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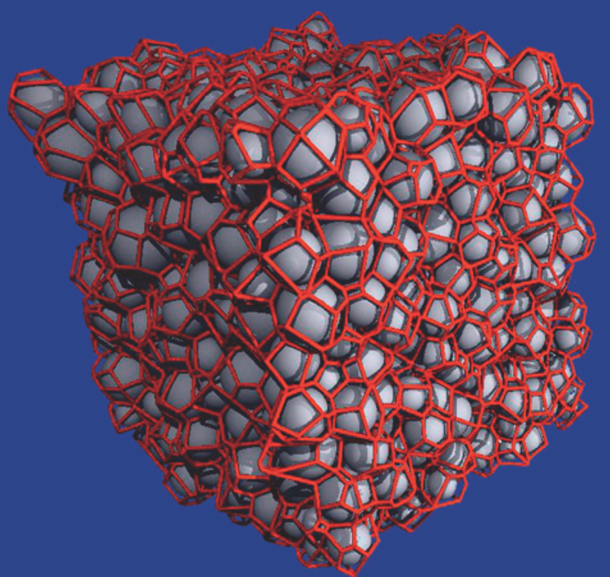
Stochastic Geometry and its Applications

Third Edition



Sung Nok Chiu • Dietrich Stoyan
Wilfrid S. Kendall • Joseph Mecke

WILEY

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Third Edition

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This edition first published 2013

Registered office
John Wiley & Sons Ltd, The Atrium, Southern Gate, Chichester, West Sussex, PO19 8SQ, United Kingdom

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Library of Congress Cataloging-in-Publication Data

Stoyan, Dietrich.
[Stochastische Geometrie. English]
Stochastic geometry and its applications. – Third edition / Sung Nok Chiu, Dietrich Stoyan, Wilfrid S. Kendall, Joseph Mecke.
pages cm – (Wiley series in probability and statistics)
Revision of: Stochastic geometry and its applications / Dietrich Stoyan, Wilfrid S. Kendall, Joseph Mecke. – 2nd ed. – ?1995.
Includes bibliographical references and indexes.
ISBN 978-0-470-66481-0 (hardback)
1. Stochastic geometry. I. Chiu, Sung Nok. II. Kendall, W. S. III. Mecke, Joseph. IV. Title.
QA273.5.S7813 2013
519.2–dc23

2013012066

A catalogue record for this book is available from the British Library.

ISBN: 978-0-470-66481-0

Set in 10/12pt Times by Thomson Digital, Noida, India.

1 2013

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Foreword to the first edition

My good friends the authors of this book have kindly invited me to add a few words describing 'how it all began'. At present one can write only an anecdotal history of stochastic geometry, and we must recognise that the anecdotes of others will certainly much extend the account given here, and will supply fresh perspectives. A serious attempt to write a history would be premature. The historian of mathematics looks always to the future rather than to the past. He hopes to find early instances of general concepts later seen to be of fundamental significance, and so if he works at a time of rapid development (such as the present) he will overlook many clues in the early record which point to a future not yet revealed.

My own first contact with *classical geometrical probability* occurred during the war, when the Superintendent of my group (Louis Rosenhead) asked me to investigate the following problem; for the sake of clarity I formulate it in the modern terminology.

> A euclidean plane with a marked origin O carries a Poisson field of unsensed lines with a uniform intensity. Almost surely the point O will lie in the interior of a unique Crofton cell C, with (unlabelled) shape $\sigma(C)$ and area $a(C)$. What probability statements can be made about C which convey information about the strength of a fabric ('a sheet of paper') consisting of the field of lines ('fibres')? Thus it would be useful to be able to calculate the rate of occurrence of splinter-shaped cells C, and the rate of occurrence of cells with large area $a(C)$.

A few moments of the $a(C)$-distribution were already known, and I managed to add one more to these. One would have preferred to be able to say something about the asymptotics of the marginal $a(C)$-distribution valid for large areas, and to throw light one way or the other on my conjecture that the conditional law for $\sigma(C)|a(C)$ converges weakly, as $a(C) \to \infty$, to the degenerate law concentrated at the circular shape.

Unfortunately nothing substantial is known about either of these questions even today, apart from the limited information that can be derived from a massive series of simulations carried out in Stanford by E. I. George (1982, 1987) in association with Herbert Solomon.

