\partial}\varphi_j + \varphi_j \overline{\partial}\left(\sum_{k=1}^{n} f_k \varphi_k\right) - \overline{\partial}\varphi_j \sum_{k=1}^{n} f_k \varphi_k$$
$$= \overline{\partial}\varphi_j + \varphi_j \overline{\partial} 1 - \overline{\partial}\varphi_j 1 = 0,$$

so the functions g_j are analytic.

Thus, finding an H^∞ solution of the Bezout equation $\sum_k f_k g_k \equiv 1$ is reduced to finding a bounded solution of the $\overline{\partial}$-equation (7). Note that, since at the end we are getting analytic functions g_k, it is sufficient to prove that the solutions $b_{j,k}$ are bounded on the circle $\mathbb{T}$.[2]

Hörmander used the fact that the equation

$$\overline{\partial} w = \mu$$

has a bounded on $\mathbb{T}$ solution whenever the variation $|\mu|$ is a Carleson measure; this fact can be easily obtained by a duality argument and using the Carleson Embedding Theorem. Note that, if the right side is a function G, then we just require that the measure $G dA(z)$ is a Carleson measure.

For the bounded solutions φ_k Hörmander chose

$$\varphi_k := \frac{1 - \mathbf{1}_{\Omega_k}}{f_k} \left(\sum_{k=1}^{n} (1 - \mathbf{1}_{\Omega_k})\right)^{-1},$$

where $\partial\Omega_k$ is the Carleson contour for f_k with δ/n for δ.

The derivative $\overline{\partial}\varphi_k$ in the definition of G_k can understood in the sense of distributions. If one wants to avoid the technical difficulties of working with distributions, one can consider smoothing out the characteristic functions $\mathbf{1}_{\Omega_k}$. Hörmander in [36] only sketched the proof, but did not give any details. A reader interested in all the details should look at J. Garnett's monograph [30, Ch. VIII, Sect 5] where Hörmander's construction was "smoothed out."

It should be mentioned that Hörmander's approach to the corona problem, and perhaps Tom Wolff's solution (discussed below) inspired Peter Jones [67] to come up with an interesting new way to attack the problem. Jones constructs, on the upper halfplane, a nonlinear solution to the $\overline{\partial}$ problem and uses that together with the Koszul complex to effect a corona solution.

[2] Of course there are some technical details about interpreting the boundary values of the function $b_{j,k}$, but one can avoid such difficulties by first assuming that the corona data is analytic in a bigger disc and then using a standard normal family argument.

3.4 Wolff's Solution

T. Wolff's solution followed Hörmander's construction with the (simplest possible) bounded non-analytic solutions φ_k given by

$$\varphi_k := \frac{\overline{f}_k}{\sum_{j=1}^n |f_j|^2}. \tag{8}$$

To prove the existence of bounded solutions of the $\overline{\partial}$-equations (7) on $\mathbb{T}$, he introduced the following theorem.

Theorem 4 (Wolff, [7, 30, 40, 42, 48]). *Suppose that $G(z)$ is bounded and smooth on the closed disc $\overline{\mathbb{D}}$. Further assume that the measures*

$$|G|^2 \log \frac{1}{|z|} dA(z) \quad \text{and} \quad |\partial G| \log \frac{1}{|z|} dA(z)$$

are Carleson measures with Carleson norms B_1 and B_2 respectively. Then the equation

$$\overline{\partial} b = G$$

has a solution $b \in C^1(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and, moreover,

$$\|b\|_{L^\infty(\mathbb{T})} \leq C_1 \sqrt{B_1} + C_2 B_2$$

where C_1 and C_2 are absolute constants.

The proof of this theorem is a clever application of Green's Theorem together with the Carleson Embedding Theorem.

If one does not care about sharp estimates, the Corona Theorem follows from Theorem 4 almost immediately. Namely, it is an easy exercise to show that, if $f \in H^\infty$, then the measure $|f'(z)|^2 \ln \frac{1}{|z|} dA(z)$ is Carleson with the Carleson norm at most $C\|f\|_\infty$. Then the Corona Theorem follows immediately if one notes that, for $G = G_{j,k} = \varphi_j \overline{\partial} \varphi_k$ with φ_k defined by (8), we have

$$|G|^2, \; |\overline{\partial} G| \leq C \sum_{l=1}^n |f_l|^2.$$

To get sharper estimates, the following lemma attributed to A. Uchiyama can be used.

