BROUWER MEETS HUSSERL

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# BROUWER MEETS HUSSERL

## ON THE PHENOMENOLOGY OF CHOICE SEQUENCES

by

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in memoriam
Gian-Carlo Rota
1932–1999

Duo sunt nimirum labyrinthi humanae mentis, unus circa compositionem continui, alter circa naturam libertatis, qui ex eodem infiniti fonte oriuntur.

Leibniz, 'De libertate'

# Contents

# Preface

This is an analysis, using Husserl's methods, of Brouwer's main contribution to the ontology of mathematics. The discussion is essentially self-contained, but, depending on one's background and purposes, one may wish to consult further literature. An introduction, from an equally phenomenological point of view, to Brouwer's intuitionism as a philosophical foundation of mathematics is [3].[1] There are many introductions to phenomenology. I mention Husserl's own [128] and [130], the latter of which Gödel considered a 'momentous lecture';[2] the wide-ranging, historiographical [203]; and the more problem-oriented [198], [201] and [248]. A short intellectual and psychological biography of Husserl is [236]; on Brouwer's life there is now the two-volume biography by Dirk van Dalen [60, 63]. There are also entries on Brouwer and on Husserl in the *Stanford Encyclopedia of Philosophy* on the internet [2, 15].

Paris, April 2006 — MvA

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Rudolf Bernet kindly provided me with a copy of the recent English translation [131] of Husserl's *Philosophie der Arithmetik* [107].

Several sections have appeared elsewhere before, in part or whole, with changes to adapt them for separate publication. I thank the respective publishers and editors for their kind permission to make use of the following material here:

Parts of sections 4.3.1 and 5.4.2 were included in 'Phenomenology's Reception of Brouwer's Choice Sequences', in *Oskar Becker und die Philosophie der Mathematik*, Wilhelm Fink Verlag: München, 2005, pp. 101–117. ©2005 V. Peckhaus.

Section 4.3.4 appeared as part of a larger paper with Dirk van Dalen and Richard Tieszen, 'Brouwer and Weyl: the Phenomenology and Mathematics of the Intuitive Continuum', *Philosophia Mathematica* 10, no. 3 (2002):203-226. ©2002 R.J. Thomas.

Sections 5.1-5.3 appeared as 'Why Husserl Should Have Been a Strong Revisionist in Mathematics', *Husserl Studies* 18, no. 1 (2002):1-18. ©2002 Kluwer Academic Publishers.

Section 5.5 appeared as 'The Irreflexivity of Brouwer's Philosophy', *Axiomathes* 13, no. 1 (2002):65-77. ©2002 Kluwer Academic Publishers.

An earlier version of section 7.3 was transformed into part of a paper with Dirk van Dalen, 'Arguments for the Continuity Principle', *The Bulletin of Symbolic Logic* 8, no. 3 (2002): 329-347. The present section 7.3 has in turn been rewritten from that.

Scattered parts of the main text found their way into a talk titled 'Brouwer, as Never Read by Husserl', published in *Synthèse* 137, no. 1 (2003):3–19, an issue that contains the proceedings of the conference 'History of Logic', Helsinki, 2000. ©2002 Kluwer Academic Publishers.

The appendix appeared, under the same title, in the *Graduate Faculty Philosophy Journal*, 25(2), 2004, pp. 205–225. 

For permission to quote from material in the Brouwer Archive (Utrecht) and the Husserl Archive (Louvain), I am much obliged to their respective directors, Dirk van Dalen and Rudolf Bernet.

# 1

# An Informal Introduction

1. *What is the aim of this book?*
   The aim is to use phenomenology to justify Brouwer's choice sequences as mathematical objects.