In 1961 Roger Miles wrote his Cambridge PhD thesis on a generalised version of this problem under the direction of Dennis Lindley and Peter Whittle and in consultation with paper technology experts from Wiggins Teape Research and Development Ltd. This initiated a long series of famous papers by Miles which provide a huge volume of information about the problem as a whole without, however, bringing us any nearer to the answer to the two questions above.*

All this work was within the classical Croftonian framework, not however without hints that a statistical theory of shape could play a useful rôle. What we now call *stochastic geometry* began for me with three papers by Maurice Bartlett: 'The spectral analysis of point processes' (*J. Roy. Statist. Soc.* B 1963), 'The spectral analysis of two-dimensional point processes' (*Biometrika* 1964), and especially 'The spectral analysis of line processes' (*5th Berkeley Symp.* 1967). These became available just when Rollo Davidson joined me as a research student in October 1965, and the impact of Bartlett's third paper can be felt in the Smith's Prize Essay which he wrote in 1967. As Bartlett's work had focused on the empirical spectral analysis of point and line processes, I encouraged Davidson to set up an appropriate theoretical framework underlying such an empirical analysis, and the second half of his PhD thesis was concerned with this; see Chapter 2.1 of *Stochastic Geometry* (ed. Harding and Kendall) for a reprint of it. The following two quotations give the flavour of the approach:

> *. . . we can talk of flat- and line-processes, meaning the point-processes that they induce on the appropriate manifolds,*
>
> *. . . my results showed that it is profitable, when considering point-processes, to observe simply whether certain sets contain points of the process or not; the usual approach is, of course, to look at the number of points in these sets.*

The first quotation, coupled with a general specification for a point process on a manifold M, takes us away from Croftonian geometric probability to the consideration of arbitrary random fields of geometric objects, while the second hints at a theory of random sets of very general character to replace the point process on the representation-manifold. Two such (closely related) theories of random sets very shortly afterwards became available; both were strongly influenced by earlier ideas due to Gustave Choquet. Stochastic Geometry thus became a reality.

When was the phrase 'stochastic geometry' first used? Klaus Krickeberg, who was to play a leading rôle in its development, thinks that perhaps he and I may have used the phrase informally in the Spring of 1969 when he was in Cambridge and we were planning the

*David Kendall died in 2007 at age 89 years. So it is the authors' duty to inform the reader about the fate of his conjecture. David Kendall posed the problem in the foreword of the first (1987) edition of the present book, but only in its second (1995) edition the problem, known as Kendall's conjecture, attracted more interest. Miles (1995) offered a heuristic proof, and surprisingly, only two years later, in 1997, a solution was given by the Ukrainian I. N. Kovalenko (1997, 1999), known until that time as a queueing theorist. He even found a similar result for large cells of the Poisson-Voronoi tessellation in Kovalenko (1998). Kovalenko's main idea is to give an upper bound for some conditional probability, by enlarging the numerator and reducing the denominator. For the former he used Bonnesen's inequality (a refined form of the planar isoperimetric inequality), for the latter an explicit construction.

In a series of papers, German stochastic geometers (Hug, Reitzner and Schneider) '*treated very general higher-dimensional versions, variants and analogy of Kendall's problem*', see Schneider and Weil (2008, p. 512) for an excellent overview. They considered the problem in d dimensions and could omit the isotropy assumption. For this, they started from Kovalenko's ideas but employed more sophisticated geometrical tools. In the anisotropic case '*the asymptotic shape of such cells was found to be that of the so-called Blaschke body of the hyperplane process. This is (up to a dilatation) the convex body, centrally symmetric with respect to the origin, that has the spherical directional distribution of the hyperplane process as its surface area measure*', see Hug and Schneider (2010). The last paper presents a solution for k-dimensional faces for $2 \leq k \leq d$, that is, for the k-volume weighted typical k-face, for example a polygonal cell face in the three-dimensional case.

Oberwolfach meeting (for June 1969) on *Integral Geometry and Geometrical Probability*. This is very possible, because 'stochastic analysis' was already in common use in the UK as part of the title of the Stochastic Analysis Group set up in December 1961 under the chairmanship of Harry Reuter, and one phrase naturally suggested the other. Certainly there was obviously no other choice for the titles of the two memorial books produced after Davidson's death in 1970. So 1969 was perhaps the year of coining. The initial group of enthusiasts could readily be identified by inspecting the *Tagungsbuch* at Oberwolfach. From the first Ruben Ambartzumian played a very important rôle and he has continued to influence the development of the subject in characteristic ways.

To some extent and to my great satisfaction there has also been a close association between those interested in stochastic geometry, and those interested in geometrical statistics, so that we now have a broad and lively subject area with abstract and empirical edges to it. I trust that this will continue, and the balance of the present book makes that seem likely.

