

Dorian Goldfeld · Jay Jorgenson  
Peter Jones · Dinakar Ramakrishnan  
Kenneth A. Ribet · John Tate *Editors*

# Number Theory, Analysis and Geometry

In Memory of Serge Lang

 Springer

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*Editors*

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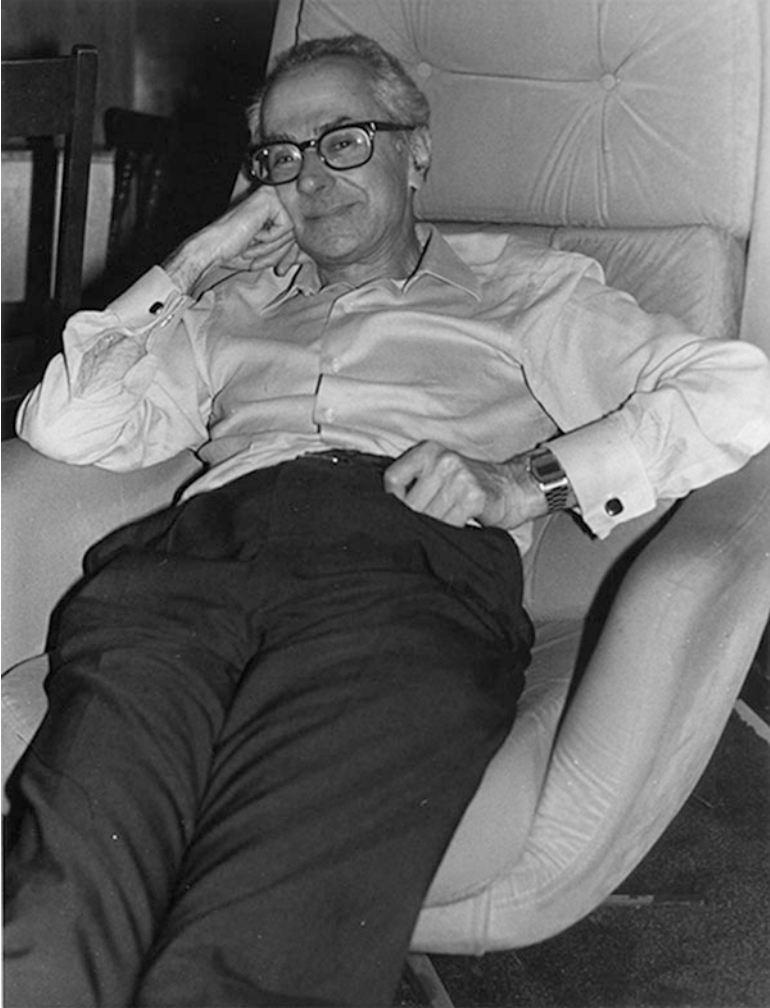
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Serge Lang (Photo provided courtesy of Kenneth A. Ribet.)



# Preface

Serge Lang was an iconic figure in mathematics, both for his own important work and, perhaps even more crucially, for the indelible impact he left on the field, and on his students and colleagues. It would be difficult to find a mathematician who came of age in the past forty years, who had not been exposed to Serge's articles, monographs, and textbooks. Serge's writing shaped the mathematical perspectives of all who came in contact with them. Many were challenged by the glimpses of open problems and conjectures that Serge interweaved with his expositions of established subjects. Serge's exposition invariably transcended our discipline's preference for brevity and perfection, which often obscures the intuition underlying the subject. Serge was never one to conform.

One of Serge's uplifting qualities was his openness to new areas of mathematics and his concurrent willingness, even eagerness, to learn novel concepts and techniques. He was constantly reinventing himself, while sharing his accumulated wisdom with students and young mathematicians. Over the course of his career, he traversed a tremendous amount of mathematical ground. As he moved from subject to subject, he found analogies that led to important questions in such areas as number theory, arithmetic geometry, and the theory of negatively curved spaces. Lang's conjectures will keep many mathematicians occupied far into the future.

This memorial volume contains articles in a variety of areas of mathematics, attempting to represent Serge's breadth of interest and impact. We are happy to publish here (for the first time) Serge's final paper, *The heat kernel, theta inversion, and zetas on  $\Gamma \backslash G/K$* , written jointly with one of us (J. Jorgenson). Except for that one article, which was left in the form it assumed just before Serge's passing, every other entry here was thoroughly refereed. We thank all the authors for their contributions to the volume and for their willingness to put up and comply with our demands for revision. Thanks also to the anonymous referees for their excellent and timely work.

We, the editors, are pleased to be a part of this production, especially since we were all fortunate enough to know Serge personally. We thank Stacey Croomes, the math administrator at Caltech, for her invaluable help in organizing the receipt of

the articles, the refereeing process, and the revisions. We are grateful to Ann Kostant and Elizabeth Loew of Springer for their enthusiasm and helpful advice during the many months of editorial preparation. It took a village to produce this volume.

Columbia University  
Yale University  
The City College of New York  
Caltech  
University of California, Berkeley  
Harvard University

Dorian Goldfeld  
Peter Jones  
Jay Jorgenson  
Dinakar Ramakrishnan  
Kenneth A. Ribet  
John Tate

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# Publications of Serge Lang: from 2000 and beyond

The five volumes of Serge Lang's *Collected Papers* from 1952 to 1999 were published by Springer-Verlag and noted below. His additional publications from 2000 can be found on the website of the American Mathematical Society's MathSciNet and are listed here.

Titles of books and journal articles are in italics.

- [2000a] *Collected Papers*. Vol. I. 1952–1970. Springer-Verlag, New York, 2000. xxiv+525 pp.
- [2000b] *Collected Papers*. Vol. II. 1971–1977. Springer-Verlag, New York, 2000. xvi+590 pp.
- [2000c] *Collected Papers*. Vol. III. 1978–1990. Springer-Verlag, New York, 2000. xvi+393 pp.
- [2000d] *Collected Papers*. Vol. IV. 1990–1996. Springer-Verlag, New York, 2000. xvi+471 pp.
- [2001a] *Collected Papers*. Vol. V. 1993–1999. With Jay Jorgenson. Springer-Verlag, New York, 2001. xvi+426 pp.
- [2001b] (with Jay Jorgenson). Guinand's theorem and functional equations for the Cramér functions. *J. Number Theory* **86** (2001), no. 2, 351–367.
- [2001b] (with Jay Jorgenson). *Spherical inversion on  $SL_n(\mathbf{R})$* . Springer Monographs in Mathematics. Springer-Verlag, New York, 2001. xx+426 pp.
- [2001c] (with Jay Jorgenson). The ubiquitous heat kernel. *Mathematics Unlimited—2001 and Beyond*. Springer, Berlin, 2001, pp. 655–683.
- [2001d] Comments on non-references in Weil's works. *Gaz. Math.* No. 90 (2001), 46–52. 01A80.
- [2002a] Short Calculus. The original edition of *A First Course in Calculus*, [Addison-Wesley, Reading, MA, 1964.] Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2002. xii+260 pp.
- [2002b] *Algebra*. Revised third edition. Graduate Texts in Mathematics, Vol. 211. Springer-Verlag, New York, 2002. xvi+914 pp.

