Selected Works in Probability and Statistics

Selected Works of Willem van Zwet

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Editors

Selected Works of Willem van Zwet

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To Willem

Preface to the Series

Springer's Selected Works in Probability and Statistics series offers scientists and scholars the opportunity of assembling and commenting upon major classical works in probability and statistics, and honors the work of distinguished scholars in probability and statistics. Each volume contains the original papers, original commentary by experts on the subject's papers, and relevant biographies and bibliographies.

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The subjects of the volumes have been selected by an editorial board consisting of Anirban DasGupta, Peter Hall, Jim Pitman, Michael Sörensen, and Jon Wellner.

Preface

With this collections volume, some of the important works of Willem van Zwet are moved to the front layers of modern statistics. We have made a selection based on discussions with Willem, and aiming at a representative sample. The result is a collection of papers that the new generations of statisticians should not be denied. They are here to stay, to enjoy and to form the basis for further research.

We have grouped the papers into six themes. The first three papers give an impression of the broad scope of statistics. One of its core business is as in all mathematics: classification, characterization, and unification. The third paper here discusses M- and Z-estimators, which have their modern face nowadays in non- and semi-parametric models.

The next theme concerns asymptotic theory. We cite Lucien Le Cam ([1]) *"If you need to use asymptotic arguments, don't forget to let your number of observations tend to infinity"*. Asymptotic statistics is indeed a subtle area involving much more than only pointwise limit theorems. The papers in this volume cover nonparametric tests as well as semi-parametric estimation, putting down the fundamentals for asymptotic efficiency in such models.

A very important, but sometimes notoriously technical topic, is second order approximations. With his co-authors, Willem deals with this topic in an impressingly elegant way. The beauty of concepts in this area is evolving further, for example by the formalization of the distance of distributions to the normal distribution. Within this theme, this volume contains the original contribution of Sergey Bobkov, Gennadiy Chistyakov and Friedrich Götze exposing the limits of near-normality.

Willem was very much intrigued by the bootstrap. It is often used without worrying about its validity, whereas Willem's intuition said its all round applicability is very questionable. This turned out to be a mind twisting and exciting issue: see the papers in this theme.

There is the modeling, the analysis of the model, and the statistical estimation. In the applications theme, we see all three aspects together. It shows that even though there are many sophisticated probabilistic models around, one still may have to start from scratch when looking at a particular real life problem. This is difficult hard work, but the final result is complete and beautiful.

Although statistics is not often associated with mathematical conjectures, it actually generates many. These are often questions in theoretical probability. The challenge to prove or disprove conjectures deserves its prominent place in statistics, and gives rise to fascinating storytelling.

This volume serves as basic reference for fundamental statistical theory, and at the same time reveals some of its history. We hope the unique mix will show the adventurous aspects of our profession, and that it will be an inspiration to all!

Zürich, *Sara van de Geer*
June 2011 *Marten Wegkamp*

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Contents

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Biography of Willem van Zwet

Marten Wegkamp

Willem R. van Zwet was born in Leiden, the Netherlands, in 1934. He obtained a Masters degree in Mathematics at the University of Leiden in 1959. After his military service, Willem decided to continue his studies in statistics. The Mathematics Centre in Amsterdam was at that time the only place in the Netherlands with a proper statistics program. This centre had been founded after the war, in 1946. The first head of the Statistics Department was the mathematician David van Dantzig. His successor, Jan Hemelrijk, appointed Willem as sous-chef of the department in 1961. During daytime, Willem taught classes and did consulting work at the Centre. During the late hours, he worked on his thesis. With Jan Hemelrijk as advisor, Willem graduated in 1964 with a Ph.D. in Mathematics at the University of Amsterdam.

In 1965, he was appointed Associate Professor of Statistics at the University of Leiden. He spent the first semester as Associate Professor at the University of Oregon. Willem was promoted to Full Professor in 1968 and he remained in Leiden until his retirement in 1999. During 1990–1996, he visited the University of North Carolina at Chapel Hill on a regular basis as the William Newman Professor. He was a frequent visitor, and Miller Professor in 1997, of the University of California at Berkeley.

Willem is known for his pertinent contributions in various areas of mathematical statistics. This book is an homage to his scientific work. But Willem is also known as a talented and tireless organizer. The interview in Beran and Fisher (2009) paints an excellent picture of his academic life, and filled with many humorous anecdotes, makes for a recommended read. Statistics was still in its infancy in the early seventies in Europe and his service for the statistics community in his native Netherlands and worldwide are truly remarkable.

