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Eitan Sayag
Editors

Representation Theory, Complex Analysis, and Integral Geometry

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Preface

This volume is an outgrowth of a special summer term on "Harmonic analysis, representation theory, and integral geometry", hosted by the Max Plank Institute for Mathematics (MPIM) and the then newly founded Hausdorff Research Institute for Mathematics (HIM) in Bonn in 2007. It was organized and led by S. Gindikin and B. Krötz with the help of O. Offen and E. Sayag. The purpose of this book is to make an essential part of the activity from the summer term available to a wider audience.

The book contains research contributions on the following themes: connecting periods of Eisenstein series on orthogonal groups and double Dirichlet series (Gautam Chinta and Omer Offen); vanishing at infinity of smooth functions on symmetric spaces (Bernhard Krötz and Henrik Schlichtkrull); a formula involving all the Rankin–Selberg convolutions of holomorphic and non-holomorphic cusp forms (Jay Jorgenson and Jürg Kramer); a scheme of a new proof for the so-called Helgason conjecture on a Riemannian symmetric space $X = G/K$ of the non-compact type (Simon Gindikin); an algorithm for the computation of special unipotent representations attached to certain regular K-orbits on a flag variety of the dual group (Dan Ciubotaru, Kyo Nishiyama, and Peter E. Trapa); applications of symplectic geometry, particularly moment maps, to the study of arithmetic issues in invariant theory (Marcus J. Slupinski and Robert J. Stanton); and restrictions of representations of $SL_2(\mathbb{C})$ to $SL_2(\mathbb{R})$ treated in a geometric way, thus providing a useful introduction to this research area (Birgit Speh and T. N. Venkataramana).

In addition, the volume contains three papers of an expository nature that should be considered a bonus. The first, by Joseph Bernstein, is a course for beginners on the representation theory of Lie algebras; experts can also benefit from this. Although Feigin and Zelevinski published an expanded version of these notes, the original from 1976, which is much more suitable for beginners, had never been published. The second contribution, by Jacques Faraut, introduces the work of Okounkov and Olshanski on the asymptotics of spherical functions on symmetric spaces of a large rank. The third, by Yuri A. Neretin, is an introduction to the Stein–Sahi complementary series.

Acknowledgments

We thank the invited lecturers and participants for creating a stimulating atmosphere of cooperation and communication without which this volume would not have been possible. We thank the referees for their efficient and helpful reports. We also express our gratitude to MPIM and HIM for providing us with a wonderful working environment.

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On Function Spaces on Symmetric Spaces

Bernhard Krötz and Henrik Schlichtkrull

Abstract Let $Y = G/H$ be a semisimple symmetric space. It is shown that the smooth vectors for the regular representation of G on $L^p(Y)$ vanish at infinity.

Keywords Smooth vectors • Decay of matrix coefficients • RiemannLebesgue lemma • Symmetric spaces

Mathematics Subject Classification (2010): 43A85, 43A90, 46E35

1 Vanishing at Infinity

Let G be a connected unimodular Lie group, equipped with a Haar measure $\mathrm{d}g$, and let $1 \leq p < \infty$. We consider the left regular representation L of G on the function space $E_p = L^p(G)$.

Recall that $f \in E_p$ is called a *smooth vector for* L if and only if the map

$$G \to E_p, \quad g \mapsto L(g)f$$

is a smooth E_p-valued map.

Write $\mathfrak{g}$ for the Lie algebra of G and $\mathcal{U}(\mathfrak{g})$ for its enveloping algebra. The following result is well known, see [3].

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Theorem 1. *The space of smooth vectors for L is*

$$E_p^\infty = \{f \in C^\infty(G) \mid L_u f \in L^p(G) \text{ for all } u \in \mathcal{U}(\mathfrak{g})\}.$$

Furthermore, $E_p^\infty \subset C_0^\infty(G)$, the space of smooth functions on G which vanish at infinity.