- [2002c] Comments on non-references in Weil's works. *Mitt. Dtsch. Math.-Ver.* 2002, no. 1, 49–56.
- [2002d] *Introduction to Differentiable Manifolds*. Second Edition. Universitext. Springer-Verlag, New York, 2002. xii+250 pp.
- [2002e] Comments on non-references in Weil's work. (Italian) Translated from *Mitt. Dtsch. Math.-Ver.* 2002, no. 1, 49–56 [MR1900622 (2003c:01058)]. *Lett. Mat. Pristem* No. **44** (2002), 29–37.
- [2002f] Comments on Chow's work [see 1410978]. Reprinted from *Notices Amer. Math. Soc.* 43 (1996), no. 10, 1117–1124. *Nankai Tracts Math.*, 5, *Contemporary Trends in Algebraic Geometry and Algebraic Topology* (Tianjin, 2000), World Sci. Publ., River Edge, NJ, 2002, pp 243–250.
- [2003a] (with Jay Jorgenson). Analytic continuation and identities involving heat, Poisson, wave and Bessel kernels. *Math. Nachr.* **258** (2003), 44–70.
- [2003b] (with Jay Jorgenson). *Spherical inversion on  $SL_2(\mathbf{C})$ . Heat kernels and analysis on manifolds, graphs, and metric spaces* (Paris, 2002), 241–270, *Contemp. Math.*, 338, Amer. Math. Soc., Providence, RI, 2003.
- [2005a] (with Jay Jorgenson). A Gaussian space of test functions. *Math. Nachr.* **278** (2005), no. 7–8, 824–832.
- [2005b] On the AMS Notices publication of Krieger's translation of Weil's 1940 letter [MR2125268]. *Notices Amer. Math. Soc.* 52 (2005), no. 6, 612–622.
- [2005c] (with Jay Jorgenson).  $Pos_n(\mathbf{R})$  and Eisenstein series. *Lecture Notes in Mathematics*, 1868. Springer-Verlag, Berlin, 2005. viii+168 pp.
- [2008] (with Jay Jorgenson). *The heat kernel and theta inversion on  $SL_2(\mathbf{C})$* . Springer Monographs in Mathematics. Springer, New York, 2008. x+319 pp.
- [2009] (with Jay Jorgenson). *Heat Eisenstein series on  $SL_n(\mathbf{C})$* . *Mem. Amer. Math. Soc.* 201 (2009), no. 946, viii+127 pp.
- [2010] (with Jay Jorgenson). *Heat kernel, theta inversions, and zetas on  $\Gamma \backslash G/K$* . In this volume.

Additionally, the following articles about Lang appeared after September 2005.

- [2006a] Marc Hindry: La géométrie diophantienne, selon Serge Lang. (French) [Diophantine geometry according to Serge Lang] *Gaz. Math.* No. **1108** (2006), 17–32.
- [2006b] David E. Rohrlich: Serge Lang. *Gaz. Math.* No. **108** (2006), 33–34.
- [2006c] Michel Waldschmidt: Les contributions de Serge Lang à la thorie des nombres transcendants. (French) [Serge Lang's contributions to the theory of transcendental numbers] *Gaz. Math.* No. **108** (2006), 35–46.
- [2006d] Jay Jorgenson and Steven G. Krantz: Serge Lang, 1927–2005. *Notices Amer. Math. Soc.* 53 (2006), no. 5, 536–553.
- [2006e] Obituary: Serge Lang (1927–2005). (Spanish) *Lect. Mat.* **27** (2006), no. 2, 166–167.
- [2007] Jay Jorgenson and Steven G. Krantz: The mathematical contributions of Serge Lang. *Notices Amer. Math. Soc.* 54 (2007), no. 4, 476–497.

# Introduction

**John Tate**

This introduction is meant as a brief account of Serge Lang's life and his enormously varied contributions to mathematics. Much more about this remarkable man can be found in two articles in the *Notices of the AMS* by Jay Jorgenson and Steven G. Krantz. The first of these, "Serge Lang, 1927–2005" (May 2006) contains a fuller account of Lang's life than we can give here, and includes memories of Serge by twenty-two of his friends, students, colleagues, and even some whom Serge might have seen as adversaries. Read together, these short pieces give a vivid image of Lang. The second article, "Mathematical Contributions of Serge Lang" (April 2007), contains an overview of his research, followed by discussions of its different aspects by seven colleagues in the various fields. These articles, and conversations with Lang's friends Dick Gross, Dinakar Ramakrishnan, and Ken Ribet, have been of great help to me in writing this introduction.

Lang spent his childhood in Saint-Germain-en-Laye, a western suburb of Paris famous for its chateau and long terrace with a view over the valley of the Seine and Paris in the distance. Lang's teen years were spent in quite different surroundings. After emigrating with his family to Los Angeles, he attended Beverly Hills High and Caltech, graduating in 1946 with a BA in physics.

He then did a year and a half of military service with the U.S. Army in Europe. This was of great help to him in his future career, for he served in a clerical position in which he learned to type at incredible speed.

Next, Serge enrolled in the graduate philosophy program at Princeton. Fortunately for mathematics he was disappointed by the quality of the philosophy seminars, and managed to switch to the math graduate program the following year. I don't know why he chose mathematics, but the Princeton math program was an outstanding one and student morale was very high.

---

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Though he was not particularly well prepared, Serge plunged right in. For example, knowing little number theory, he attended Emil Artin's seminar on class field theory during his first year and was fascinated. Math, especially algebra and number theory, were the subjects for him! He soon became one of Artin's Ph.D. students along with me and a few others that Artin had brought with him from Indiana. We all felt very fortunate to have Artin as our advisor. In the foreword to his collected works, Lang writes, "I take this opportunity to express once more my appreciation for having been Artin's student. I could not have had a better start in my mathematical life."