Shape-theory is generally viewed as part of stochastic geometry and I think that this is as it should be. I have already mentioned one early hint at the need for a theory of shape, and here is a much earlier one. *The Ladies' Diary* (1706–1840), *The Gentleman's Diary* (1741–1840), and *The Lady's and Gentleman's Diary* (1841–1871) are mathematical periodicals known now perhaps only to a few specialists, but are worth very serious study because of the frequency with which important ideas first found explicit mention in their pages. In *TLGD* (1861) there is the following challenge by the London-based mathematician Wesley Stoker Woolhouse (1809–1893):

> *Problem 1987. Three points being taken at random in space as the corners of a plane triangle, determine the probability that it shall be acute.*

A solution by Stephen Watson of Haydonbridge, Northumberland, appeared in the 1862 edition of the diary. It begins with the comment:

> *'Space' is equivalent to a sphere of infinite radius, and it is obvious that the chance will be the same whatever be the radius of the sphere within which the three points may lie; hence we may suppose them to always lie within a sphere of radius unity.*

He then continues with an integration argument yielding the probability 33/70, and the probability $4/\pi^2 - 1/8$ for three points in a plane is added in a comment by Woolhouse.

Watson's solution provoked a strong reaction from Augustus de Morgan, who argued in *Trans. Cambridge Philos. Soc.* **11** (1871) 145–189 that 'it is very easily shown that the chance of an acute-angled triangle must be infinitely small'. The controversy raged for many years in the journal *Mathematical Questions with their Solutions from the 'Educational Times'*, where Woolhouse's challenge had been reprinted as Question 1333. The article on it by M. W. Crofton in 1867 is particularly interesting. In the pages of these journals we can see classical geometrical probability taking shape under our eyes.

But from our present point of view a most interesting comment is that Watson could very well have approximated to 'infinite space' by expanding an arbitrary compact convex set K about its centroid, and he would then have obtained a different answer based on the shape measure induced on Σ_2^3 (a hemisphere) by i. i. d.-uniform sampling from the interior of K, so that the resulting probability would depend on the shape of K itself, but not at all upon its size. Of course Watson's choice was a very natural one, because one feels that one is required to respect the isotropy of ‘infinite space’ – though this is a delusion, for the isotropy is only maintained at the centre of the sphere. Meaningless though Woolhouse’s problem is, without further specification, any reader of this book will probably find it instructive as well as amusing to read through these old polemics. As the same protagonists occur again and again in the pages of the journals mentioned, and as they express themselves with considerable freedom, one soon becomes familiar with them on a personal basis, and the early history of our subject comes to life in a most vivid way.

It only remains to say that, given the spherical assumption, the numerical answers obtained by Watson and by Woolhouse are identical with those derived from the recent solution to the general problem in n dimensions given by G. R. Hall in *J. Appl. Prob.* **19**, 712–15 (1982).

David Kendall

From the preface to the first edition

Complicated geometrical patterns occur in many areas of science and technology and often require statistical analysis. Examples include the structures studied in geology, sections of porous media, solid bodies, biological tissues, and patterns formed by the distinction between wood and field in a landscape. Analyses of such data sets require suitable mathematical models and appropriate statistical methods. The area of mathematical research that seeks to provide such models and methods is called *Stochastic Geometry*. The oldest part of this subject considers problems concerning a finite number of geometrical objects of fixed form, whose positions are completely random and (in some sense) uniformly distributed. The famous question of Buffon's needle is the prototype of these problems, which form the subject of *Geometrical Probability*. The modern theory of stochastic geometry (initiated by D. G. Kendall, K. Krickeberg, and R. E. Miles) considers random geometrical patterns (which may be infinite in extent) of more complicated distribution. *Stereology* is that branch of stochastic geometry which studies the problem of recovering information on three-dimensional structures when the only information available is two- or one-dimensional, obtained by planar or linear section.

This monograph grew out of a book originally published in German (Stoyan and Mecke, 1983b), but has undergone considerable expansion and reorganisation. Its aim is to make the results and methods of stochastic geometry more generally accessible to applied scientists, but also to provide an exposition which is mathematically exact and general, and which takes into account the current state of research in order to serve as an introduction to stochastic geometry for mathematicians. Of course these aims conflict and the resulting compromises have strongly influenced the form of the book. In most parts of the monograph proofs are omitted. The level of exposition is uneven and the subjects are treated with varying thoroughness: some topics are illustrated by numerical examples, some results are stated without much comment, others are accompanied by heuristic arguments, and sometimes substantial issues are dismissed with only a few remarks and a few references to the literature. Throughout the text attempts are made to explain the plausible nature and the underlying ideas of mathematical concepts, in order to facilitate the reader's understanding and to pave the way for a deeper study of the literature. Our hope is that those readers who do not wish to invest much effort in following mathematical arguments will nevertheless be able to interpret and to use most of the formulae.