Our concern is with the corresponding result for a homogeneous space Y of G. By that we mean a connected manifold Y with a transitive action of G. In other words,

$$Y = G/H$$

with $H \subset G$ a closed subgroup. We shall require that Y carries a G-invariant positive measure dy. Such a measure is unique up to scale and commonly referred to as Haar measure. With respect to $\mathrm{d}y$, we form the Banach spaces $E_p := L^p(Y)$. The group G acts continuously by isometries on E_p via the left regular representation:

$$[L(g)f](y) = f(g^{-1}y) \qquad (g \in G, y \in Y, f \in E_p).$$

We are concerned with the space E_p^∞ of smooth vectors for this representation. The first part of Theorem 1 is generalized as follows, see [3], Theorem 5.1.

Theorem 2. *The space of smooth vectors for L is*

$$E_p^\infty = \{f \in C^\infty(Y) \mid L_u f \in L^p(Y) \text{ for all } u \in \mathcal{U}(\mathfrak{g})\}.$$

We write $C_0^\infty(Y)$ for the space of smooth functions vanishing at infinity. Our goal is to investigate an assumption under which the second part of Theorem 1 generalizes, that is,

$$E_p^\infty \subset C_0^\infty(Y). \tag{1}$$

Notice that if H is compact, then we can regard $L^p(G/H)$ as a closed G-invariant subspace of $L^p(G)$, and (1) follows immediately from Theorem 1.

Likewise, if $Y = G$ regarded as a homogeneous space for $G \times G$ with the left×right action, then again (1) follows from Theorem 1, since a left×right smooth vector is obviously also left smooth.

However, (1) is false in general as the following class of examples shows. Assume that Y has finite volume but is not compact, e.g. $Y = \mathrm{Sl}(2,\mathbb{R})/\mathrm{Sl}(2,\mathbb{Z})$. Then the constant function $\mathbf{1}_Y$ is a smooth vector for E^p, but it does not vanish at infinity.

2 Proof by Convolution

We give a short proof of (1) for the case $Y = G$, based on the theorem of Dixmier and Malliavin (see [2]). According to this theorem, every smooth vector in a Fréchet representation (π, E) belongs to the Gårding space, that is, it is spanned by vectors of the form $\pi(f)v$, where $f \in C_c^\infty(G)$ and $v \in E$. Let such a vector $L(f)g$, where $g \in E_p = L^p(G)$ be given. Then by unimodularity

$$[L(f)g](y) = \int_G f(x)g(x^{-1}y)\,\mathrm{d}x = \int_G f(yx^{-1})g(x)\,\mathrm{d}x. \tag{2}$$

For simplicity, we assume $p = 1$. The general case is similar. Let $\Omega \subset G$ be compact such that $|g|$ integrates to $< \epsilon$ over the complement. Then for y outside of the compact set $\operatorname{supp} f \cdot \Omega$, we have

$$yx^{-1} \in \operatorname{supp} f \Rightarrow x \notin \Omega,$$

and hence

$$|L(f)g(y)| \le \sup|f| \int_{x\notin\Omega} |g(x)|\,\mathrm{d}x \le \sup|f|\,\epsilon.$$

It follows that $L(f)g \in C_0(G)$.

Notice that the assumption $Y = G$ is crucial in this proof, since the convolution identity (2) makes no sense in the general case.

3 Semisimple Symmetric Spaces

Let $Y = G/H$ be a semisimple symmetric space. By this, we mean:

- G is a connected semisimple Lie group with finite center.
- There exists an involutive automorphism τ of G such that H is an open subgroup of the group $G^\tau = \{g \in G \mid \tau(g) = g\}$ of τ-fixed points.

We will verify (1) for this case. In fact, our proof is valid also under the more general assumption that G/H is a reductive symmetric space of Harish–Chandra's class, see [1].

Theorem 3. *Let $Y = G/H$ be a semisimple symmetric space, and let $E_p = L^p(Y)$ where $1 \le p < \infty$. Then*

$$E_p^\infty \subset C_0^\infty(Y).$$

Proof. A little bit of standard terminology is useful. As customary we use the same symbol for an automorphism of G and its derived automorphism of the Lie algebra $\mathfrak{g}$. Let us write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ for the decomposition in τ-eigenspaces according to eigenvalues $+1$ and -1.

Denote by K a maximal compact subgroup of G. We may and shall assume that K is stable under τ. Write θ for the Cartan-involution on G with fixed point group K, and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the eigenspace decomposition for the corresponding derived involution. We fix a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$.