2. *First of all, what is a choice sequence?*
   Imagine that you have a collection of mathematical objects at your disposal, let's say the natural numbers. Pick out one of them, and note the result. Put it back into the collection, and choose again. You may choose a different one, or the same. Note the result, and put it back. For example, perhaps you chose

   $$12, 3$$

   Making further choices, you may arrive at

   $$12, 3, 81, 12, 221$$

   and you can continue from there. A choice sequence is what you get if you think of the sequence you are making as potentially infinite. The two sequences given above are initial segments of the choice sequence. Initial segments are always finite. We cannot make an actually infinite number of choices, but we can always extend an initial segment by making a further choice. This potential infinity of the choice sequence we indicate by three dots:

   $$12, 3, 81, 12, 221, \ldots$$

3. *How are they used?*
   There is an age-old problem in mathematics how to analyse the straight line ('the continuum'). Traditional mathematics thinks of the straight line as a large number of isolated points lying next to each other, like grains of sand. As Aristotle already pointed out, the problem is that this isolation breaks the line's continuity. A line is continuous through and through; a continuum is not made up from grains of sand but rather from strings of

melted cheese. The mathematician L.E.J. Brouwer was the first to show how to rectify the situation mathematically: his choice sequences provide a means to give a mathematical form to the strings of cheese.

4. *You say you want to give a phenomenological justifation of choice sequences as mathematical objects. Is there a need, then, for a justification?*
   Yes. Most mathematicians refuse to accept Brouwer's choice sequences as mathematical objects: these sequences depend on the individual's choices and they grow in time, but none of the objects that traditional mathematics talks about are like that. They are too strange. A small group of mathematicians however has accepted choice sequences, and they continue to develop Brouwer's ideas.

5. *So mathematics is not a unified science and mathematicians actually disagree among each other as to what objects they are talking about?*
   Yes, this is the situation (and choice sequences are not even the only disputed objects). The reason that this is not generally known is perhaps that the mathematical objects that non-mathematicians use in daily life (such as finite numbers, fractions, real numbers generated by an algorithm, and geometrical shapes) are not among the bones of contention. But specific views on the nature of mathematical objects may introduce (or, alternatively, rule out) specific constraints on what mathematical objects can exist. Hence, with different philosophical views may come different kinds of mathematics. Therefore, even someone whose primary interest is doing mathematics rather than philosophising about it will at some point have to engage in some philosophy, or at least to acknowledge that there is a philosophical question to be answered.

6. *Didn't Brouwer have a justification of his own? If so, why not use that one?*
   Brouwer indeed had a justification of his own, but it was based on a background philosophy that is defective in such a way that it cannot be used to justify the introduction of choice sequences.

7. *If Brouwer's justification doesn't work, why turn specifically to Husserl for another one?*
   It seems natural to me to look to Husserl for an alternative, most of all because I believe that phenomenology in general grants the power to understand, see into, see insightfully, and so think and justify cogently. But there are also specific circumstances that bring Brouwer and Husserl close together. First, Husserl's general philosophy is very similar to Brouwer's, without suffering from the defect I referred to above. Second, Husserl was very interested in phenomena that Brouwer also studied, such as time, which is itself an example of a continuum (think of the idiom 'a timeline').

Third, in Husserl one finds analyses of the philosophical aspects of the notions of object and sequence as such, which will be helpful.

8. *But if the philosophies of Brouwer and Husserl were that close, then why didn't Husserl himself come up with choice sequences?*
Apart from the fact that one does not always come up with a good idea when one has all its ingredients in hand, from various things that Husserl said about mathematics it follows that he would not have accepted choice sequences when asked, let alone have been led to consider their possibility himself. For example, he claims that mathematical objects exist eternally and never change; choice sequences, on the other hand, come into being (at the moment one begins to make choices) and change over time (as with each new choice they grow longer). Husserl's claims are characteristic of the tradition that he, orginally a mathematician, was trained in, and which is still dominant today. But a closer look at the motives that led Husserl (and others) to make these claims will show that these motives can also be honoured by somewhat weaker claims which, in contrast, do not rule out choice sequences. A crucial part of the argument will consist in showing that the fundamental tenets of Husserl's phenomenology did not force him to make the strong claims that he in fact made.