There is some redundancy in the exposition but we believe this will help most readers. At some points it has been appropriate to use formulae and notation in anticipation of their introduction. In any case, readers may prefer to turn directly to the chapters concerning the topics that interest them the most, rather than to read though the book consecutively. Generally we have not sought to use the most elegant possible mathematical style but rather to strike a balance between generality and concrete special cases. For example, we use the theory of marked point processes, and this frees us from the need to consider point processes in abstract spaces.

At places we refer to ideas of mathematical physics that are related to the techniques of stochastic geometry. A closer collaboration between mathematical physicists and stochastic geometers might be very fruitful; the two subjects meet at several points but use different languages.

Mathematical terminology is used throughout the book. This exhibits some peculiarities due to historical accident. The word 'process' as in 'point process' and in 'line process' does not imply any dependence on time (with the possible exception of point processes on the real line). A more logical terminology would use the phrases 'random point field' and 'random line field'. . . .

The examples in the book are for the most part concerned with the analysis and description of images by numbers and functions, and are drawn from various branches of science. Generally the theoretical basis of the statistical methods is not discussed. In some cases statistical methods are given and these enable the fitting of models to empirical data. Much work remains to be done on statistical theory for geometrical structures. For example, little is known of the distribution theory for most estimators appearing in this book.

A brief summary of the contents of the book will illustrate the way in which theory and practice are intertwined. The first chapter briefly introduces areas of mathematics with which most scientists and engineers are not familiar. . . . We assume a basic knowledge of probability theory and statistics.

In the remaining chapters the development of the exposition does not proceed from the general to the particular but rather in the reverse direction. Thus Chapters 2 and 3 discuss the Poisson process and the Boolean model, which are simple cases of the random structures to be discussed in the remainder of the text. Chapters 4 and 5 continue the subject of point processes and give a general discussion; Chapter 6 expounds the general theory of random sets. Chapter 7 briefly introduces the important concept of a random measure, which arises throughout the subject at a more theoretical level. The theory of random processes of geometrical objects is introduced in Chapter 8, which leads on to the discussion of fibre processes in Chapter 9[†] and tessellations in Chapter 10. The final Chapter 11 is on stereology, which is of great importance in practice and uses results and ideas from all of the preceding discussion. . . .

[†]In the third edition Chapters 8 and 9 are combined to form a chapter on line, fibre and surface processes.

Preface to the second edition

We the authors present a second edition of our book. The first edition met with a kind reception and has become a standard reference in its field . This has encouraged us to retain its style and conception. As before this book has an applied character, presents the matter in a less than strictly sequential form and admits inhomogeneities in the presentation. Our personal taste and interests played an important rôle in choosing the topics.

We have tried to present many of the new ideas and developments in the fields of stochastic geometry and spatial statistics since 1987. They seem to us particularly prominent in the fields of Boolean models, stereology, random shapes, Gibbs processes, and random tessellations. The progress of these years is also visible in the jacket of this book: a figure in Chapter 10 of the old edition presented a small part of a Johnson–Mehl tessellation (drawn by hand by H. Stoyan); this has been replaced by a computer-generated figure containing many cells, and we have used a similar figure to decorate the cover of the new edition.

We hope very much that our readers will find the style and presentation of the second edition better than that of its predecessor. It was a pleasure to eliminate a series of misprints and (we have to confess) errors; and also the poor texture of the paper of the first edition can now be forgotten. We also hope that the many minor additions will be noticed, which arose from many discussions with colleagues. On the other hand we have to warn our readers that at a few points notation has been changed; we hope that the number of new misprints and errors is small.

As in the first edition, we do not present all which may go under the names 'stochastic geometry' and 'spatial statistics'. This is quite appropriate since there are already specialised books on spatial statistics (Cressie, 1993), fractals (Falconer, 1990; Stoyan and Stoyan, 1994), random shapes (Stoyan and Stoyan, 1994; Barden, Carne, Kendall, and Le, 1996[‡]), and integral geometry (Schneider, 1993).

Producing the manuscript of the second edition was not an easy task for us because of our various other professional duties. It was only possible with the help of many friends and colleagues. They read whole chapters or parts of them and suggested many corrections and additions. We are very grateful to them: S. N. Chiu, L. M. Cruz-Orive, L. Heinrich, D. G. Kendall, M. N. M. van Lieshout, U. Lorz, K. V. Mardia, I. Molchanov, L. Muche, W. Nagel, J. Ohser, R. Schneider, and E. Schüle.