The simultaneous eigenspace decomposition of $\mathfrak{g}$ under ad $\mathfrak{a}$ leads to a (possibly reduced) root system $\Sigma \subset \mathfrak{a}^* \backslash \{0\}$. Write $\mathfrak{a}_{\rm reg}$ for $\mathfrak{a}$ with the root hyperplanes removed, i.e.:

$$\mathfrak{a}_{\rm reg} = \{X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma)\, \alpha(X) \neq 0\}.$$

Let $M = Z_{H \cap K}(\mathfrak{a})$ and $W_H = N_{H\cap K}(\mathfrak{a})/M$.

Recall the polar decomposition of Y. With $y_0 = H \in Y$ the base point of Y it asserts that the mapping

$$\rho : K/M \times \mathfrak{a} \to Y, \quad (kM, X) \mapsto k \exp(X) \cdot y_0$$

is differentiable, onto and proper. Furthermore, the element X in the decomposition is unique up to conjugation by W_H, and the induced map

$$K/M \times_{W_H} \mathfrak{a}_{\rm reg} \to Y$$

is a diffeomorphism onto an open and dense subset of Y.

Let us return now to our subject proper, the vanishing at infinity of functions in E_p^∞. Let us denote functions on Y by lowercase roman letters, and by the corresponding uppercase letters their pull backs to $K/M \times \mathfrak{a}$, for example $F = f \circ \rho$. Then f vanishes at infinity on Y translates into

$$\lim_{\substack{X \to \infty \\ X \in \mathfrak{a}}} \sup_{k \in K} |F(kM, X)| = 0. \tag{3}$$

We recall the formula for the pull back by ρ of the invariant measure dy on Y. For each $\alpha \in \Sigma$ we denote by $\mathfrak{g}^\alpha \subset \mathfrak{g}$ the corresponding root space. We note that $\mathfrak{g}^\alpha$ is stable under the involution $\theta\tau$. Define p_α, resp. q_α, as the dimension of the $\theta\tau$-eigenspace in $\mathfrak{g}^\alpha$ according to eigenvalues $+1, -1$. Define a function J on $\mathfrak{a}$ by

$$J(X) = \left| \prod_{\alpha \in \Sigma^+} [\cosh \alpha(X)]^{q_\alpha} \cdot [\sinh \alpha(X)]^{p_\alpha} \right|.$$

With $d(kM)$ the Haar-measure on K/M and $\mathrm{d}X$ the Lebesgue-measure on $\mathfrak{a}$ one then gets, up to normalization:

$$\rho^*(\mathrm{d}y) = J(X)\, d(k, X) := J(X)\, d(kM)\, \mathrm{d}X.$$

We shall use this formula to relate certain Sobolev norms on Y and on $K/M \times \mathfrak{a}$. Fix a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$. For an n-tupel $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$, we define elements $X^{\mathbf{m}} \in \mathcal{U}(\mathfrak{g})$ by

$$X^{\mathbf{m}} := X_1^{m_1} \cdot \ldots \cdot X_n^{m_n}.$$

These elements form a basis for $\mathcal{U}(\mathfrak{g})$. We introduce the L^p-Sobolev norms on Y,

$$S_{m,\Omega}(f) := \sum_{|\mathbf{m}| \leq m} \left[\int_\Omega |L(X^{\mathbf{m}}) f(y)|^p \, \mathrm{d}y \right]^{1/p}$$

where $\Omega \subset Y$, and where $|\mathbf{m}| := m_1 + \ldots + m_n$. Then a function $f \in C^\infty(Y)$ belongs to E_p^∞ if and only if $S_{m,Y}(f) < \infty$ for all m.

Likewise, for $V \subset \mathfrak{a}$ we denote

$$S^*_{m,V}(F) := \sum_{|\mathbf{m}| \leq m} \left[\int_{K \times V} |L(Z^{\mathbf{m}}) F(kM, X)|^p \, J(X)\, d(k, X) \right]^{1/p}.$$

Here Z refers to members of some fixed bases for $\mathfrak{k}$ and $\mathfrak{a}$, acting from the left on the two variables, and again $\mathbf{m}$ is a multiindex.