Lang got his Ph.D in 1951 with a thesis on quasi-algebraic closure in which he proved that a field complete in a discrete valuation with algebraically closed residue field is quasi-algebraically closed, and that the same is true for several kinds of dense subfields of such a field. Another result in his thesis is a key ingredient in the proof of the Ax–Kochen theorem,<sup>1</sup> which can be viewed as a corrected version of Artin's conjecture that  $p$ -adic fields have property  $C_2$ . For each degree  $d$ , this is true for all but a finite set of primes  $p$ , but not necessarily for all, as Artin had guessed.

Lang stayed in Princeton for two more years, with postdoc positions at the university and at the Institute for Advanced Study. Then, after two years of an instructorship at the University of Chicago, where he interacted with Weil and his circle, he accepted a permanent position at Columbia University. He stayed there for the next fifteen years except for a Fulbright Fellowship year in Paris during 1957–58. In 1970, Lang resigned his position at Columbia in protest of the university's handling of student unrest during 1968–69. After visiting professorships at Princeton and Harvard, he accepted a permanent position at Yale. He retired from Yale in 2005, a few months before his death. That is a bare-bones account of Lang's life and the way he got into mathematics.

Serge Lang contributed to mathematics in so many ways that it's hard to know where to begin. Let's start with some remarks on his research, with no attempt to cover it completely. He published his *Collected Papers* in five volumes with Springer in 2000. They contain all of his research papers through 1999, together with reprints of a few of his Springer Lecture Notes and two of his books that were out of print. There are also some interesting accounts of some special topics on which Lang held strong views, especially in Volume IV.

In a brief foreword in Volume 1, Lang gives his own classification of his work into periods as follows:

1. 1951–1954                      Thesis on quasi-algebraic closure and related matters.
2. 1954–1962                      Algebraic geometry and abelian (or group) varieties; geometric class field theory.<sup>2</sup>

---

<sup>1</sup>As Deligne pointed out, I misstated the Ax–Kochen theorem in my *Notices* article on Lang's early work, by interchanging "prime  $p$ " and "degree  $d$ ." I hope this "senior moment" misled no one.

<sup>2</sup>This work was the beginning of higher-dimensional class field theory and earned Lang a Cole Prize in 1959.

3. 1963–1975 Transcendental numbers and Diophantine approximation on algebraic groups.
4. 1970 First paper on analytic number theory—jump to Jorgenson–Lang.
5. 1975  $SL_2(\mathbf{R})$ —jump to Jorgenson–Lang.
6. 1972–1977 Lang–Trotter Frobenius distributions.
7. 1973–1981 Modular curves, Kubert–Lang modular units.
8. 1974, 1982–1991 Diophantine geometry, complex hyperbolic spaces, and Nevanlinna theory.
9. 1985, 1988 Riemann–Roch and Arakelov theory.
10. 1992–2000+ Jorgenson–Lang (analytic number theory and connections with spectral analysis, heat kernel, differential geometry, Lie groups, and symmetric spaces).

The arrangement of the 2007 *Notices* article fits quite well with Lang’s scheme. Here is a list of the authors in the order of their appearance, followed by the periods of Lang’s work they discuss. Tate 1,2; Buium 1,2,3; Waldschmidt 3; Rohrlich 6,7; Vojta 8,9; Jorgenson 10; and Kim, who wrote on the theme of the fundamental group in Lang’s work, rather than on a specific period.

Lang wrote over 70 research papers and proved many important theorems, but of at least equal significance were his conjectures, his points of view, and his way of looking at things. Waldschmidt expresses this well (*loc. cit.*), writing about Lang’s work on transcendental numbers:

With his outstanding insight and his remarkable pedagogical gifts, Lang comes into the picture and contributes to the subject in at least two very different ways: on the one hand, he simplifies the arguments (sometimes excessively) and produces the first very clear proofs which can be taught easily; on the other hand, he introduces new tools, like group varieties, which put the topic closer to the interests of many a mathematician.

Waldschmidt concludes his article as follows:

Among the contributions of Lang to transcendental number theory (also to Diophantine geometry), the least are not his many conjectures which shed a new light on the subject. On the contrary, he had a way of considering what the situation should be, which was impressive. Indeed, he succeeded in getting rid of the limits from the existing results and methods. He made very few errors in his predictions, especially if we compare them with the large number of conjectures he proposed. His description of the subject will be a guideline for a very long time.

As Vojta points out, the title of Lang’s magnum opus, *Fundamentals of Diophantine Geometry* suggests that Serge’s outlook on number theory was decidedly geometric. Mazur puts this beautifully in concluding his memory of Serge article in the *Notices*:

Over the decades of mathematics, Lang was led, more specifically, by an over-arching vision, which he pursued through the agency of various fields of mathematics. The vision, baldly put, is that *geometry* is an extraordinarily striking dictator of qualitative *dipohantine* behavior. The still open *Conjecture of Lang* in higher dimensions continues to serve as a

guiding principle to the way in which the grand subjects of geometry and number theory meet, just as Serge himself served as an inspirer of generations of mathematicians, and a spokesman for intellectual honesty.

The conjecture of Lang to which Mazur refers is easy to state. A projective algebraic variety  $V$  defined over a number field  $F \subset \mathbf{C}$  is *Mordellic* if and only if the corresponding complex space  $V(\mathbf{C})$  is *hyperbolic*. Here Mordellic means that for each finite extension field  $E$  of  $F$ , the set  $V(E)$  of points of  $V$  with coordinates in  $E$  is finite. Hyperbolic meant for Lang, when he made the conjecture in 1974, that the Kobayashi semidistance on  $V(\mathbf{C})$  is actually a distance, but we now know, thanks to Brody (1978), that this property is equivalent to there being no nonconstant holomorphic map  $\mathbf{C} \rightarrow V(\mathbf{C})$ . A Riemann surface is hyperbolic if and only if its genus is  $\geq 2$ , so that for curves, Lang's conjecture is equivalent to the famous Mordell conjecture proved by Faltings in 1983. In higher dimensions the conjecture is still open, though it has been proved for closed subvarieties of abelian varieties.

In the 1980s Lang thought deeply about the Mordellic–hyperbolic relationship and introduced plausible variants of the above conjecture which have turned out to have very interesting unexpected implications, such as the existence of a bound  $B(g, F)$  depending only on  $g$  and  $F$  for the number of rational points on a curve of genus  $g \geq 2$  defined over a number field  $F$ .