The hard technical work was done by H. Stoyan, assisted by I. Gugel and R. Pohlink. She did this work with incredible care and patience and also suggested many corrections and improvements of a scientific nature.

We have also to thank two collections of electronic software. LaTeX proved to be an excellent tool for the production of our manuscript: in common with very many other mathematical scientists, we owe an almost incalculable debt to D. E. Knuth and L. Lamport. The first edition

[‡]This actually refers to D. G. Kendall *et al.* (1999).

was still produced in the classical way using lead type, and so W.S.K. may be one of the last Englishmen to have seen in his proofs a ‘Zwiebelfisch’ (the German word for a letter standing on the head).[§] The existence of *e-mail* made the correspondence between Warwick, Chichester and Freiberg easy and very fast. (It would have seemed incredible to us ten years ago, but the authors did not have any personal meeting during the work for the manuscript.) We also thank Stuart Gale of John Wiley & Sons Ltd. for his work as an editor; his predecessor of the first edition, Dr R. Höppner, is now Ministerpräsident of the German Bundesland Sachsen-Anhalt.

July 1995 *The authors*

[§]In some sense, this was not true. In the 1995 edition the name ‘Hansen’ was written as ‘Hausen’. The ‘u’ can be seen as a upside-down ‘n’ and hence can be regarded as a Zwiebelfisch, but in fact it was a typo.

Preface to the third edition

It is perhaps unusual to make a third edition of a book 18 years after its second. However, the authors remained active in the field of the book and observed with pleasure that the second edition, abbreviated as 'SKM95' in the text, became a standard reference for (applied) stochastic geometry and wished to maintain this status for the future. Finally and crucially, the original authors found a younger new co-author, so being competent to produce a modernised book.

In the years since 1995 many other books on stochastic geometry have been published, but all have been of a nature different from SKM95. There are now excellent books of a high theoretical level such as Schneider and Weil (2008) and Kendall and Molchanov (2010). The present book uses them as references and source for proofs of complicated mathematical facts and by no means aims to compete with them. Then there are now specialised books which present the methods of image analysis and processing of lattice data coming from modern imaging techniques, such as Ohser and Schladitz (2009). On the other hand there are now various books which present ideas of stochastic geometry to physicists, engineers and others, such as Ohser and Mücklich (2000), Torquato (2002) and Buryachenko (2007). However, none of these books plays the rôle of SKM95, as a book which is accessible for a broad readership of applied mathematicians, physicists and engineers, but also presents mathematical foundations and in some cases mathematical proofs. By the way, the selection of the statements which are proved was made according to the following considerations: proofs are included when they show how the mathematical tools work, where the argument is not too complex and where somehow unexpected results are derived. The book by Schneider and Weil (2008) clearly demonstrates how large a book may become if it aims to give nearly 'all' proofs.

In the process of modernising (and correcting errors in) SKM95, which started in 2010, the authors learned which areas of stochastic geometry have been particularly active. The first such area is the theory of random sets, where new books such as Molchanov (2005) and Nguyen (2006) were published and many new models have been developed, for example in Baccelli and Błaszczyszyn (2009a,b). The second is the theory of tessellations. The corresponding Chapter 9 of this book was enlarged by new sections on networks and random graphs since these areas are becoming more and more important. Of course, the classical branches such as the theory of point processes also developed new ideas, and so the important theory of point-stationarity and balanced partitions appears in Chapter 4. Unfortunately, in order to limit the volume of the new book the section on random shape theory had to be omitted. A reason for this omission is that there are now excellent books on random shape theory with which the present book could never compete. Nevertheless, this edition is still much thicker than its predecessor and has about 700 new references, though about 300 outdated references have been deleted.

In the years since 1995 stochastic geometry further developed as a mathematical discipline. This has resulted in simplifying and generalising its theories and making its notation more elegant. For example, the Minkowski functionals intensively used by Matheron have been replaced by the intrinsic volumes. The description of tessellations has been refined to include not necessarily face-to-face tessellations in the theory, following R. Cowan and V. Weiß.

In discussing changes in the notation, it may be interesting to add some words about the notation used in this book. Unfortunately, there is no unique notation system in stochastic geometry. Even the various book authors have different personal notations. And the notations used by mathematicians and physicists differ greatly.

This book commits to a consistently mathematical notation as exemplified by the use of $\mathbf{E}(X)$ for the mean of X instead of $\langle X \rangle$ as physicists would write. As geometers do, the multidimensional space is denoted by $\mathbb{R}^d$, with d as 'dimension'.