Observe that for $Z \in \mathfrak{a}$ we have for the action on $\mathfrak{a}$,

$$[L(Z)F](kM, X) = [L(Z^k) f](k \exp(X) \cdot y_0),$$

where $Z^k := \mathrm{Ad}(k)(Z)$ can be written as a linear combination of the basis elements in $\mathfrak{g}$, with coefficients which are continuous on K. It follows that for every m there exists a constant $C_m > 0$ such that for all $F = f \circ \rho$,

$$S^*_{m,V}(F) \leq C_m S_{m,\Omega}(f), \tag{4}$$

where $\Omega = \rho(K/M, V) = K \exp(V) \cdot y_0$.

Let $\epsilon > 0$ and set

$$\mathfrak{a}_\epsilon := \{X \in \mathfrak{a} \mid (\forall \alpha \in \Sigma)\ |\alpha(X)| \geq \epsilon\}.$$

Observe that there exists a constant $C_\epsilon > 0$ such that

$$(\forall X \in \mathfrak{a}_\epsilon) \quad J(X) \geq C_\epsilon. \tag{5}$$

We come to the main part of the proof. Let $f \in E_p^\infty$. We shall first establish that

$$\lim_{\substack{X \to \infty \\ X \in \mathfrak{a}_\epsilon}} F(eM, X) = 0. \tag{6}$$

It follows from the Sobolev lemma, applied in local coordinates, that the following holds for a sufficiently large integer m (depending only on p and the dimensions of K/M and $\mathfrak{a}$). For each compact symmetric neighborhood V of 0 in $\mathfrak{a}$, there exists a constant $C > 0$ such that

$$\begin{aligned} &|F(eM, 0)| \\ &\leq C \sum_{|\mathbf{m}| \leq m} \left[\int_{K/M \times V} |[L(Z^{\mathbf{m}})F](kM, X)|^p \, d(k, X) \right]^{1/p} \end{aligned} \tag{7}$$

for all $F \in C^\infty(K/M \times \mathfrak{a})$. We choose V such that $\mathfrak{a}_\epsilon + V \subset \mathfrak{a}_{\epsilon/2}$.

Let $\delta > 0$. Since $f \in E^p$, it follows from (4) and the properness of ρ that there exists a compact set $B \subset \mathfrak{a}$ with complement $B^c \subset \mathfrak{a}$, such that

$$S^*_{m,B^c}(F) \leq C_m S_{m,\Omega}(f) < \delta, \tag{8}$$

where $\Omega = K \exp(B^c) \cdot y_0$.

Let $X_1 \in \mathfrak{a}_\epsilon \cap (B + V)^c$. Then $X_1 + X \in \mathfrak{a}_{\epsilon/2} \cap B^c$ for $X \in V$. Applying (7) to the function

$$F_1(kM, X) = F(kM, X_1 + X),$$

and employing (5) for the set $\mathfrak{a}_{\epsilon/2}$, we derive

$$\begin{aligned} &|F(eM, X_1)| \\ &\leq C \sum_{|\mathbf{m}| \leq m} \left[\int_{K/M \times V} |[L(Z^{\mathbf{m}})F_1](kM, X)|^p \, d(k, X) \right]^{1/p} \\ &\leq C' \sum_{|\mathbf{m}| \leq m} \left[\int_{K/M \times B^c} |[L(Z^{\mathbf{m}})F](kM, X)|^p \, J(X) \, d(k, X) \right]^{1/p} \\ &= C' S^*_{m,B^c}(F) \leq C'\delta, \end{aligned}$$

from which (6) follows.

In order to conclude the theorem, we need a version of (6) which is uniform for all functions $L(q)f$, for q in a fixed compact subset Q of G.

Let $\delta > 0$ be given, and as before let $B \subset \mathfrak{a}$ be such that (8) holds. By the properness of ρ, there exists a compact set $B' \subset \mathfrak{a}$ such that

$$QK \exp(B) \cdot y_0 \subset K \exp(B') \cdot y_0.$$

We may assume that B' is W_H-invariant. Then for each $k \in K$, $X \notin B'$ and $q \in Q$ we have that

$$q^{-1}k \exp(X) \cdot y_0 \notin K \exp(B) \cdot y_0, \tag{9}$$

since otherwise we would have

$$k \exp(X) \cdot y_0 \in qK \exp(B) \cdot y_0 \subset K \exp(B') \cdot y_0$$

and hence $X \in B'$.