For instance, Willem served as member and chair of the European Regional Committee of the Institute of Mathematical Statistics (1969–1980) that organized the European Meetings of Statisticians. In 1972, he organized the first Lunteren Stochastics conference and he remained, until 1999, an organizer of this successful meeting, that continues to be held each Fall in Lunteren, the Netherlands. He was president of the Institute of Mathematical Statistics (1991–1992) and the Bernoulli Society for Mathematical Statistics and Probability (1987–1989), and vice-president (1985–1989) and president (1997–1999) of the International Statistical Institute. Willem was Associate Editor (1972–1980) and Editor (1986–1988) of the Annals

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of Statistics and Editor-in-Chief of Bernoulli (2000–2003). He was the director of the Thomas Stieltjes Institute of Mathematics in the Netherlands (1992–1999), and founding director of the European research institute EURANDOM (1997–2000). Other activities included Dean of the School of Mathematics and Natural Sciences of the University of Leiden (1982–1984), chair of the scientific council and member of the board of the Mathematics Centre at Amsterdam (1983–1996) and the Leiden University Fund (1993–2005), member of the Board of Directors of the American Statistical Association (1993–1995) and member of the Corporation and the Board of NISS (1993–2002).

Fortunately, Willem's many scientific and organizational efforts are well recognized. He is a Fellow of the Institute of Mathematical Statistics (1972) and the American Statistical Association (1988), Honorary Fellow Royal Statistical Society (1978) and Honorary member of the International Statistical Institute (1999) and Netherlands Statistical Society (2000). He presented the Hotelling Lectures at the University of North Carolina (1988), the Wald Memorial Lectures (1992), and the Bahadur Lectures at the University of Chicago (2005). He is a member of the Royal Netherlands Academy of Sciences (1979) and the Academia Europaea (1990), and an honorary doctor of Charles University at Prague (1997). He received the Van Dantzig Medal of the Netherlands Society for Statistics and Operations Research (1970), the Bernoulli Medal (Tashkent, 1986), the Peace Medal of Charles University (1988), the Médaille de la Ville de Paris (1989), the Adolphe Quételet Medal of the International Statistical Institute (1993), the Certificate of Appreciation of the American Statistical Association (1995), the AKZO-Nobel Award (1996), and the Alexander von Humboldt Research Prize (2006). Perhaps the most prominent recognition happened in 1996, when Queen Beatrix of the Netherlands made him a Knight in the Order of the Netherlands Lion.

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Part I
Three Fundamental Statistics Papers

Convex transformations: A new approach to skewness and kurtosis *)

by W. R. van Zwet **)

UDC 519.2

S a m e n v a t t i n g

In dit artikel worden een tweetal orde-relaties voor waarschijnlijkheidsverdelingen voorgesteld, die – beter dan de klassieke maten gebaseerd op derde en vierde momenten – aangeven wanneer een verdeling een grotere scheefheid of kurtosis bezit dan een andere verdeling. Voorts wordt een aantal karakteriseringen en toepassingen van deze orde-relaties behandeld. Bewijzen worden in dit artikel niet gegeven; deze zijn te vinden in een meer uitgebreide, aan dit onderwerp gewijde studie [5].

1. Introduction

Every statistician will have at least an intuitive idea of what is meant by the concepts of „skewness" and „kurtosis" of a probability distribution and he will be aware of the fact that these should play an important role in applications. He will also probably feel vaguely dissatisfied with the existing measures for these concepts, i.e. the standardized third and fourth central moments, and indeed there are at least two perfectly good reasons for this uneasy feeling.

The first one is that, according to these measures, any pair of probability distributions that possess finite fourth moments may be compared as to skewness and kurtosis, whereas one feels that pairs of such distributions exist that are quite incomparable in these respects. The second reason is that, to the author's knowledge, very few interesting applications of any generality exist. It is fairly obvious that both disadvantages are closely related: the reason for the apparent lack of applications is precisely the fact that comparison of probability distributions on the basis of these measures is so often meaningless.

From the above it will be clear that at the root of the trouble lies the fact that these measures impose a simple ordering – i.e. an ordering where every pair of elements are comparable – on too large a class of probability distributions. Rather than restricting ourselves to considering smaller classes of distributions we shall try and find a more satisfactory approach by considering partial orderings – i.e. orderings where not every pair of distributions are necessarily comparable – to replace the classical measures.