Serge led a regular life. During the winter holidays he visited his sister in Los Angeles. He spent the early summer in Europe and July–August in Berkeley, where he had an apartment. In Europe he spent a month in one place, Paris in the early years, Bonn later in his life, but also visited regularly other mathematical centers, Zurich, Berlin, Moscow.... In Berkeley he interacted with the large community of resident and visiting mathematicians.

Lang was an effective communicator, an excellent source of mathematical news. Dick Gross likens his gathering and distributing information to the cross-pollination of a bumblebee. Serge kept in touch with friends in many places, not only in person, by traveling, but by phone. If he had a question or thought for anyone anywhere in the world, he just picked up the phone. When Yale was considering an offer to Serge, I remember warning the department that if he accepted, its phone bill would at least double, but that in fact, the phoning he would be doing was just one more reason for making the offer.

A more important reason for Yale's doing so is that Lang was an excellent and caring teacher. This was recognized by his being awarded the Dylon Hixon Prize for Teaching Excellence in the Natural Sciences at Yale. We were reminded of the esteem in which his former students held him by the testimony of so many of them at the memorial meeting for Lang at Yale in February 2006. One of them, Anthony Petrello, announced the establishment of a Yale fund for an annual prize in Lang's honor which he was launching with a large seed contribution, and the promise of matching funds.

Another memorial to Lang is the Serge Lang Undergraduate Lecture Series at Berkeley. There, when students returned to classes at the end of August, Serge often gave talks to the Math Undergraduate Student Association (MUSA). His

talks over the years were incorporated into his 1999 volume “Math talks for Undergraduates.” These talks became formalized, and each year from 1999 to 2005 Serge gave a MUSA lecture at 4pm on the first day of classes. The lecture on August 25, 2005, “Weierstrass–Dirac Families,” shortly before his death, was one of Serge Lang’s last mathematics talks. In response to MUSA’s request to somehow continue this tradition, the Department of Mathematics inaugurated the Serge Lang Undergraduate Lecture Series, each fall inviting someone to give a lecture for undergraduates. Ken Ribet made this happen and Anthony Petrello contributed a major part of the initial funding.

Serge was known not only for his support of his students, but also of his younger colleagues at the start of their careers. John Coates thought one of Serge’s most remarkable qualities was his unstinting support of young mathematicians. Barry Mazur, after recounting his first encounter with Serge, writes:

And Serge did this sort of thing through the decades, with many of the young; he would proffer to them gracious, yet demanding, invitations to engage as a genuine colleague—not teacher to student, but mathematician to mathematician; he did all this naturally, and with extraordinary generosity and success.

Lang was awarded the 1999 AMS Leroy P. Steele Prize for Mathematical Exposition “for his many mathematics books.” The amount of mathematical knowledge that has been made accessible to students of all ages all over the world by Lang’s more than 40 books is amazing to contemplate. Their range both in subject and in level is astonishingly broad. Most were new and modern for their time with Lang’s insistence on functoriality and axiomatization. He was almost unique in the way he regularly learned new topics throughout his life, topics often not close to his main interests (algebra and number theory), and wrote textbooks on them (such as “Differentiable Manifolds” and  $SL_2(\mathbf{R})$ ), thereby influencing new generations of students pursuing those fields. He brought excitement to his books that challenged readers to rise above themselves by tackling them.

Lang’s *Algebra* is a classic, still the best reference book in algebra in print today. The first edition in 1965 has been kept up-to-date with new editions and revisions. The latest is the corrected fourth printing of the revised third edition (2004). For Lang, the important thing in a book is its timeliness, and its global aspects, such as arrangement of topics, and degree of abstraction. He did not worry much about an occasional error in a proof, and was widely criticized for this. Given the short time he spent writing a book, there are relatively few of these oversights, and when he became aware of one he was highly attentive to its correction in the next edition or the next printing. These oversights could be useful. I remember a few times first recognizing a student who turned out to be very strong when s/he came for help in understanding one of Lang’s faulty arguments.

Lang was driven to publish. In addition to his own writing, he saw to the publication of at least two books which were not his own, namely *Class Field Theory* by Artin and me, and the *Collected Papers of Emil Artin*. He should really have been included among the authors of the former, for the main part of it is essentially Serge’s rewriting of his own notes from the 1951–52 Princeton seminar, and the

book's publication would never have happened without Serge's persistent prodding and attending to the details. In the case of Artin's papers, edited by Serge Lang and John T. Tate, it was Lang who did all the work in collecting the material and preparing it for publication. He assigned me the trivial task of writing a paragraph on Artin's conjectures to justify my sharing the editorship. In the first printing, Addison-Wesley made the mistake of including Lang's and my names on the cover, below Artin's. Lang hit the ceiling, insisting that the whole printing be redone. Fortunately a solution was found. Our names are masked by a decorative strip which was added to the cover, although they are discernible if you know they are there.

To touch on some of Serge's personal characteristics: He did things fast. He typed fast, ate fast, drove fast, walked fast. I still remember trying to keep up with him on walks across campus. On a wintry day he used only earmuffs and gloves (no coat) to keep warm. He seemed to have a hummingbird-like metabolism. Besides being fast, he did not waste time. He hated small talk.

Serge cared about quality. His possessions and gifts were chosen thoughtfully. He had impeccable taste and sought out fine things. I think of his leather jacket, his gloves, his silk scarf and clothing generally, but also of his appliances, his collection of rugs, and his furniture. In the summer of 1958, after a year in Paris, we each went through Copenhagen to purchase Danish furniture. Serge chose from the best collections, I from the mid-range.

Except for an occasional sip of a vintage wine, Lang did not drink, but he insisted on supplying a fine cognac for Yale's (and often Harvard's as well) winter holiday party eggnog. The batter for the crepes suzettes he served was beaten by hand with a whisk, never with a mechanical mixer.

It is hard to imagine that Serge's many mathematical activities left time for much else, but he was an accomplished musician, playing piano and lute, and in later years he spent much time and energy exposing cases of what he saw as scientific, editorial or bureaucratic irresponsibility, by compiling and distributing, at considerable personal expense for photocopying and postage, collections<sup>3</sup> of relevant original documents, in his aim to create transparency and promote clear understanding of the facts of a case. He also supported many people and causes he found to be worthy, in ways ranging from enthusiastic encouragement to assistance in funding.