9. *Let me see if I get this right. You base a defense of Brouwer's mathematical innovation on a philosophy, Husserl's, that Brouwer himself did not embrace, while Husserl, in turn, had a conception of mathematics that would not embrace Brouwer's idea?*
Yes. Brouwer's idea was good, but his background philosophy is not capable of justifying it; Husserl's background philosophy can justify it, but this has always been obscured by various of Husserl's specific claims about mathematics. These specific claims however can be shown to be unwarranted by his own standards. So I defend Brouwer's idea by Husserl's means, even though Husserl himself would have said this cannot be done. In other words, I exploit the possibility that there could be a difference between Husserl's utterances on a certain subject and what his philosophy actually implies about it. In the case of mathematics, I argue, there is indeed such a difference, which, moreover, opens up sufficient space for choice sequences to find a place in Husserl's philosophy.

10. *Aha. Now, does the phenomenological analysis of choice sequences you give provide us with sufficient insight so that we not just accept them as mathematical objects, but also see how to go ahead with developing out of them a better mathematical theory of the continuum?*
Yes. For technical reasons, it would not be possible to develop much actual mathematics from choice sequences unless a certain crucial principle (called 'the continuity principle', but in a sense which is different from the continuity of a line) holds for them. Without this principle, choice

sequences would in a mathematical sense just be curiosities. Brouwer freely used the principle, but for its validity only plausibility arguments were advanced; however, it will turn out that the phenomenological analysis of our processes of thinking when we create choice sequences provides a justification of the principle in question.

11. *The motto of this book is taken from Leibniz. Would he have accepted choice sequences?*
    No. According to Leibniz, the objects of pure mathematics exist in God's mind, God exists outside of time, and mathematics in no way depends on God's will. In such a setting, growing objects that depend on choices would not have been recognisable as mathematical objects. There is no evidence that Brouwer knew the passage in Leibniz from which the motto is taken, or similar ones; but, as it happens, Brouwer's mathematical model of the one notion that Leibniz mentions, the continuum, depends precisely on the other one, freedom.

# 2

# Introduction

## 2.1 The Aim

The aim is to use phenomenology to justify Brouwer's choice sequences[3] as mathematical objects.[4]

## 2.2 The Thesis

One correct, phenomenological argument on the issue whether mathematical objects can be dynamic (e.g., choice sequences) is not Husserl's (negative) argument, but a reconstruction of Brouwer's (positive) one.

## 2.3 Motivation

The thesis involves a meeting of the thoughts of Brouwer (1881-1966) and Husserl (1859-1938); as their careers overlapped for some thirty years, this naturally suggests the question whether they ever met in person. They did, in April 1928, when Husserl came to the Netherlands to deliver his 'Amsterdamer Vorträge' [103]. On the 30th of that month, Brouwer wrote to a German friend, 'Here, at the moment, Husserl is darting around, which strongly draws me in' [63, p. 567, trl. Dirk van Dalen].[5] That the appreciation was mutual is clear from Husserl's report to Heidegger of May 5:

> Among the most interesting things in Amsterdam were the long conversations with Brouwer, who made a quite distinguished impression on me, that of a wholly original, radically sincere, genuine, entirely modern man. [128, IV:p. 156, trl. mine][6]

However, nothing is known about the content of these conversations, nor, for that matter, about possible further exchanges between them.[7] They have never discussed each other's work,[8] yet there has always been a close (conceptual and factual) link between Husserl's phenomenology and Brouwer's intuitionism (or constructivism in general). In one sense, this is not surprising [141, p. 99]: in both strands of thought the main principle is that all genuine knowledge refers back, directly or indirectly, to intuitions: experiences in which objects are given as themselves. Cases in point are Becker [11], Heyting [88] and Weyl [238, 239] who have applied phenomenology to argue in favour of (parts of) intuitionism. Heyting's well-known interpretation of the logical constants [85, 87], for instance, uses the phenomenological concepts of intention and fulfilment to analyse intuitionistic ideas about meaning.

