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Topics in Percolative and Disordered Systems

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Preface

The Pan American Advanced Studies Institute (PASI), *Topics in Percolative and Disordered Systems*, took place in January 2012 in Santiago de Chile and Buenos Aires. It brought together mathematicians, physicists and advanced students from Latin America, North America and beyond for an intense 2-week period focused on current research problems in some of the mainstream areas of Probability Theory and Statistical Physics, such as the stochastic Ising model, random walks in random media, the KPZ universality class and interacting particle systems. This volume contains a selection of five peer-reviewed articles that are representative of the topics discussed in the PASI. Two survey articles are presented—one concerns the KPZ universality class (Quastel and Remenik) and the other treats random walks in random media (Drewitz and Ramírez). Other articles present new results about the scaling limit of the stochastic Ising model (Lacoin) and about its coarsening behaviour (Damron, Kogan, Newman and Sidoravicius) and a review of exact computational methods to compute the current of particles through a given site in the asymmetric simple exclusion process (Corwin).

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Two Ways to Solve ASEP

Ivan Corwin

Abstract The purpose of this chapter is to describe two approaches to compute exact formulas (which are amenable to asymptotic analysis) for the probability distribution of the current of particles past a given site in the asymmetric simple exclusion process (ASEP) with step initial data. The first approach is via a variant of the coordinate Bethe Ansatz and was developed in work of Tracy and Widom in 2008–2009, while the second approach is via a rigorous version of the replica trick and was developed in work of Borodin, Sasamoto and the author in 2012.

1 Introduction

Exact formulas in probabilistic systems are exceedingly important, and when a new one is discovered, it is worth paying attention. This is a lesson that I first learned in relation to the work of Tracy and Widom on the asymmetric simple exclusion process (ASEP) and through my subsequent work on the Kardar–Parisi–Zhang (KPZ) equation. New formulas can enable asymptotic analysis and uncover novel (and universal) limit laws. Comparing new formulas to those already known can help lead to the realization that certain structures or connections exist between disparate areas of study (or at least can suggest such a possibility and provide a guidepost).

The purpose of this chapter is to describe the synthesis of exact formulas for ASEP. There are presently two approaches to compute the current distribution for ASEP on $\mathbb{Z}$ with step initial condition. The first (called here the *coordinate approach*) is due to Tracy and Widom [26–28] in a series of three papers from 2008–2009, while the second (called here the *duality approach*) is due to Borodin, Sasamoto and the author [5] in 2012.

The duality approach is parallel to an approach (also developed in [5]) to study current distribution for another particle system, called q-TASEP. Via a limit transition, the duality approach becomes the replica trick for directed polymers. In fact, ASEP and q-TASEP should be considered as integrable discrete regularizations of the directed polymer model in which the replica trick (famous for being non-rigorous)

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becomes mathematically rigorous. Underlying the solvability of q-TASEP and directed polymers is an integrable structure recently discovered by Borodin and the author [4] called Macdonald processes (which in turn is based on the integrable system surrounding Macdonald symmetric polynomials). It is not presently understood where ASEP could fit into this structure, but the fact that the duality approach applies in parallel for ASEP and q-TASEP compels one to look for a higher structure which encompasses both.

2 Current Distribution for ASEP

ASEP is an interacting particle system introduced by Spitzer [24] in 1970 (though arising earlier in biology in the work of MacDonald, Gibbs and Pipkin [18] in 1968). Since then, it has become a central object of study in interacting particle systems and non-equilibrium statistical mechanics. Each site of the lattice $\mathbb{Z}$ may be inhabited by at most one particle. Each particle attempts to jump left at rate q and right at rate p ($p + q = 1$), except that jumps which would violate the 'one particle per site rule' are suppressed. We will assume $q > p$, and for later use call $q - p = \gamma$ and $p/q = \tau$ (note that $\gamma > 0$ and $\tau < 1$).