In Serge's battles with the establishment, his positions were almost always fundamentally sound, though the extremes to which he went and the vehemence with which he pursued his points of view may have prevented some from realizing this. Serge could be difficult. To him things were black or white. To compromise was not his way.

Serge Lang devoted his life to advancing mathematics, to teaching, and to fighting for honesty in science and politics.

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<sup>3</sup>These collections were known as "files" to the people on Lang's extensive "cc lists."

# Raynaud's group-scheme and reduction of coverings

Dan Abramovich with an Appendix by Jonathan Lubin

*In grateful memory of Serge Lang*

**Abstract** Let  $Y_K \rightarrow X_K$  be a Galois covering of smooth curves over a field of characteristic 0, with Galois group  $G$ . We assume  $K$  is the fraction field of a discrete valuation ring  $R$  with residue characteristic  $p$ . Assuming  $p^2 \nmid |G|$  and the  $p$ -Sylow subgroup of  $G$  is normal, we consider the possible reductions of the covering modulo  $p$ . In our main theorem we show the existence, after base change, of a twisted curve  $\mathcal{X} \rightarrow \text{Spec}(R)$ , a group scheme  $\mathcal{G} \rightarrow \mathcal{X}$  and a covering  $Y \rightarrow \mathcal{X}$  extending  $Y_K \rightarrow X_K$ , with  $Y$  a stable curve, such that  $Y$  is a  $\mathcal{G}$ -torsor.

In case  $p^2 \mid |G|$  counterexamples to the analogous statement are given; in the appendix a strong counterexample is given, where a non-free effective action of  $\alpha_p^2$  on a smooth 1-dimensional formal group is shown to lift to characteristic 0.

**Key words** Algebraic curves • Galois coverings

**Mathematics Subject Classification (2010):** 14H25, 14H30

## 1 Introduction

### 1.1 Reduction of coverings of degree divisible by $p$

Let  $R$  be a discrete valuation ring of mixed characteristics, with spectrum  $S = \text{Spec } R$ . Denote the generic point  $\eta$  with fraction field  $K$ , and the special

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point  $s$  with residue field  $k$  of characteristic  $p > 0$ . Consider a generically smooth, stable pointed curve  $Y \rightarrow S$  with an action of a finite group  $G$  of order divisible by  $p$ . Denote  $X = Y/G$ . We assume that  $G$  acts freely on the complement of the marked points in  $Y_\eta$ ; it then follows that  $G$  respects the branches of all nodes of  $Y_s$ .

In situations where the order of  $G$  is prime to the residue characteristic, the reduced covering  $Y_s \rightarrow X_s$  is an admissible  $G$ -covering, and a nice complete moduli space of admissible  $G$ -coverings exists. An extensive literature exists describing that situation, see e.g., [H-M, Mo, Ek, W,  [\$\mathfrak{N}\$ -C-V](#)]. However, in our case where the residue characteristic divides the order of  $G$ , interesting phenomena occur (see e.g., [ [\$\mathfrak{N}\$ -Oo](#)]). The situation was studied by a number of people; we will concern ourselves with results of Raynaud [Ra] and, in a less direct way, Henrio [He] and more recently Maugeais [Ma]. Related work of Saidi [Sa1, Sa2, Sa3], Wewers and Bouw [W1, W2, W3, Bo, B-W1, B-W2], Romagny [Ro] and others provides additional inspiration. In [ [\$\mathfrak{N}\$ -O-V2](#), Section 5] the curve  $Y$  is replaced by something which could be much more singular, and therefore the results are somewhat orthogonal to the situation here.

Thus, in our case where  $p \mid |G|$ , the covering  $Y \rightarrow X$  is no longer generically étale on each fiber. It is natural to consider some sort of group-scheme degeneration  $\mathcal{G} \rightarrow X$  of  $G$ , in such a way that  $Y$  might be considered something like an admissible  $\mathcal{G}$ -covering. In our main theorem we show this is the case under appropriate assumptions:

**Theorem 3.2.2.** Assume  $p^2 \nmid |G|$  and the  $p$ -Sylow subgroup of  $G$  is normal.

There exist

- (1) a twisted curve  $\mathcal{X} \rightarrow X$ ,
- (2) a finite flat group scheme  $\mathcal{G} \rightarrow \mathcal{X}$ ,
- (3) a homomorphism  $G_{\mathcal{X}} \rightarrow \mathcal{G}$  which is an isomorphism on  $\mathcal{X}_k$ ,
- (4) a lifting  $Y \rightarrow \mathcal{X}$  of  $Y \rightarrow X$ , and
- (5) an action of  $\mathcal{G}$  on  $Y$  through which the action of  $G$  factors,

such that  $Y \rightarrow \mathcal{X}$  is a principal  $\mathcal{G}$ -bundle.

The formation of  $\mathcal{G}$  commutes with any flat and quasi-finite base change  $R \subset R'$ .

It is important to note that, unlike the characteristic 0 case,  $X$  is not a stable pointed curve in general.

## 1.2 Background

Raynaud ([Ra], Proposition 1.2.1) considered such a degeneration locally at the generic points of the irreducible components of  $X_s$ , in the special case where  $|G| = p$ ; our first goal, see Theorem 3.1.1 below, is to work out its extension to the smooth locus of  $X$ , and slightly more general groups, where as above  $p^2 \nmid |G|$  and the  $p$ -Sylow subgroup of  $G$  is normal. The case where  $p^2 \mid |G|$  remains a question which I find very interesting. See Example 2.1.7 and the appendix by J. Lubin for a

negative result in general, the discussion of question 2.1.5 for positive results in the literature, and Remark 2.1.8 for a positive result for small ramification.

One still needs to understand the structure of  $Y \rightarrow X$  at the nodes of  $X_s$  and  $Y_s$ . Henrio, working  $p$ -adic analytically, derived algebraic data along  $X_s$ , involving numerical combinatorial invariants and differential forms, which in some sense classify  $Y_s \rightarrow X_s$ . Our second goal in this note is to present a different approach to such degenerations at a node, modeled on *twisted curves*, i.e. curves with algebraic stack structures. Borrowing a metaphor from A. Ogus, these twisted curves have served well in the past as a sort of “magic powder” one sprinkles over the “bad locus” of certain structures, which brings about a hidden good property. The point here is that, just as in [N-C-V], the introduction of twisted curves allows one to replace  $Y \rightarrow X$  by something that is actually a principal bundle. Unlike the case of residue characteristics prime to  $|G|$ , the twisted curves will in general be Artin stacks rather than Deligne–Mumford stacks. See [OI], [N-O-V2].