Various traditions come together in the notation of the present book. Queueing theorists played a significant rôle in the early development of point process theory and stochastic geometry. As a consequence, the intensity or density is conventionally denoted by λ. This follows the queueing tradition, which denotes the arrival and service rates of queueing systems by λ and μ. Product densities are denoted by $\varrho^{(n)}$, following an old notation system of physicists, to which, by the way, the symbol ϱ for the intensity belongs. The authors considered replacement of λ by ϱ, but the tradition was stronger and even the youngest author argued in favour of old λ; moreover a capital Λ is needed, whereas the capital counterpart to ϱ is P, which has many other uses in the book. The Lebesgue measure is sometimes denoted by λ, which does not fit into this scheme; and also μ is too often used for means. Thus also in this edition the Lebesgue measure is again denoted by ν.

There is some influence of the now classical book Matheron (1975). The French 'fermé' has led to $\mathbb{F}$ for the system of all closed subsets of $\mathbb{R}^d$ and the related $\mathcal{F}$ and $\mathcal{F}_K$. And b_d with 'b' as 'ball' is used for the volume of the unit ball in $\mathbb{R}^d$, for which other authors use κ_d and ω_d, and the ball with centre x and radius r is $B(x, r)$.

Finally, the use of Φ to denote a point process goes back to Klaus Matthes.

It is a pleasure to thank here all the colleagues who helped us in producing the manuscript of the third edition. The list is so long that we may speak of a 'collective work'. In alphabetic order we name R. Adler, A. Baddeley, F. Baccelli, F. Ballani, S. Bernstein, B. Błaszczyszyn, P. Calka, S. Ciccariello, R. Cowan, D. Dereudre, W. Gille, P. Grabarnik, P. Hall, L. Heinrich, H. Hermann, J. Janáček, S. Kärkkäinen, M. Kiderlen, M. Lang, G. Last, T. Mattfeldt, N. Medvedev, I. Molchanov, J. Møller, L. Muche, W. Nagel, A. Penttinen, P. Ponížil, C. Redenbach (née Lautensack), V. Schmidt, R. Schneider, D. Schuhmacher, V. Weiß, K. Y. Wong and S. Zuyev.

A particular rôle as readers and suppliers of mathematical criticism of whole chapters was played by P. Grabarnik, L. Muche, W. Nagel and J. Ohser. G. Last read Chapter 4 and wrote for us the Sections 4.4.9 and 4.4.10. K. Y. Wong offered technical support. Finally, R. Schneider answered with great patience many questions from D.S.

This book contains an accompanying website. Please visit www.wiley.com/go/cskm

Autumn 2012 *The authors*

Notation

This index contains only the notation used throughout the book. Symbols with localised usage are omitted, as are ‘standard’ symbols such as e and π.

Symbols

Symbol	Usage	Page
$\| \cdot \|$	Euclidean metric	3
$\mathbf{1}_A$	indicator function of A	17
$\# \{\ldots\}$	number of elements of set $\{\ldots\}$	46

Operations

Symbol	Usage	Page
$\circ$	opening	8
$\bullet$	closing	8
$\oplus$	Minkowski-addition	5
$\ominus$	Minkowski-subtraction	5
$*$	convolution	132
conv	convex hull	11
∂	boundary	4

Greek letters

Symbol	Usage	Page
$\alpha^{(n)}$	factorial moment measure	46, 47, 121
χ	connectivity number	25
$\gamma_K, \overline{\gamma}_K(r)$	set covariance function	17, 18
$\kappa(\mathbf{r}), \kappa(r)$	correlation function	75, 218
λ	intensity, intensity function	41, 51, 113, 280
Λ	intensity measure	45, 51, 112, 118, 280
$\mu^{(n)}$	moment measure	44, 47, 120
ν_d	Lebesgue measure	30
Ω	generic sample space	33
Φ	point process, fibre (surface) process, random measure	109, 280, 315, 336
φ, ϕ	realisation of Φ	108, 280, 315
$\Phi_k(K, B)$	curvature measure of K, where B is a Borel set	290
Ψ	marked point process, marked random measure	116, 283
ψ	realisation of Ψ	283
Φ_x	translated point process	42, 112
$\varrho^{(n)}$	product density	46, 47, 122
$\sum^{\neq}$	summation over all distinct pairs (distinct n-tuples)	46, 121
Ξ	random closed set	65, 206