There are two ways of constructing ASEP as a Markov process. The 'occupation process' keeps track of whether each site in $\mathbb{Z}$ is occupied or unoccupied. The state space is $Y = \{0, 1\}^{\mathbb{Z}}$ and for a state $\eta = \{\eta_x\}_{x\in\mathbb{Z}} \in Y$, $\eta_x = 1$ if there is a particle at x and 0 otherwise. This Markov process is denoted $\eta(t)$.

The 'coordinate process' keeps track of the location of each particle. Assume there are only k particles in the system, then the state space $X_k = \{x_1 < \cdots < x_k\} \subset \mathbb{Z}^k$ and for a state $\vec{x} = \{x_1 < \ldots < x_k\} \in X_k$, the value of x_j is the location of particle j. We call X_k a Weyl chamber. Because particles cannot hop over each other, the ASEP dynamics preserve particle ordering. This Markov process is denoted $\vec{x}(t)$.

In this chapter, we will be concerned with the 'step' initial condition for ASEP in which every positive integer site is initially occupied and every other site is initially unoccupied. In terms of the occupation process, this corresponds to having $\eta_x(0) = \mathbf{1}_{x>0}$ (here and throughout $\mathbf{1}_E$ is the indicator function for event E). Let $N_x(\eta) = \sum_{y\leq x} \eta_y$ and note that $N_0(\eta(t))$ records the number of particles of ASEP which, at time t are to the left of, or at the origin—that is to say, it is the net current of particles to pass the bond 0 and 1 in time t.

Theorem 1 *For ASEP with step initial condition and $q > p$,*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_0(t/\gamma) - t/4}{2^{-1/3}t^{1/3}} \geq -s\right) = F_{\mathrm{GUE}}(s),$$

where $F_{\mathrm{GUE}}(s)$ is the GUE Tracy-Widom distribution.

Remark 1 The distribution function $F_{\mathrm{GUE}}(s)$ can be defined via a Fredholm determinant as

$$F_{\mathrm{GUE}}(s) = \det(I - K_{\mathrm{Ai}})_{L^2(s,\infty)}$$

where Airy kernel K_{Ai} acts on $L^2(s,\infty)$ with integral kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt.$$

For $q = 1$ and $p = 0$, result was proved in 1999 by Johansson [13] and for general $q > p$, it was proved by Tracy and Widom [26–28] in 2009, and then reproved via a new formula by Borodin, Sasamoto and the author [5] in 2012. This result confirms that for all $q > p$, ASEP is in the KPZ universality class [15] (see also the review [6]).

In order to prove an asymptotic result (such as above), it is very useful to have a pre-asymptotic (finite t) formula to analyze. If the formula does not increase in complexity as t goes to infinity, there is hope to compute its asymptotics. Presently, there are two approaches to computing manageable formulas for the distribution of $N_0(t)$.

3 The Coordinate Approach

In [26], Tracy and Widom start by considering the ASEP coordinate process $\vec{x}(t)$ with only k particles. In 1997, Schütz [22] computed the transition probabilities (i.e. Green's function) for ASEP with $k = 2$ particles. The first step in [26] is a generalization to arbitrary k. Let $P_{\vec{y}}(\vec{x};t)$ represent the probability that in time t, a particle configuration $\vec{y}$ will transition to a second configuration $\vec{x}$. As long as $p \neq 0$, it was proved in [26] that

$$P_{\vec{y}}(\vec{x};t) = \sum_{\sigma\in S_k} \int \cdots \int A_\sigma \prod_{i=1}^{k} \xi_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} e^{\epsilon(\xi_j)t} d\xi_j, \tag{1}$$

where the contour of integration is a circle centered at zero with radius so small as to not contain any poles of A_σ. Here, $\epsilon(\xi) = p\xi^{-1} + q\xi - 1$ and

$$A_\sigma = \prod \left\{ S_{\alpha\beta} : \{\alpha,\beta\} \text{is an inversion in}\, \sigma \right\}, \qquad S_{\alpha\beta} = -\frac{p + q\xi_\alpha\xi_\beta - \xi_\alpha}{p + q\xi_\alpha\xi_\beta - \xi_\beta}.$$