However, another aspect of intuitionism is completely at odds with Husserl's philosophy of mathematics, and this aspect concerns the nature of the mathematical universe.

According to Husserl, the mathematical universe is static: its objects are finished (or complete) and mathematical truths and objects are omnitemporal ('allzeitlich', e.g., in *Experience and Judgement* [124, section 64]).

Brouwer, in contrast, regards the universe as a construction of the mathematician. Hence it is not omnitemporal and, moreover, it is dynamic in the sense that some objects, namely, choice sequences, are open-ended and are developed in time. He showed how, if choice sequences are accepted as genuine mathematical objects, one can develop a rich and constructive theory of the continuum.[9]

Brouwer's argument is based on a background philosophy that, as I attempt to show in chapter 5, is in fact incapable of justifying anything. I look to phenomenology for an alternative foundation of parts of intuitionism. But Husserl emphatically denies what Brouwer affirms, i.e., the possibility of dynamic mathematical objects. Who is right?

A glance at the reasons Brouwer and Husserl give for their respective positions leads to the following observation.

Brouwer appeals to acts of construction and free choice, against the background of his mystical theory of mind. These ideas can be found in Brouwer's writings from the early *Life, Art and Mysticism* [22] to the mature and elaborate 'Consciousness, philosophy and mathematics' [39].

Husserl, on the other hand, often states without further argument, and never even mentioning choice sequences, that mathematical objects are static. He considers it simply part of the meaning of mathematical statements that mathematical objects have this property. Examples from respectively the early and later Husserl can be found in his *Logical Investigations* [113, p. 134] and the already mentioned *Experience and Judgement* [124, section 64].

But this is precisely what Brouwer contests, and he does give arguments for doing so. Moreover, the appeal in these arguments to certain acts makes Brouwer seem, in this matter, the real phenomenologist of the two. This suggests that, even if we shift the background from Brouwer's own specific philosophy to phenomenology, an argument can be found for a dynamic universe by reconstructing Brouwer's argument (where 'to reconstruct' means following an argument closely, changing it where necessary, trying to preserve as much of the conclusion as possible).

Note that, logically, there are three possible conclusions:

1. All mathematical objects are omnitemporal. (Husserl)
2. No mathematical objects are omnitemporal. (Brouwer)
3. Some mathematical objects are omnitemporal, some are not.

My argument here concerns the third of these. In particular, I argue that choice sequences are an example of dynamic, and hence not omnitemporal, objects in mathematics. Whether there are other dynamic mathematical objects I will leave an open question. As an argument against Husserl's thesis, one counterexample suffices. This is why the thesis speaks of *one* rather than *the* correct argument.

That choice sequences might be justified on phenomenological grounds is yet just an idea. To put it in a metaphor that Husserl liked to use: if the suggestion is of any real value, we should be able to get small change for its large banknotes. The aim is to see if that can be done. If it can, we will have a way to do justice to some of Brouwer's ideas without compromising a phenomenological point of view.

## 2.4 Method, and an Assumption

The framework I will adopt is that of Husserl's transcendental phenomenology, as outlined in his *Formal and Transcendental Logic* [112] and the *Cartesian Meditations* [126]. It suggests that ontological questions in a priori sciences are to be settled by attempting a constitution analysis; this determines my method. I will assume without further argument that this framework is, by and large, correct.[10]

## 2.5 The Literature

In the early literature (Weyl [238, 239], Becker [10, 11], Kaufmann [139]) choice sequences are discussed extensively. But none of these discussions relates them to Husserl's ideas about temporal aspects of mathematical objects. (It is true that Becker [11] investigates temporal aspects of mathematics in depth, but he does so within the context of Heideggers's *Being and Time*, which is rather different from Husserl's framework, which I adopt here.)