*) Rapport S 328 van de afdeling Mathematische Statistiek van het Mathematisch Centrum te Amsterdam; Lezing gehouden op de Statistische Dag 1964.

**) Sous-chef voor mathematische statistiek van het Mathematisch Centrum te Amsterdam.

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It will be shown in this paper that two partial order relations exist that seem to cover our intuitive ideas about skewness and kurtosis. These order relations will not only be seen to imply the ordering according to the classical measures, but also to be so much stronger than the classical orderings as to permit meaningful applications. No proofs will be given in this paper; they may be found in [5], where a more extensive study of the subject is made. Part of the material presented here and in [5] was previously discussed in [4].

2. Notation

Let $\underline{x}$ be a non-degenerate real-valued random variable[1]) and let I be the smallest interval for which $P(\underline{x} \varepsilon I) = 1$. We define the distribution function F of $\underline{x}$ by

$$F(x) = \tfrac{1}{2} P(\underline{x} < x) + \tfrac{1}{2} P(\underline{x} \leqslant x)$$

and the expectation and central moments of $\underline{x}$ by

$$\mathscr{E}\underline{x} = \int_I x \, dF(x),$$

$$\sigma^2(\underline{x}) = \mu_2(\underline{x}) = \int_I (x - \mathscr{E}\underline{x})^2 \, dF(x), \text{ and}$$

$$\mu_k(\underline{x}) = \int_I (x - \mathscr{E}\underline{x})^k \, dF(x), \quad k = 3,4,\ldots,$$

where the right-hand sides denote STIELTJES integrals. We shall say that these expectations exist only if they are finite. The distribution given by F is said to be symmetrical about $x_0 \, \varepsilon \, I$ if

$$F(x_0 - x) + F(x_0 + x) = 1 \quad \text{for all real } x.$$

Let $\underline{x}_{1:n} \leqslant \underline{x}_{2:n} \leqslant \ldots \leqslant \underline{x}_{n:n}$ denote an ordered sample of size n from the distribution F; $\underline{x}_{i:n}$ is called the $i - th$ order statistic of a sample of size n from F. In the greater part of this paper we shall confine our attention to the class $\mathscr{F}$ of distribution functions F satisfying

(a) F is twice continuously differentiable on I;
(b) $F'(x) > 0$ on I;
(c) There exist integers i and n, $1 \leqslant i \leqslant n$, such that $\mathscr{E}\underline{x}_{i:n}$ exists.

For $F \, \varepsilon \, \mathscr{F}$ the inverse function G is uniquely defined on (0,1) by[2])

$$GF(x) = x \qquad \text{for } x \, \varepsilon \, I.$$

[1]) We denote random variables by underlining their symbols.

[2]) We shall usually not use brackets to denote composite functions and write GF and $GF(x)$ rather than $G(F(.))$ and $G(F(x))$.

We shall also be concerned with the subclass $\mathscr{S} \subset \mathscr{F}$ of symmetric distributions in $\mathscr{F}$.

When we consider simultaneously two random variables, $\underset{\sim}{x}$ and $\underset{\sim}{x}^*$, with distribution functions F and F^*, we shall adopt similar conventions and notations with regard to $\underset{\sim}{x}^*$ and F^*, and write: I^*, $\underset{\sim}{x}^*_{i:n}$ and G^*.

A real-valued function φ defined on I is said to be convex on I if for all $x_1, x_2 \varepsilon I$ and $0 \leqslant \lambda \leqslant 1$

$$\varphi(\lambda x_1 + (1 - \lambda) x_2) \leqslant \lambda \varphi(x_1) + (1 - \lambda) \varphi(x_2),$$

i.e. the graph of φ lies below any chord. We note that this definition implies continuity of φ on I, except perhaps at its endpoints, if these exist. A real-valued function φ on I is said to be antisymmetrical and concave-convex on I about $x_0 \varepsilon I$, if for all $x_0 - x \varepsilon I$, $x_0 + x \varepsilon I$,

$$\varphi(x_0 - x) + \varphi(x_0 + x) = 2\varphi(x_0),$$

and if φ is concave for $x \leqslant x_0$ and convex for $x \geqslant x_0$, $x \varepsilon I$; x_0 will be called a central point of φ.