This result is proved by showing that that $P_{\vec{y}}(\vec{x};t)$ solves the master equation for k-particle ASEP

$$\frac{d}{dt}u(\vec{x};t) = ((L^k)^* u)(\vec{x};t), \qquad u(\vec{x};0) = \mathbf{1}_{\vec{x}=\vec{y}}.$$

Here $(L^k)^*$ is the adjoint of the generator of the k-particle ASEP coordinate process (this just means that the role of p and q are switched in going between L^k and $(L^k)^*$). For $k = 1$, L^1 and $(L^1)^*$ act on function $f : \mathbb{Z} \to \mathbb{R}$ as

$$(L^1 f)(x) = q\,[f(x-1) - f(x)] + p\,[f(x+1) - f(x)],$$
$$\big((L^1)^* f\big)(x) = p\,[f(x-1) - f(x)] + q\,[f(x+1) - f(x)].$$

For $k > 1$, the generator L^k and its adjoint depend on the location of $\vec{x}$ in the Weyl chamber, reflecting the fact that certain particle jumps are not allowed near the boundary of the Weyl chamber.

Quoting a footnote in [26]:

> The idea in Bethe Ansatz (see, e.g. [16, 25, 30]), applied to 1-D k-particle quantum mechanical problems, is to represent the wave function as a linear combination of free particle eigenstates and to incorporate the effect of the potential as a set of $k-1$ boundary conditions. The remarkable feature of models amendable to Bethe Ansatz is that the boundary conditions for $k \geq 3$ introduce no more new conditions ... The application of Bethe Ansatz to the evolution equation (master equation) describing ASEP begins with Gwa and Spohn [9] with subsequential developments by Schütz [22].

To see this in practice, assume that one wants to solve

$$\frac{d}{dt} u(\vec{x};t) = \big((L^k)^* u\big)(\vec{x};t), \qquad u(\vec{x};0) = u_0(\vec{x})$$

for $\vec{x}$ in the Weyl chamber X_k.

Proposition 1 1 *If $v : \mathbb{Z}^k \times \mathbb{R}_+ \to \mathbb{R}$ solves the 'free evolution equation with boundary condition':*

(1) *For all $\vec{x} \in \mathbb{Z}^k$*

$$\frac{d}{dt} v(\vec{x};t) = \sum_{j=1}^{k} \big([L^1]^*_j v\big)(\vec{x};t);$$

(2) *For all $\vec{x} \in \mathbb{Z}^k$ such that $x_{j+1} = x_j + 1$ for some $1 \leq j \leq k-1$,*

$$p v(x_1, \dots, x_j, x_{j+1} - 1, \dots, x_k; t) + q v(x_1, \dots, x_j + 1, x_{j+1}, \dots, x_k; t)$$
$$- v(\vec{x};t) = 0;$$

(3) *For all $\vec{x} \in X_k$, $v(\vec{x};0) = u_0(\vec{x})$;*

Then, for all $t \geq 0$ and $\vec{x} \in X_k$, $u(\vec{x};t) = v(\vec{x};t)$.

In (1) above, $[L^1]^*_j$ means to apply $(L^1)^*$ in the x_j variable. In fact, some growth conditions must be imposed to ensure that u and v match (see Propositions 4.9 and 4.10 of [5]) but we will not dwell on this presently.

This reformulation of the master equation involves only $k-1$ boundary conditions and is amendable to Bethe Ansatz—hence one is led to postulate Eq. (1). It remains to check the Ansatz (i.e. $P_{\vec{y}}(\vec{x};t)$ solves the reformulated equation). The A_σ is just right to enforce the boundary condition. The only challenge (which requires an involved residue calculation) is to check the initial data, since there are a total of $k!$ integrals.