Lemma 1 (Uchiyama's lemma). *Let u, $0 \le u \le 1$, be a subharmonic function. Then the measure $\Delta u(z) \ln \frac{1}{|z|} dxdy$ is a Carleson measure with Carleson norm at most $2\pi e$, meaning that*

$$\int_{\mathbb{D}} |f(z)|^2 \Delta u(z) \ln \frac{1}{|z|} dxdy \le 2\pi e \|f\|_{H^2}^2 \qquad \forall f \in H^2.$$

The proof of the lemma is, again, a clever application of Green's formula. It can be found, for example, in the monograph [48, Lemma 6 in Appendix 3].

3.5 Estimates and Corona Theorems with infinitely many generators

In 1980 M. Rosenblum [55] and V. Tolokonnikov [64], using a modification of T. Wolff's proof, independently proved that with correct normalization the Corona Theorem holds for infinitely many generators f_k.

Theorem 5 (M. Rosenblum [55], V. Tolokonnikov [64]). *Let $f_k \in H^\infty$, $k \in \mathbb{N}$ satisfy*

$$1 \ge \left(\sum_{k=1}^{\infty} |f_k(z)|^2 \right)^{1/2} \ge \delta > 0 \qquad \forall z \in \mathbb{D}. \tag{9}$$

Then there exist functions $g_k \in H^\infty$, such that $\sum_k f_k g_k \equiv 1$ and such that

$$\left(\sum_{k=1}^{\infty} |g_k(z)|^2 \right)^{1/2} \le C(\delta)$$

In both [55] and [64], $C(\delta) = C\delta^{-4}$.

Later A. Uchiyama, in an unpublished but extremely influential preprint [81], proved the above theorem with $C(\delta) = C\delta^{-2} \ln \delta^{-1}$ for small δ, which remains the best known to date (even in the case of 2 generators). For the proof of this result the reader could look at [48, Appendix 3]. The main idea is to use Wolff's method, estimating $a_{j,k} = b_j - b_k$ not separately, but estimating instead the Hilbert–Schmidt norm of the matrix $(a_{j,k})_{j,k=1}^{\infty}$. To do that one treats the system of $\overline{\partial}$-equations as a vector-valued equation and uses Lemma 1 with appropriately chosen functions u. As in the proof of Theorem 4 the norm of the solution is estimated by duality.

In [81] A. Uchiyama also proved that the Corona Theorem with the ℓ^∞ normalization of the corona data as in (1) also holds with infinitely many generators. Namely, he proved that if the functions $f_k \in H^\infty$ satisfy the condition

$$1 \geq \max_{k=1,\dots,n} |f_k(z)| \geq \delta > 0 \qquad \forall z \in \mathbb{D},$$

then there exist $g_k \in H^\infty$ such that $\sum_{k=1}^\infty f_k g_g \equiv 1$ and

$$\sum_{k=1}^\infty |g_k(z)| \leq C(\delta) \qquad \forall z \in \mathbb{D}.$$

To prove his result he used a modification of the Carleson–Hörmander construction with Carleson contours.

Note that, using the above result and Theorem 5 it is easy to show, see [39], that for any $p \geq 2$ and for any sequence of functions $f_k \in H^\infty$ satisfying

$$1 \geq \left(\sum_{k=1}^\infty |f_k(z)|^p \right)^{1/p} \geq \delta > 0 \qquad \forall z \in \mathbb{D},$$

there exist $g_k \in H^\infty$ such that $\sum_k f_k g_k \equiv 1$ and

$$\left(\sum_{k=1}^\infty |g_k(z)|^{p'} \right)^{1/p'} \leq C(\delta),$$

where $1/p + 1/p' = 1$.

Uchiyama's estimate $C(\delta) = C\delta^{-2} \ln(1/\delta)$ in Theorem 5 is close to optimal, if not optimal. Namely, V. Tolokonnikov [65] has shown that the estimate cannot be better than $C\delta^{-2}$, even in the case of two generators. This result was later improved by S. Treil [69], who had shown that the estimate cannot be better than $C\delta^{-2} \ln \ln(1/\delta)$. This looks like a silly "improvement," but it allowed the author to solve T. Wolff's problem [34, Problem 11.10] about ideals of H^∞; for more details see Section 7 below.