### 1.3 Towards a proper moduli space

The main theorem should be thought of as a first step in constructing a nice proper moduli space of degenerate coverings in mixed characteristics - it gives a special case of the valuative criterion for properness. In joint work with M. Romagny we plan to complete this task. Foundations have only recently been developed in [N-O-V, N-O-V2].

### 1.4 Brief introduction to twisted curves

A twisted pointed curve over a base scheme  $S$  is a diagram as follows:

$$\begin{array}{ccc}
 \Sigma_i^C & \hookrightarrow & \mathcal{C} \\
 \downarrow & & \downarrow \\
 \Sigma_i^C & \hookrightarrow & \mathcal{C} \\
 & & \downarrow \\
 & & S.
 \end{array}$$

Here we follow [N-O-V2, Section 2]:

- $\mathcal{C} \rightarrow S$  is a usual  $n$ -pointed nodal curve, with sections  $\Sigma_i^C, i = 1, \dots, n$ ;
- $\mathcal{C}$  is an algebraic stack with finite diagonal having  $\mathcal{C}$  as its coarse moduli space;

- $\Sigma_i^C \subset C$  are its markings, each of which a gerbe banded by some  $\mu_{r_i}$  over  $\Sigma_i^C \simeq S$ ;
- $C \rightarrow C$  is an isomorphism away from nodes and markings of  $C$ ;
- at a marking of  $C$ , where the strict henselization  $C^{\text{sh}}$  is described by  $(\text{Spec}_S \mathcal{O}_S[x])^{\text{sh}}$  and  $\Sigma_i^C$  is the vanishing locus of  $x$ , the twisted curve  $\mathcal{C}$  is described as

$$\left[ (\text{Spec}_S \mathcal{O}_S[u])^{\text{sh}} / \mu_{r_i} \right],$$

where  $\mu_{r_i}$  acts on  $u$  via the standard character and  $u^{r_i} = x$ , and  $\Sigma_i^C$  is the quotient of the vanishing locus of  $u$ ;

- at a node of  $C$ , where the strict henselization  $C^{\text{sh}}$  is described by  $(\text{Spec}_S \mathcal{O}_S[x, y]/(xy - f))^{\text{sh}}$  with  $f \in (\mathcal{O}_S)^{\text{sh}}$ , the twisted curve  $\mathcal{C}$  is described as

$$\left[ (\text{Spec}_S \mathcal{O}_S[u, v] / (uv - g))^{\text{sh}} / \mu_r \right]$$

for some  $r$ , where  $\mu_r$  acts via

$$(u, v) \mapsto (\zeta_r u, \zeta_r^{-1} v),$$

and  $u^r = x, v^r = y$  and  $g^r = f$ .

Of course the description on the level of strict henselization descends to some étale neighborhoods. In case  $p$  divides  $r_i$  or  $r$ , the twisted curve  $\mathcal{C}$  is not a Deligne–Mumford stack, and it is a little bit of a miracle, following from [N-O-V2, Proposition 2.3], that one can use such a nice description locally in the étale topology (or on strict henselizations) rather than the f.p.p.f. topology. The reader is warned that transition isomorphisms between the étale local charts are in general not given in étale neighborhoods but rather in smooth or f.p.p.f. charts.

Near a marking  $\Sigma_i^C$ , the twisted curve is determined, uniquely up to a unique isomorphism, by the choice of  $r_i$ . In fact locally in the Zariski topology of  $C$  we can write  $\mathcal{C} = \sqrt[r_i]{(C, \Sigma_i^C)}$ , see [N-G-V] for notation and proof. Near a node,  $\mathcal{C}$  is still uniquely determined by  $r$ , but not up to a unique isomorphism - in fact  $\text{Aut}_C \mathcal{C}$  has a factor  $\mu_r$  for each such twisted node of index  $r$ , see [N-C-V].

## 1.5 Acknowledgements

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## 2 Extensions of group-schemes and their actions in dimension 1 and 2

### 2.1 Raynaud's group-scheme

Raynaud (see [Ra], Proposition 1.2.1, see also Romagny, [Ro]) considers the following construction: let  $U$  be integral and let  $V/U$  be a finite flat  $G$ -invariant morphism of schemes, with  $G$  finite. Assume that the action of  $G$  on the generic fiber of  $V/U$  is faithful. We can view this as an action of the constant group scheme  $G_U$  on  $V$ , and we consider the schematic image  $\mathcal{G}$  of the associated homomorphism of group-schemes

$$G_U \rightarrow \mathrm{Aut}_U V.$$

Since, by definition, the image  $G_U \rightarrow U$  is finite, we have that  $\mathcal{G} \rightarrow U$  is finite as well. The scheme  $\mathcal{G} \rightarrow U$  can also be recovered as the closure of the image of the generic fiber of  $G_U$ , which is, by the faithfulness assumption, a subscheme of  $\mathrm{Aut}_U V$ . By definition  $\mathcal{G}$  acts faithfully on  $V$ .

**Definition 2.1.1.** We call the scheme  $\mathcal{G}$  the effective model of  $G$  acting on  $V/U$ .

Note that a priori we do not know that  $\mathcal{G}$  is a group-scheme. It is however automatically a flat group-scheme if  $U$  is the spectrum of a Dedekind domain. This follows because, in that case, the image of  $\mathcal{G} \times_U \mathcal{G} \rightarrow \mathrm{Aut}_U V$  is also flat, and therefore must coincide with  $\mathcal{G}$ .

Also note that, if  $s$  is a closed point of  $U$  whose residue characteristic is prime to the order of  $G$ , then the fiber of  $\mathcal{G}$  over  $s$  is simply  $G$ . So this effective model is only of interest when the residue characteristic divides  $|G|$ .

The following is a result of Raynaud, see [Ra], Proposition 1.2.1. The statement here is slightly extended as Raynaud assumes  $|G| = p$ :

**Proposition 2.1.2.** *Let  $U$  be the spectrum of a discrete valuation ring, with special point  $s$  of residue characteristic  $p$  and generic point  $\eta$ . Let  $V \rightarrow U$  be a finite and flat morphism, and assume that the fiber  $V_s$  of  $V$  over  $s$  is reduced (but not assuming geometrically reduced). Assume given a finite group  $G$ , with normal  $p$ -Sylow subgroup, such that  $p^2 \nmid |G|$ , and an action of  $G$  on  $V$  such that  $V \rightarrow U$  is  $G$ -invariant, and such that the generic fiber  $V_\eta \rightarrow \{\eta\}$  is a principal homogeneous space. Let  $\mathcal{G} \rightarrow U$  be the effective model of  $G$  acting on  $V/U$ .*

*Then  $V/U$  is a principal bundle under the action of  $\mathcal{G} \rightarrow U$ .*

**Remark 2.1.3.** An analogous construction in a wider array of cases is given in Romagny’s [Ro1, Theorem A]. Romagny does not aim to construct a principal bundle; on the other hand he shows that an effective model for an action exists even if  $V/U$  and  $G/U$  are not finite, under very mild hypotheses.