Blackboard bold letters

Symbol	Usage	Page
$C(\mathbb{K})$	system of compact convex subsets of $\mathbb{R}^d$	11
$\mathbb{F}$	system of closed subsets of $\mathbb{R}^d$	4
$\mathbb{G}, \mathbb{G}_n, \mathbb{G}(n, p)$	random graph	402, 404
$\mathbb{K}$	system of compact subsets of $\mathbb{R}^d$	4
$\mathbb{L}_k$	set of all k-subspaces	14
$\mathbb{M}$	space of marks, set of measures	116, 166, 280
$\mathbb{N}$	system of all simple and locally finite point sequences	108
$\mathbb{R}$	real line	1
$\mathbb{R}^d$	d-dimensional Euclidean space	2
$\mathbb{S}$	extended convex ring	25
$\mathbb{W}$	space of marks	283

Script letters

Symbol	Usage	Page
$\mathcal{A}$	σ-algebra of Ω	33
$\mathcal{B}^d$	Borel σ-algebra of $\mathbb{R}^d$	28
$\mathscr{C}$, $\mathscr{C}^!$	Campbell measure	123, 281
$\mathcal{F}$	hitting σ-algebra	206
$\mathscr{K}$	reduced second moment measure	140
$\mathcal{M}$	σ-algebra of $\mathbb{M}$	116, 280
$\mathcal{N}$	σ-algebra of $\mathbb{N}$	109
$\mathcal{R}$	convex ring	24
$\mathscr{R}$	rose of directions	307
$\mathscr{V}(\varphi)$	Voronoi tessellation relative to φ	347
$\mathcal{W}$	σ-algebra of $\mathbb{W}$	283

Bold letters

Symbol	Usage	Page
E	expectation	33
P	probability	33
cov	covariance	34
var	variance	34
$\boldsymbol{m}$	Euclidean isometry, rigid motion	9, 112
$\boldsymbol{r}$	rotation of $\mathbb{R}^d$ around o	10

Roman letters

Symbol	Usage	Page
$\check{A}$	reflection $-A$	5
A_A	area fraction	217, 414
$A(K)$	area of K	12
$A_{\oplus r}$	$A \oplus B(o, r)$	6
$A_{\ominus r}$	$A \ominus B(o, r)$	6
A_x	translation $A + x$	5
a_V	integral range	219
$B(a, r)$	ball of radius r centred at a	3
$\overline{b}$	average breadth	12
b_k	volume of unit ball in $\mathbb{R}^k$	14
$C(\mathbf{r})$, $C(r)$	covariance, two-point probability function	74, 218
$C_0(\Theta)$, C_0	zero cell of a tessellation	359
C^o	typical cell of a tessellation	364

Symbol	Usage	Page
$D(r)$	nearest-neighbour distance distribution	132
$g(r)$	pair correlation function of a point process	47, 141
h_k	k-dimensional Hausdorff measure	30
$H_B(r)$, $H_d(r)$, $H_l(r)$, $H_q(r)$, $H_s(r)$	contact distribution function	77, 81, 82, 83, 223
$k_{mm}(r)$	mark correlation function	123
$k(\mathbf{r})$, $k(r)$	covariance function	160, 218, 259
$K(r)$	Ripley's K-function	51, 141
L_A, L_V	intensity of fibre process, length density	289, 308, 316, 331, 415
$l(K)$	length of K in $\mathbb{R}^1$	12
$L(K)$	perimeter, boundary length of K	12
$L(r)$	L-function of a point process,	57, 141
	chord length distribution of a random closed set	276, 418
M	integral of mean curvature	14
$M(L)$	mark distribution	119, 283
M_V	specific integral of mean curvature	80
N_A, N_V	specific connectivity number	80, 294
N_A^+, N_V^+	specific convexity number	80, 294
$\overline{n}_{kl}$	mean number of l-faces adjacent to the typical k-face of a tessellation	368
o	origin	3
p	volume fraction	70, 74, 216
P	distribution of point process, random set, etc.	109, 207
P_o, $P_o^!$, P_x, $P_x^!$	Palm distribution	129, 130, 131, 132
S^{d-1}	unit sphere in $\mathbb{R}^d$	4
$S(K)$	surface area of K	12
$s(K, u)$	support function of K	11
$s_m(K, \ell)$	modified support function of K	11
S_V	intensity of surface process, specific surface area	289, 312, 334, 415
$T_\Xi(K)$	capacity functional of Ξ, where K is a compact set	71
$V(K)$	volume of K	12
v_k	specific k^{th} intrinsic volume	79, 249
$\overline{V}_k$	mean k^{th} intrinsic volume	67, 78
$V_k(K)$	k^{th} intrinsic volume of K	15
V_V	volume fraction	217, 414
W	window of observation	55, 145, 230
$W_k(K)$	k^{th} Minkowski functional of K	15
z_α	quantile of the standard normal distribution	55, 421
$Z(x)$	Gaussian random field	260

1

Mathematical foundations

1.1 Set theory

The language of naïve set theory is ubiquitous in geometry and even more so in stochastic geometry. The reader will find a thorough introduction in specialised textbooks. The following briefly summarises notation and defines important sets and operations which will often be employed later.