The transition probabilities for k-particle ASEP is only the first step towards Theorem 1. The next step is to integrate out the locations of all but one particle, so as to compute the transition probability for a given particle x_m. The formula for the location of the m^{th} particle at time t involves a summation (indexed by certain subsets of $\{1, \ldots, k\}$) of contour integrals. These formulae are a result of significant residue calculations and combinatorics.

At this point we are only considering k particles, whereas for the asymptotic problem, we want to consider step initial conditions. This is achieved by taking $y_j = j$ for $1 \leq j \leq k$ and taking k to infinity. After further manipulations, the m^{th} particle location distribution formula has a clear limit as k goes to infinity. This is the first formula for step initial condition and it is given by an infinite series of contour integrals.

In [27], this infinite series is recognized as equal to a transform of a Fredholm determinant. By the simple relationship between the location of the m^{th} particle of ASEP and $N_0(t)$ (defined earlier), this shows that

$$\mathbb{P}(N_0(t) = m) = \frac{-\tau^m}{2\pi \mathrm{i}} \int \frac{\det(I - \zeta K_1)}{(\zeta;\tau)_{m+1}} d\zeta, \tag{2}$$

where the integral in ζ is over a contour enclosing $\zeta = q^{-k}$ for $0 \leq k \leq m-1$ and $(a;\tau)_n = (1-a)(1-\tau a)\cdots\left(1-\tau^{n-1}a\right)$. Here, $\det(I - \zeta K_1)$ is the Fredholm determinant with the kernel of $K : L^2(C_R) \to L^2(C_R)$ given by

$$K_1(\xi, \xi') = q \frac{e^{\epsilon(\xi)t}}{p + q\xi\xi' - \xi},$$

and the contour C_R a sufficiently large circle centered at zero.

There remains, however, a significant challenge to proving Theorem 1 from the above formula. As m increases, the kernel K_1 has no clear limit, and the denominator term $(\zeta;\tau)_{m+1}$, behaves widely as ζ varies on its contour of integration. Much of [28] is devoted to reworking the above formula into one for which asymptotics can be performed. This is done through significant functional analysis. The final formula, from which Theorem 1 is proved by asymptotics is (leaving off the contours of integration),

$$\mathbb{P}(N_0(t) \geq m) = \int \frac{d\mu}{\mu} (\mu;\tau)_\infty \det(I + \mu J), \tag{3}$$

where the kernel of J is given by

$$\begin{aligned}
J(\eta, \eta') &= \int \exp\left\{\Psi_{t,m,x}(\zeta) - \Psi_{t,m,x}(\eta')\right\} \frac{f(\mu, \zeta/\eta')}{\eta'(\zeta - \eta)} d\zeta, \\
f(\mu, z) &= \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu} z^k, \\
\Psi_{t,m,x}(\zeta) &= \Lambda_{t,m,x}(\zeta) - \Lambda_{t,m,x}(\xi), \\
\Lambda_{t,m,x}(\zeta) &= -x \log(1 - \zeta) + \frac{t\zeta}{1 - \zeta} + m \log \zeta.
\end{aligned}$$

4 The Duality Approach

Duality is a powerful tool in the study of Markov processes. It reveals hidden structures and symmetries of the process, as well as leads to non-trivial systems of ODEs (ordinary differential equation), which expectations of certain observables satisfy. In 1997, Schütz [23] observed that ASEP is self-dual (in a sense which will be made clear below). The fact that duality gives a useful tool for computing the moments of ASEP was first noted by Imamura and Sasamoto [12] in 2011. In 2012, Borodin, Sasamoto and the author [5] used this observation about duality, along with an Ansatz for solving the duality ODEs (which was inspired by the work of Borodin and the author on Macdonald processes [4]) to derive two different formulae for the probability distribution of $N_0(t)$. The first was new and readily amendable to asymptotic analysis necessary to prove Theorem 1, while the second was equivalent to Tracy and Widom's formula (2).