*Proof.* As in Raynaud’s argument, it suffices to show that the stabilizer of the diagonal of  $V_s \times_U V_s$  inside the group-scheme  $V_s \times_U \mathcal{G}$  is trivial. Since  $G$  acts transitively on the closed points  $t_i$  of  $V_s$  sending the stabilizer on  $t_i$  to that over  $t_j$ , it is enough to show that one of these stabilizers, say over  $t \in V_s$ , is trivial. But this stabilizer  $P$  is a group-scheme over the residue field  $k(t)$  with degree  $\deg P \mid p$ , and if nontrivial it is of degree exactly  $p$ . In such a case it must coincide with the pullback of the unique  $p$ -Sylow group-subscheme of  $\mathcal{G}$ ; therefore that  $p$ -syllow acts trivially, contradicting the fact that  $\mathcal{G}$  acts effectively.  $\square$

**Remark 2.1.4.** In case the inertia group is not normal, Raynaud passes to an auxiliary cover, which encodes much of the behavior of  $V \rightarrow U$ .

**Question 2.1.5.** *What can one say about the action of  $\mathcal{G}$  on  $V$  in case the order of  $G$  (and the degree of  $V \rightarrow U$ ) is divisible by  $p^2$ , but the inertia group is still normal? Specifically, what happens if  $|G| = p^2$ ?*

In the latter case, consider a subgroup  $P \subset G$  of order  $p$ . It can be argued, as in Raynaud’s proof, that the effective model  $\mathcal{P} \rightarrow U$  of  $P$  acts freely on  $V$ , and thus  $V \rightarrow V/P$  is a principal  $\mathcal{P}$ -bundle. Similarly, if  $\mathcal{Q}$  is the effective model of  $G/P$  acting on  $V/P$ , then  $V/P \rightarrow U$  is a principal  $\mathcal{Q}$ -bundle. At the same time, we have an action of the effective model  $\mathcal{G}$  of  $G$  on  $V/P$ , but it is not necessarily the case that  $\mathcal{G}/\mathcal{P} \rightarrow \mathcal{Q}$  is an isomorphism.

While the statement of Question 2.1.5 is somewhat vague, two definite answers can already be given. First, if one concentrates on effective models of the action in the sense of Romagny, a great deal can be said. The recent work of Tossici [To1, To2] concentrates on the case where  $G_K = \mathbb{Z}/p^2\mathbb{Z}$  and  $\mathcal{O}_U$  contains a primitive root of unity of order  $p^2$ . The paper [To1] describes explicitly the possible models  $\mathcal{G}$  of  $G_K$ ; in [To2] an explicit description of the effective model of  $G_K$  acting on  $V$  is provided. I think it would be of interest to see if results as in Theorems 3.1.1 and 3.2.2 can be obtained for more general effective models such as these.

Second, in general no model of  $\mathcal{G}$  will act freely on  $V$ . This is the case even for some of the prettiest actions one can consider. This makes giving a complete answer to the previous Question 2.1.5 tricky, and underscores the importance of work such as Tossici’s.

As probably the simplest example, consider an action of  $\mathcal{G}_0 = (\alpha_p)^2 = \text{Spec } k[a, b]/(a^p, b^p)$  on  $k(t)$ . Examples of liftings of a non-free action of the type

$$t \mapsto t + a + f(t)b$$

for any residue characteristic have been written down by Romagny (personal communication) and Saidi (see [Sa4]). The case of

$$t \mapsto t + a + t^p b \quad (1)$$

is particularly appealing, as it involves torsion and endomorphisms of a formal group. I therefore ask

**Question 2.1.6.** *Can one lift the action (1) to characteristic 0?*

A formal positive answer in arbitrary residue characteristics is given by Jonathan Lubin in the appendix. Here I discuss explicitly the case of residue characteristic 2, where this action can be obtained as a reduction of an action of  $(\mathbb{Z}/2\mathbb{Z})^2$  on a smooth curve. I concentrate on the local picture (making it global is not difficult):

*Example 2.1.7.* Let  $R = \mathbb{Z}_2[\sqrt{2}]$ . Consider the group-scheme  $Y/R$  defined by

$$t * t' = t + t' + \sqrt{2} t t'$$

This is an additive reduction of the multiplicative group. The reduction of the subgroup  $\mu_2$  is given as

$$\text{Spec } R[a] / (a(a + \sqrt{2})),$$

reducing to  $\alpha_2$ . It acts on  $Y$  by translation via the addition law as above:

$$t \mapsto t + a + \sqrt{2} a t.$$

The reduction of the action of  $\mathbb{Z}/2\mathbb{Z}$  by inversion is the same group-scheme, again reducing to  $\alpha_2$ , which we write as

$$\text{Spec } R[b] / (b(b + \sqrt{2})).$$

This time the action is given by

$$t \mapsto (1 + \sqrt{2}b)t - \frac{bt^2}{1 + \sqrt{2}t}.$$

Since 2-torsion is fixed by inversion, these actions commute. Explicitly, the action of the product is given by

$$\begin{aligned} t \mapsto & a + (1 + \sqrt{2}b)t - \frac{bt^2}{1 + \sqrt{2}t} \\ & + \sqrt{2}a \left( (1 + \sqrt{2}b)t - \frac{bt^2}{1 + \sqrt{2}t} \right). \end{aligned}$$

The reduction modulo  $\sqrt{2}$  is given by

$$t \mapsto t + a + t^2 b,$$

as required.

**Remark 2.1.8.** Raynaud's arguments do work when  $p^2 \mid |G|$  if the  $p$ -Sylow group-scheme of  $\mathcal{G}$  has only étale and cyclotomic Jordan–Hölder factors. This is because, in that case, there are no nonconstant group subschemes in the reduction. In particular this works whenever the absolute ramification index over  $\mathbb{Z}_p$  is  $< p$ .