A *set* is a collection of mathematical objects taken from a suitable domain of discourse. If x is an *element* of a set S this is written as $x \in S$. All sets appearing in this book are constructed from two fundamental sets, which are the set of the natural numbers $\{1, 2, \ldots\}$ and the set of the real numbers (the *real line*) $\mathbb{R} = (-\infty, \infty)$. All the constructions here are suitably regular, and the more profound aspects of mathematical logic and set theory are ignored.

The notation for the sets of natural and real numbers illustrates two useful conventions for the description of sets. The braces $\{\,\}$ in the example above enclose a description of the set of natural numbers by (implicit, infinite) enumeration. The notation (u, v) for two real numbers, perhaps equal to $-\infty$ or $+\infty$, describes the set of all real numbers x such that $u < x < v$. This set (u, v) is an *open interval* of the real line $\mathbb{R}$. *Closed* and *half-open intervals* are given by

$$
\begin{aligned}
(u, v] &= \{x \in \mathbb{R} : u < x \le v\} \quad \text{(half-open)},\\
[u, v) &= \{x \in \mathbb{R} : u \le x < v\} \quad \text{(half-open)},\\
[u, v] &= \{x \in \mathbb{R} : u \le x \le v\} \quad \text{(closed)}.
\end{aligned}
$$

Here the braces $\{\,\}$ enclose a description of a set as the collection of elements of another set satisfying some property. Contraction of this notation is often used, as for example:

$$
\begin{aligned}
&\{x \in \mathbb{R} : x = y + z \text{ with } 0 < y < 1 \text{ and } 0 < z < 1\}\\
&\quad = \{x : x = y + z \quad \text{with } 0 < y < 1 \text{ and } 0 < z < 1\}\\
&\quad = \{y + z : 0 < y < 1 \text{ and } 0 < z < 1\}.
\end{aligned}
$$

Stochastic Geometry and its Applications, Third Edition.
Sung Nok Chiu, Dietrich Stoyan, Wilfrid S. Kendall and Joseph Mecke.

Note that this set is actually the open interval $(0, 2)$. Call A a *subset* of a set S (and S a *superset* of A), and write $A \subset S$, if all elements of A are also elements of S. (This book does not use the symbol $\subseteq$, thus $\subset$ includes also the case that $A = S$.) If $A, B \subset S$ for some set S then their *union, intersection*, and *difference* are

$$\begin{array}{ll} \text{union} & A \cup B = \{x \in S : x \in A \text{ or } x \in B\}, \\ \text{intersection} & A \cap B = \{x \in S : x \in A \text{ and } x \in B\}, \\ \text{difference} & A \setminus B = \{x \in S : x \in A \text{ and } x \notin B\}. \end{array}$$

Also define the *complement* A^{c} of A in S as

$$\begin{aligned} A^{\mathrm{c}} &= \{x \in S : \; x \notin A\} \\ &= S \setminus A. \end{aligned}$$

Notice that the definition of A^{c} depends on the – usually implicit – choice of superset S. The empty set $\emptyset$ is the set that contains no elements. Formally, it is

$$\emptyset = S \setminus S = A \setminus A$$

for any A.

Special collections of sets (*σ-algebras*) are considered in Section 1.9.

Two sets A and B can be used to form the *Cartesian product* $A \times B$ given by the ordered pairs (a, b), that is,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

More generally, the Cartesian product of n sets $A_1, \ldots, A_n$ is

$$A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_1 \in A_1, \ldots, a_n \in A_n\}.$$

An important example is given by

$$\begin{aligned} \mathbb{R}^2 &= \mathbb{R} \times \mathbb{R} \\ &= \{(x_1, x_2) : x_1, x_2 \in \mathbb{R}\} \end{aligned}$$

which is the Cartesian plane. The higher-dimensional counterparts are

$$\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

and

$$\mathbb{R}^d = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \mathbb{R}\}.$$

The spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are often referred to as the *plane* and *space*, respectively, and $\mathbb{R}^d$ as the *d-dimensional space*. Because of additional structures such as topology (see Section 1.2) and linearity (Section 1.3), the term *Euclidean space* is used. An element $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ is usually referred to as a *point* in geometry. However, in stochastic geometry a distinction must be drawn. The study of stochastic geometry frequently concerns random collections of points, referred to as *point processes*. It is convenient to refer to members of such processes as *points of the process*, or simply *points*. Therefore points that are merely locations