To define the general concept of duality, consider two Markov processes, $\eta(t)$ with state space Y and $\vec{x}(t)$ with state space X (for the moment, we think of these as arbitrary, though after the definition of duality, we will take these as before). Let $\mathbb{E}^{\eta}$ and $\mathbb{E}^{\vec{x}}$ represent the expectation of these two processes (respectively) started from $\eta(0) = \eta$ and $\vec{x}(0) = \vec{x}$. Then, $\eta(t)$ and $\vec{x}(t)$ are dual with respect to a function $H : Y \times X \to \mathbb{R}$, if for all $\eta \in Y$, $\vec{x} \in X$ and $t \geq 0$,

$$\mathbb{E}^{\eta}\left[H(\eta(t), \vec{x})\right] = \mathbb{E}^{\vec{x}}\left[H(\eta, \vec{x}(t))\right].$$

One immediate consequence of duality is that if we define $u_\eta(\vec{x};t)$ to be the expectations written above, then

$$\frac{d}{dt}u_\eta(\vec{x};t) = L u_\eta(\vec{x};t),$$

where L is the generator of $\vec{x}(t)$ and where the initial data is given by $u_\eta(\vec{x};0) = H(\eta,\vec{x})$.

Schütz [23] observed that if $\eta(t)$ is the ASEP occupation process and $\vec{x}(t)$ is the k-particle ASEP coordinate process with p and q switched from the earlier definition, then these two Markov processes are dual with respect to

$$H(\eta,\vec{x}) = \prod_{j=1}^{k} \tau^{N_{x_j-1}(\eta)} \eta_{x_j}.$$

The generator of the p, q reversed particle process $\vec{x}(t)$ is equal to $(L^k)^*$, as discussed earlier. Schütz demonstrated this duality in terms of a spin-chain encoding of ASEP by using a commutation relation along with the $U_q[SU(2)]$ symmetry of the chain. A direct proof can also be given in terms of the language of Markov processes [5]. When $p = q$, $\tau = 1$ and this duality reduces to the classical duality of correlation functions for the symmetric simple exclusion process (see [17] Chap. 8, Theorem 1).

As before, we focus on step initial condition, so that $\eta_x = \mathbf{1}_{x\geq 1}$. Duality implies that $u_{\text{step}}(\vec{x};t) := \mathbb{E}^{\eta}\left[H(\eta(t),\vec{x})\right]$ solves

$$\frac{d}{dt}u_{\text{step}}(\vec{x};t) = L^k u_{\text{step}}(\vec{x};t), \qquad u_{\text{step}}(\vec{x};0) = \mathbf{1}_{x_1\geq 1}\prod_{i=1}^{k}\tau^{x_i-1}. \tag{4}$$

The above system is solved by

$$u_{\mathrm{step}}(\vec{x};t) = \frac{\tau^{k(k-1)/2}}{(2\pi \mathrm{i})^k} \int \cdots \int \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{j=1}^{k} h_{x_j,t}(z_j) dz_j, \tag{5}$$

where

$$h_{x,t}(z) = e^{\epsilon'(z)t} \left(\frac{1+z}{1+z/\tau} \right)^{x-1} \frac{1}{\tau + z}, \qquad \epsilon'(z) = -\frac{z(p-q)^2}{(1+z)(p+qz)},$$

and where the contour of integration for each z_j is a circle around $-\tau$, so small as to not contain -1. In order to see this, we use the reformulation of the system (4) in terms of the free evolution equation with boundary condition with ASEP given earlier in Proposition 3.1. Condition (1) is trivially checked since for each z, $\frac{d}{dt} h_{x,t}(z) = L^1 h_{x,t}(z)$. Condition (3) is checked via a simple residue calculation. Condition (2) reveals the purpose of the $\frac{z_A - z_B}{z_A - \tau z_B}$ factor. Applying the boundary condition to the integrand above brings out a factor of $z_j - \tau z_j$. This cancels the corresponding term in the denominator and the resulting integral is simultaneous symmetry and antisymmetry in z_j and z_{j+1}. Hence, the integral must equal zero, which is the desired boundary condition (2).