3. Convex and concave-convex transformations

Suppose that φ is non-decreasing and convex on I and consider the random variables $\underset{\sim}{x}$ and $\varphi(\underset{\sim}{x})$. Apart from an overall linear change of scale such a transformation of the random variable $\underset{\sim}{x}$ to the random variable $\varphi(\underset{\sim}{x})$ effects a contraction of the lower part of the scale of measurement and an extension of the upper part. As, moreover, this deformation increases towards both ends of the scale, the transformation from $\underset{\sim}{x}$ to $\varphi(\underset{\sim}{x})$ produces what one intuitively feels to be an increased skewness to the right. The following theorem holds:

Theorem 3.1

If φ is a non-decreasing convex function on I, which is not constant on I, and if $\mu_{2k+1}(\underset{\sim}{x})$ and $\mu_{2k+1}(\varphi(\underset{\sim}{x}))$ exist, then

$$\frac{\mu_{2k+1}(\underset{\sim}{x})}{\sigma^{2k+1}(\underset{\sim}{x})} \leqslant \frac{\mu_{2k+1}(\varphi(\underset{\sim}{x}))}{\sigma^{2k+1}(\varphi(\underset{\sim}{x}))}, \quad \text{for } k = 1,2, \ldots .$$

It is intuitively equally appealing that a non-decreasing, antisymmetric and concave-convex transformation of a symmetrically distributed random variable should lead to an increased kurtosis of the distribution. We have:

Theorem 3.2

Let φ be a non-decreasing, antisymmetrical, concave-convex function on I, which is not constant on I, and let the distribution given by F be symmetrical about x_0, where x_0 denotes a central point of φ. Then, if $\mathscr{E}\varphi^{2k}(\underset{\sim}{x})$ exists,

$$\frac{\mu_{2k}(\underset{\sim}{x})}{\sigma^{2k}(\underset{\sim}{x})} \leqslant \frac{\mu_{2k}(\varphi(\underset{\sim}{x}))}{\sigma^{2k}(\varphi(\underset{\sim}{x}))} \quad \text{,for } k = 2,3, \ldots .$$

4. Two weak-order relations

In the remaining part of this paper we shall confine our attention to distribution functions $F \varepsilon \mathscr{F}$; part of the results, however, remain valid without this restriction.

Returning to the theorems of section 3 we remark that they obviously continue to hold if one replaces $\varphi(\underset{\sim}{x})$ by any other random variable with the same distribution, i.e. they hold for any $\underset{\sim}{x}^*$ with distribution function F^* satisfying

$$F^*\varphi(x) = P(\underset{\sim}{x}^* \leqslant \varphi(x)) = P(\varphi(\underset{\sim}{x}) \leqslant \varphi(x)) = P(\underset{\sim}{x} \leqslant x) = F(x),$$

or

$$\varphi(x) = G^*F(x) \quad \text{on } I.$$

We therefore define the following order relations on $\mathscr{F}$ and $\mathscr{S}$ respectively:

Definition 4.1

If $F, F^* \varepsilon \mathscr{F}$, then $F \underset{c}{<} F^*$ (or equivalently $F^* \underset{c}{>} F$) if and only if G^*F is convex on I.

Definition 4.2

If $F, F^* \varepsilon \mathscr{S}$, then $F \underset{s}{<} F^*$ (or equivalently $F^* \underset{s}{>} F$) if and only if G^*F is convex for $x > x_0$, $x \varepsilon I$, where x_0 denotes the point of symmetry of F.

We shall say in this case that F *c*-precedes or *s*-precedes F^*, or that F^* *c*-follows or *s*-follows F, and that the two are *c*-comparable or *s*-comparable. We shall also speak of *c*-ordering, *s*-ordering, *c*-comparison, *s*-comparison, etc., where the letters *c* and *s* stand for convex and symmetrical. According to the above the meaning of these definitions is clear: $F \underset{c}{<} F^*$ if and only if a random variable with distribution F may be transformed into one with distribution F^* by an increasing and convex transformation; for symmetrical distributions, $F \underset{s}{<} F^*$ if and only if this can be done by an increasing, antisymmetrical, concave-convex transformation. From the theorems of the preceding section the implications are also obvious: we have every right to say that $F \underset{c}{<} F^*$ implies that F^* has greater skewness to the right than F, whereas for symmetrical distributions $F \underset{s}{<} F^*$ implies that F^* has greater kurtosis than F.