## 2.2 Extension from codimension 1 to codimension 2

Consider now the case where  $U$  a Gorenstein noetherian scheme,  $\dim U = 2$ , and  $V/U$  finite flat and  $G$ -invariant as above. Consider the  $S_2$ -saturation  $\mathcal{G}' \rightarrow \mathcal{G}$  of the effective model  $\mathcal{G}$  of the  $G$  action on  $V/U$ . In Section 6.1.2 of [Va] Vasconcelos considers such saturation ( $S_2$ -ification in his terminology). His Proposition 6.21 on page 318 applies in our situation, and gives the existence and a characterization of the  $S_2$ -saturation. We have

**Lemma 2.2.1.** *If  $\mathcal{G}' \rightarrow U$  is flat, then  $\mathcal{G}'$  is a group-scheme acting on  $V$ .*

*Proof.* We claim that the rational map  $\mathcal{G}' \times_U \mathcal{G}' \dashrightarrow \mathcal{G}'$  induced by multiplication in  $\text{Aut}_U V$  is everywhere defined. Indeed the graph of this map is finite over  $\mathcal{G}' \times_U \mathcal{G}'$  and isomorphic to it over the locus where  $\mathcal{G}' \rightarrow \mathcal{G}$  is an isomorphism, whose complement has codimension  $\geq 2$ . Now  $\mathcal{G}'$  is  $S_2$  and of dimension 2, hence Cohen–Macaulay. Pulling back to  $\mathcal{G}'$  the flat Cohen–Macaulay  $\mathcal{G}' \rightarrow U$  we get that  $\mathcal{G}' \times_U \mathcal{G}'$  is Cohen–Macaulay, in particular  $S_2$ . This implies that the graph of  $\mathcal{G}' \times_U \mathcal{G}' \dashrightarrow \mathcal{G}'$  is isomorphic to  $\mathcal{G}' \times_U \mathcal{G}'$ , and therefore the map is regular. The same works for the map defined by the inverse in  $\text{Aut}_U V$ . This makes  $\mathcal{G}'$  a group-scheme, and the map  $\mathcal{G}' \rightarrow \text{Aut}_U V$  into a group-homomorphism.  $\square$

This applies, in particular, when  $U$  is regular:

**Lemma 2.2.2.** *If  $U$  is regular, the  $S_2$ -saturation  $\mathcal{G}'$  of the effective model  $\mathcal{G}$  is a finite flat group-scheme acting on  $V$ .*

*Proof.* Again  $\mathcal{G}'$ , being 2-dimensional and  $S_2$ , is Cohen–Macaulay, and being finite over the nonsingular scheme  $U$ , it is finite and flat over  $U$  (indeed its structure sheaf, being saturated, is locally free over the nonsingular 2-dimensional scheme  $U$ ). The result follows from Lemma 2.2.1.  $\square$

When the action on the generic fiber is free, we have more:

**Proposition 2.2.3.** *Let  $U$  be a Cohen–Macaulay integral scheme with  $\dim U = 2$ . Let  $V \rightarrow U$  be a  $G$  invariant, finite, flat and Cohen–Macaulay morphism, and*

assume the action of  $G$  on the generic fiber is free. Let  $\mathcal{G} \rightarrow U$  be the effective model of the action. Assume that for every codimension  $-1$  point  $\xi$ , the action of the fiber  $\mathcal{G}_\xi$  on  $V_\xi$  is free.

Then

- (1)  $\mathcal{G} \rightarrow U$  is a flat group-scheme, and
- (2) The action of  $\mathcal{G}$  on  $V$  is free.

Note that, by Raynaud's result 2.1.2, the assumptions hold when  $U = V/G$  is local of mixed characteristics  $(0, p)$ , the fibers  $V_\xi$  are reduced, the  $p$ -Sylow of  $G$  is normal and  $p^2 \nmid |G|$ .

*Proof.* Consider the S2-saturation  $\mathcal{G}'$  of  $\mathcal{G}$ . Since  $V \rightarrow U$  is flat and Cohen–Macaulay, the same is true for  $V \times_U V \rightarrow V$  and for  $\mathcal{G}' \times_U V \rightarrow \mathcal{G}'$ . Since  $\mathcal{G}'$  and  $V$  are Cohen–Macaulay, we have that  $V \times_U V$  and  $\mathcal{G}' \times_U V$  are Cohen–Macaulay, hence S2, as well. The morphism  $\mathcal{G}' \times_U V \rightarrow V \times_U V$  induced by the action  $\mathcal{G}' \rightarrow \text{Aut}_U V$  is finite birational and restricts to an isomorphism in codimension 1. By the S2 property it is an isomorphism. In particular we have that  $\mathcal{G}' \times_U V \rightarrow V$  is flat, and since  $V \rightarrow U$  is faithfully flat we have that  $\mathcal{G}' \rightarrow U$  is flat. By Lemma 2.2.1 we have that  $\mathcal{G}' \rightarrow U$  is a finite flat group-scheme acting on  $V$ , and the isomorphism  $\mathcal{G}' \times_U V \rightarrow V \times_U V$  shows that the action is free, in particular  $\mathcal{G}' \rightarrow \mathcal{G}$  is an isomorphism.  $\square$

## 3 Curves

### 3.1 The smooth locus

The main case of interest for us is the following:

Let  $R$  be a complete discrete valuation ring of mixed characteristic, with fraction field  $K$  of characteristic 0, residue field  $k$  of characteristic  $p > 0$ , and spectrum  $S$ . Assume  $Y \rightarrow S$  is a stable pointed curve with smooth generic fiber,  $G$  a finite group acting on  $Y$  over  $S$ , and denote

$$X = Y/G.$$

We assume that the closure of the locus of fixed points of  $G$  in  $Y_K$  forms a disjoint union of marked sections of the smooth locus  $Y_{\text{sm}}$ . Hence for every node  $y \in Y$ , the stabilizer in  $G$  of  $y$  keeps the branches of  $Y$  at  $y$  invariant. We denote the complement of the closure in  $Y_{\text{sm}}$  of the fixed locus of the generic fiber by  $Y_{\text{gen}}$ , and the image in  $X$  by  $X_{\text{gen}}$  — the so called *general locus*.

Note that the morphism  $Y_{\text{sm}} \rightarrow X_{\text{sm}}$  is flat.

The propositions above give:

**Theorem 3.1.1.** *Assume  $p^2 \nmid |G|$  and the  $p$ -Sylow subgroup of  $G$  is normal.*