On the one hand this should not be surprising, Husserl published very little (and only late) on these matters in the period in which Weyl, Becker and Kaufmann wrote.[11] On the other, he was in close contact with these authors, so one would have expected the matter to come up in private communications. (See note 8.)

The question of dynamic objects has not received due attention in recent phenomenological literature: Tieszen [214] and Lohmar [152] discuss aspects of intuitionism, but not this question.[12] Lohmar [153] mentions that the eternity (i.e., omnitemporality or atemporality) of mathematical objects would pose a problem for intuitionism, but he in no way questions this property himself. Bachelard [9] makes only a few passing remarks on intuitionism, and objects to its reformist pretentions. Schmit [189] confines his discussion of constructivism to its use within classical mathematics. Rosado Haddock [183], in spite of his book's title (*Edmund Husserl's Philosophy of Mathematics in the Light of Modern Logic and Foundational Research*), hardly mentions intuitionism at all.

The exception is Tragesser's discussion of choice sequences [218, ch. 4]; he makes the connection with the idea of different ontological regions, each with their own appropiate logic. Even though the conflict between Husserl's and Brouwer's views is not brought out, his exposition sets the stage to do so.

# 3

# The Argument

## 3.1 Presentation

The argument for the thesis runs as follows:

1. According to Husserl, mathematical objects are static. (Premise)
2. According to Brouwer, there is at least one kind of mathematical object, the choice sequence, that is not static. It is dynamic. (Premise)
3. Husserl's and Brouwer's conclusions are contradictory. (From 1 and 2)
4. Transcendental phenomenology provides the full ontology for the a priori sciences. (Assumption)
5. In transcendental phenomenology, ontological questions in the a priori sciences are decidable. (Elucidation of 4)
6. Measured by phenomenological standards, Husserl's argument is not correct. (Premise)
7. Measured by phenomenological standards, Brouwer's argument is not correct. (Premise)
8. There must be a third, phenomenological argument for either Husserl's or Brouwer's conclusion. (From 3, 5, 6, and 7)
9. Brouwer's argument can be reconstructed in phenomenology. (Premise)
10. One correct, phenomenological argument on the issue whether mathematical objects can be dynamic is not Husserl's (negative) argument, but a reconstruction of Brouwer's (positive) one. (From 8 and 9)

## 3.2 Comments

The intended meanings of various terms in this argument are specified as we go along. The assumption 4 and its elucidation 5 are explained in chapter 5. For 'reconstruction', I refer to the introduction, 2.3; for '(transcendental) phenomenology', to the introduction, 2.4, and chapter 4.

The main work is done in chapters 4–6, in which I argue for the truth of the premises in my argument. Each of these chapters is of a different nature.

Chapter 4 (concerning steps 1 and 2 of the argument) is expository.

Chapter 5 (steps 6 and 7) is critical and presents two argumenta ad hominem. The term is not meant pejoratively, but expresses that the original arguments of Husserl and Brouwer are attacked in their own terms. The Brouwer case has the form: 'According to his own principles, he cannot argue for his possibly true conclusion'. The Husserl case has the form: 'According to his own principles, his conclusion is not drawn correctly'. In terms of Johnstone's analysis of such arguments [135], I argue that in Brouwer's case, the mismatch between argument and conclusion is implied by the content of his (general) position ('charge of self-disqualification' [135, p. 91]). Husserl's, on the other hand, is of a more accidental kind: it can be remedied without giving up his (general) position ('charge of dogmatism' [135, p. 86]).

Chapter 6 (step 9) is constructive. Its positive result is that choice sequences are, from a phenomenological point of view, acceptable mathematical objects.

Chapter 7 presents an application of my analysis to one of the key questions about choice sequences once they are admitted. It is a phenomenological justification of the continuity principle.