The inspiration for this simple solution to the system of ODEs came from analogous formulas which solve free evolution equations with boundary condition for various versions of the delta Bose gas (see Sect. 5 for a brief discussion). For the delta Bose gas and certain integrable discrete regularizations, the formulas arose directly from the structure of Macdonald processes [4]. ASEP does not fit into that structure, but the existence of similar formulas suggests the possibility of a yet higher structure.

A change of variables reveals some similarities to the integrand in (1). Letting

$$\xi_j = \frac{1+z_j}{1+z_j/\tau} \tag{6}$$

we have

$$\frac{z_A - z_B}{z_A - \tau z_B} = q \frac{\xi_A - \xi_B}{p + q\xi_A \xi_B - \xi_B}, \qquad h_{x_j,t}(z_j) dz_j = e^{\epsilon(\xi_j)t} \xi_j^{x_j - 1} \frac{d\xi_j}{\tau - \xi_j}.$$

The system (4) could also be solved via Tracy and Widom's formula (see formula 1 earlier) for the Green's function for $(L^k)^*$ (as suggested in [12]) but the resulting formula would involve the sum of $k!$ k-fold contour integrals. Symmetrizing (5) via combinatorial identities, and making the above change of variables, one does recover that formula. The reversal of this procedure is a rather unnatural anti-symmetrization, which explains why (5) was not previously known.

A suitable summation of $H(\eta, \vec{x})$ over $\vec{x}$ gives $\tau^{kN_x(\eta)}$. Using this, and formula (5), [5] proves that for ASEP with step initial condition,

$$\mathbb{E}\left[\tau^{kN_0(t)}\right] = \frac{\tau^{k(k-1)/2}}{(2\pi \mathrm{i})^k} \int \cdots \int \prod_{1\le A<B\le k} \frac{z_A - z_B}{z_A - \tau z_B} \prod_{j=1}^{k} e^{\epsilon'(z_j)t} \frac{dz_j}{z_j}, \tag{7}$$

where $N_0(t) = N_0(\eta(t))$ and where the contour of integration for z_j includes $0, -\tau$ but not -1 or τ times the contours for z_{j+1} through z_k. This is to say, that the contours of integration respect a certain nesting structure.

At this point, the utility of having a single k-fold nested contour integral formula for the moments of $\tau^{N_0(t)}$ becomes clear. There are two ways to deform the contours of integration in (7) so that all coincide with each other. The first involves expanding them all to be a circle containing $-\tau$ and 0, but not -1. There are many poles encountered in the course of this deformation and the residues can be indexed by a partition. This leads to

$$\mathbb{E}\left[\tau^{kN_0(t)}\right] = k_\tau! \sum_{\substack{\lambda \vdash k \\ \lambda = 1^{m_1} 2^{m_2} \cdots}} \frac{1}{m_1! m_2! \cdots} \frac{(1-\tau)^k}{(2\pi\iota)^{\ell(\lambda)}} \int \cdots \int \det\left[\frac{-1}{w_i \tau^{\lambda_i} - w_j}\right]_{i,j=1}^{\ell(\lambda)}$$
$$\times \prod_{j=1}^{\ell(\lambda)} e^{t \sum_{i=0}^{\lambda_j - 1} \epsilon'(\tau^i w_j)} dw_j, \tag{8}$$

where $k_\tau! = (\tau;\tau)_k (1-\tau)^{-k}$ is the τ-deformed factorial, and $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ is a partition of k (i.e. $\sum \lambda_i = k$) with $\ell(\lambda)$ nonzero parts, and multiplicity m_j of the value j. The structure of these residues is very similar to the string states indexing the eigenfunctions of the attractive delta Bose gas (see Sect. 5).