3.6 Matrix and Operator Corona Theorems

The Corona Problem admits the following interpretation/generalization. Let F be a bounded $n \times m$ matrix, $n > m$ with H^∞ entries, which has a bounded left inverse. The question is whether F has a bounded and *analytic* left inverse? The left invertibility (in H^∞) of bounded analytic matrix- or operator-valued functions, play an important role in operator theory (such as the angles between invariant subspaces, unconditionally convergent spectral decompositions, computation of spectrum, etc.).

The condition that F has a bounded left inverse means that

$$F^*(z)F(z) \geq \delta^2 I > 0, \qquad \forall z \in \mathbb{D}. \tag{C}$$

In the case when F is a column ($m = 1$), this is exactly the assumption (9) of Theorem 5. Using a simple linear algebra argument. P. Fuhrmann [27] deduced the positive answer to the above question (for the case of finite matrices) from the Corona Theorem. The generalization of this problem is the so-called *Operator Corona Problem*, dealing with the space $H^\infty_{E_*\to E}(\mathbb{D})$ of bounded analytic functions on $\mathbb{D}$ whose values are bounded operators acting between separable Hilbert spaces E_* and E.

The question is that, given $F \in H^\infty_{E_*\to E}(\mathbb{D})$ satisfying (C), does there exist $G \in H^\infty_{E_*\to E}(\mathbb{D})$ such that $GF \equiv I$? This problem was posed by Sz.-Nagy in in 1978, [62], in connection with problems in operator theory (see also [33, Problem S4.11] for this problem and some comments).

Fuhrmann's result gives the positive answer in the case $\dim E_*, \dim E < \infty$. Theorem 5 says that it is also true if $\dim E_* = 1$, $\dim E = \infty$. Using Theorem 5 and a modification of Fuhrmann's proof, V. Vasyunin was able to extend this result to the case $\dim E_* < \infty$, $\dim E = \infty$; see also a paper [78] by T. Trent, where better estimates were obtained.

In the general situation $\dim E_* = \dim E = \infty$, the condition (C) does not imply the existence of a left inverse in H^∞: a corresponding counterexample, showing that the estimates on the norm of the left inverse G blow up as $\dim E_* \to \infty$, was constructed by S. Treil see [71] or [70]. Later in [68] he presented a different counterexample, giving better lower bounds for the norm of the solution. Note that the lower bounds in [68], obtained for the case $\dim E_* = n$, $\dim E = n + 1$ were very close to the upper bounds obtained by T. Trent in [78] for the general case $\dim E_* = n$, $\dim E > n$.

While, as we discussed above, the Corona Theorem (i.e., the fact that the condition (C) implies the existence of a bounded analytic left inverse) fails in the general (infinite dimensional) case, it still holds in some particular cases. For example, the Operator Corona Theorem holds if the range $F(\mathbb{D})$ is relatively compact. Some particular results in this direction were obtained by P. Vitse [83, 84]; in full generality it was proved recently by A. Brudnyi [17], using tools of complex geometry.

Another partial result belongs to S. Treil [66], who proved in particular that the Operator Corona Theorem holds for functions F which are "small" perturbation of left invertible H^∞ functions. For example, it holds if $F = F_0 + F_1$, where $F_0, F_1 \in H^\infty_{E_*\to E}(\mathbb{D})$, F_0 is left invertible in H^∞, and the Hilbert–Schmidt norm of $F(z)$, $z \in \mathbb{D}$ is uniformly bounded (by an arbitrary large constant).

We can also mention a result of S. Treil and B. Wick [72], who introduced a curvature condition which together with (C), guarantees the existence of an H^∞ left inverse. Namely, let $\Pi(z)$ be the orthogonal projection onto Ran $F(z)$. It was proved in [72] that if $\|\partial\Pi(z)\| \le C/(1-|z|)$ and the measure

$$\|\partial\Pi(z)\|^2 \log\frac{1}{|z|} dA(z)$$

is Carleson, then condition (C) implies the left invertibility in H^∞.

In the case $\dim E_* < \infty$, the above curvature conditions easily follow from (C) (in fact they are equivalent to (C) under an extra assumption that the function F is co-outer). In the case codim Ran $F(z) < \infty$, the curvature conditions follow from the left invertibility (in H^∞) of F, so [72] solves the Operator Corona Problem in the case of finite codimension.