The final step in the duality approach is to use these moment formulas to recover the distribution of $N_0(t)$. This is done via the τ-deformed Laplace transform Hahn [10] introduced in 1949. The left-hand side of the below equation is the transform of $\tau^{N_0(t)}$ with spectral variable ζ.

$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_0(t)};\tau)_\infty}\right] = \sum_{k=0}^{\infty} \frac{\zeta^k \mathbb{E}\left[\tau^{kN_0(t)}\right]}{(\tau;\tau)_k}. \tag{9}$$

The right-hand side above comes from the left-hand side by expanding the τ-deformed exponential inside the expectation (using the τ-deformed Binomial theorem) and then interchanging the summation over k with the expectation. This interchange of summation and integration is justified here for ζ small enough because $|\tau^{kN_0(t)}| \leq 1$ deterministically (in contrast to (15) Sect. 5).

Substituting (8) into the series on the right-hand side of (9), one recognizes a Fredholm determinant. The kernel of the determinant can be rewritten using a Mellin-Barnes integral representation and the result is (leaving off the contours of integration):

$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_0(t)};\tau)_\infty}\right] = \det\left(I + K_\zeta\right), \tag{10}$$

where the kernel of K_ζ is

$$K_\zeta(w,w') = \frac{1}{2\pi \mathrm{i}} \int \frac{\pi}{\sin(-\pi s)} (-s)^\zeta \frac{g(w)}{g(\tau^s w)} \frac{ds}{w' - \tau^s w}, \qquad g(w) = e^{\gamma t \frac{\tau}{\tau + w}}.$$

The τ-Laplace transform can easily be inverted to give the distribution of $N_0(t)$ and asymptotics of the above formula are readily performed (see Sect. 9 of [5]) resulting in Theorem 1.

There is a second choice for how to deform the nested contours in (8) to all coincide. The terminal contour of this deformation is a small circle around $-\tau$, and again there are certain poles encountered during the deformation. The combinatorics of the residues here is simpler than in the first case, and one finds the following Fredholm determinant formula,

$$\mathbb{E}\left[\frac{1}{\left(\zeta\tau^{N_0(t)};\tau\right)_\infty}\right] = \frac{\det\left(I - \zeta K_2\right)}{(\zeta;\tau)_\infty} \tag{11}$$

where the kernel of K_2 is

$$K_2(w,w') = \frac{e^{\epsilon'(w)t}}{\tau w - w'}.$$

Performing the change of variables (6) and inverting this τ-Laplace transform, one recovers Tracy and Widom's formula (2). As in Tracy and Widom's work, this formula is not yet suitable for asymptotics and must be manipulated significantly to get to the form of (3).

5 Duality Approach as a Rigorous Replica Trick

Besides the Schütz duality, Borodin, Sasamoto and the author discovered that ASEP is also self-dual with respect to

$$H(\eta,\vec{x}) = \prod_{j=1}^{k} \tau^{N_{x_j}(\eta)},$$

for $k = 1$. This shows that $\mathbb{E}\left[\tau^{N_x(\eta(t))}\right]$ solves the heat equation with generator L^1. In fact, this is essentially Gärtner's 1988 observation [8] that $\tau^{N_x(\eta(t))}$ solves a certain discrete multiplicative stochastic heat equation. A multiplicative stochastic heat equation has a Feynman-Kac representation which shows that the solution can be interpreted as a partition function for a directed polymer in a disorder given by the noise of the stochastic heat equation.

In 1997, Bertini and Giacomin [2] showed that under a certain 'weakly asymmetric' scaling, $\tau^{N_x(\eta(t))}$ converges to the solution to the continuum multiplicative stochastic heat equation (SHE) with space–time white noise $\xi(x,t)$:

$$\frac{d}{dt}Z(x,t) = \frac{1}{2}\frac{d^2}{dx^2}Z(x,t) + Z(x,t)\,\xi(x,t).$$

This convergence result did not include when $\eta(0)$ is step initial condition and was extended to that case by Amir, Quastel and the author [1]. The corresponding initial