The proofs in [72] used the following surprising lemma discovered by N. Nikolski, which connects the solvability of the Corona Problem (in a general complex manifold Ω) with the geometry of the family of subspaces Ran $F(z)$, $z \in \Omega$.

Lemma 2 (Nikolski's Lemma). *F Let $F \in H^\infty_{E_*\to E}(\Omega)$ satisfy*

$$F^*(z)F(z) \ge \delta^2 I, \qquad \forall z \in \Omega.$$

Then F is left invertible in $H^\infty_{E_*\to E}(\Omega)$ (i.e., there exists $G \in H^\infty_{E\to E_*}(\Omega)$ such that $GF \equiv I$) if and only if there exists a function $\mathscr{P} \in H^\infty_{E\to E}(\Omega)$ whose values are projections (not necessarily orthogonal) onto $F(z)E$ for all $z \in \Omega$.

Moreover, if such an analytic projection $\mathscr{P}$ exists, then one can find a left inverse $G \in H^\infty_{E\to E_*}(\Omega)$ satisfying $\|G\|_\infty \le \delta^{-1}\|\mathscr{P}\|_\infty$.

4 Other Domains

The situation with the Corona Theorem on domains other than the unit disc can be summarized as follows:

> *There is no domain in the complex plane for which the H^∞ Corona Theorem is known to fail. There is no domain in $\mathbb{C}^n$, $n > 1$, for which the H^∞ Corona Theorem is known to hold.*

4.1 Several Complex Variables

In several variables it is trivial to construct a counterexample to the Corona Theorem, because of the phenomena of forced analytic continuation. Of course, such a trivial "counterexample" would be cheating; the natural question to ask is whether the Corona Theorem holds for domains of holomorphy.[3]

In 1973, Nessim Sibony [59] produced a startling result in the context of several complex variables. He gave an elementary construction to produce a domain of holomorphy U in $\mathbb{C}^2$ with the property that any *bounded* holomorphic function f on U analytically continues to a strictly larger open domain $\hat{U}$. Here $\hat{U}$ does *not* depend on f. The domain U does *not* have smooth boundary. Sibony's ideas were later developed and generalized by Berg [15], Jarnicki and Pflug [37], and Krantz [41]. In the paper [60], Sibony modified his construction so as to produce a counterexample to the corona in $\mathbb{C}^3$ with smooth boundary. It is notable that this domain is strongly pseudoconvex except at one boundary point. In the paper [26], Fornæss and Sibony produce an example in $\mathbb{C}^2$.

However, if one requires Ω to be a strictly pseudoconvex domain, no counterexamples are known; no positive results are known either, so the question of whether the H^∞ Corona Theorem holds for such domains remains wide open. Nothing is known even in the simplest case when Ω is the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. One might think that the product structure of the polydisc $\mathbb{D}^n$ could provide us with a better understanding of the Corona Problem there, but again, nothing is known in this case.

As indicated before, N. Varopoulos [82] crushed the hopes of easily transferring the techniques for the disc to the case of several complex variables by showing that in the unit ball in $\mathbb{C}^2$ the condition that the measure μ is Carleson does not imply existence of a bounded solution of the $\bar{\partial}$-equation $\bar{\partial}u = \mu$. In the positive direction, however, he was able to get BMO solutions of the Bezout equation $f_1 g_1 + f_2 g_2 \equiv 1$ with 2 generators in bounded pseudoconvex domains in $\mathbb{C}^n$ (assuming, of course, that the Corona data satisfy the Corona condition $|f_1| + |f_2| \geq \delta > 0$). Recently, Costea, Sawyer, and Wick [24] have extended this result to the case of k generators, or even infinitely many generators (but only in the ball in $\mathbb{C}^n$).

Other positive results in several variables include the solution of the so-called H^p Corona Problems, see Section 5 below.

4.2 Planar domains and Riemann surfaces

For planar domains the situation is in a sense opposite to the one in the case of several variables: there are some positive results, but no counterexamples.

Of course, the Riemann Mapping Theorem implies that the Corona Theorem holds for any simply connected domain. As we mentioned above, Edgar Lee Stout [61] and Norman Alling [3] proved in the mid-1960s that the Corona Theorem remains true on a finitely-connected Riemann surface (so in particular for finitely

[3]That is, a domain supports a (unbounded) holomorphic function that cannot be analytically continued to any larger domain.