We proceed as before, with B replaced by B', and with f, F replaced by $f_q = L_q f$, $F_q = f_q \circ \rho$. We thus obtain for $X_1 \in \mathfrak{a}_\epsilon \cap (B' + V)^c$,

$$|F_q(eM, X_1)| \leq CS^*_{m,(B')^c}(F_q) \leq C\, C_m S_{m,\Omega'}(f_q)$$

where $\Omega' = K \exp((B')^c) \cdot y_0$.

Observe that for each X in $\mathfrak{g}$ the derivative $L(X) f_q$ can be written as a linear combination of derivatives of f by basis elements from $\mathfrak{g}$, with coefficients which are uniformly bounded on Q. We conclude that $S_{m,\Omega'}(f_q)$ is bounded by a constant times $S_{m,Q^{-1}\Omega'}(f)$, with a uniform constant for $q \in Q$. By (9) and (8), we conclude that the latter Sobolev norm is bounded from the above by δ.

We derive the desired uniformity of the limit (6) for $q \in Q$,

$$\lim_{\substack{X \to \infty \\ X \in \mathfrak{a}_\epsilon}} \sup_{q \in Q} |F_q(eM, X)| = 0. \tag{10}$$

Finally, we choose an appropriate compact set Q. Let $C_1, \ldots, C_N \subset \mathfrak{a}$ be the closed chambers relative to Σ. For each chamber C_j, we choose $X_j \in C_j$ such that $X_j + C_j \subset \mathfrak{a}_\epsilon$. It follows that

$$\mathfrak{a} = \bigcup_{j=1}^{N} (-X_j + \mathfrak{a}_\epsilon). \tag{11}$$

Set $a_j = \exp(X_j) \in A$ and define

$$Q := \bigcup_{j=1}^{N} a_j K.$$

Note that for $q = a_j k$ we have

$$F_q(eM, X) = F(k^{-1}M, X - X_j).$$

Let $\delta > 0$ be given. It follows from (10) that there exists $R > 0$ such that $|F_q(eM, Y)| < \delta$ for all $q \in Q$ and all $Y \in \mathfrak{a}_\epsilon$ with $|Y| \geq R$. For every $X \in \mathfrak{a}$ with $|X| \geq R + \max_j |X_j|$, we have $X \in -X_j + \mathfrak{a}_\epsilon$ for some j and $|X + X_j| \geq R$. Hence for all $k \in K$,

$$|F(kM, X)| = |F_q(eM, X + X_j)| < \delta,$$

where $q = a_j k^{-1}$. Thus,

$$\lim_{X \to \infty} F(kM, X) = 0,$$

uniformly over $k \in K$, as was to be shown. □

Remark. Let $f \in L^2(Y)$ be a K-finite function which is also finite for the center of $\mathcal{U}(\mathfrak{g})$. Then it follows from [4] that f vanishes at infinity. The present result is more general, since such a function necessarily belongs to E_2^∞.

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A Relation Involving Rankin–Selberg L-Functions of Cusp Forms and Maass Forms

Jay Jorgenson and Jürg Kramer

Abstract In previous articles, an identity relating the canonical metric to the hyperbolic metric associated with any compact Riemann surface of genus at least two has been derived and studied. In this article, this identity is extended to any hyperbolic Riemann surface of finite volume. The method of proof is to study the identity given in the compact case through degeneration and to understand the limiting behavior of all quantities involved. In the second part of the paper, the Rankin–Selberg transform of the noncompact identity is studied, meaning that both sides of the relation after multiplication by a nonholomorphic, parabolic Eisenstein series are being integrated over the Riemann surface in question. The resulting formula yields an asymptotic relation involving the Rankin–Selberg L-functions of weight two holomorphic cusp forms, of weight zero Maass forms, and of nonholomorphic weight zero parabolic Eisenstein series.

Keywords Automorphic forms • Eisenstein series • L-functions • Rankin–Selberg transform • Heat kernel

Mathematics Subject Classification (2010): 11F30, 32N10, 20H10, 11M36

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1 Introduction

1.1 Background

Beginning with the article [13], we derived and studied a basic identity, stated in (1) below, coming from the spectral theory of the Laplacian associated with any compact hyperbolic Riemann surface. In the subsequent papers, this identity was employed to address a number of problems, including the following: Establishing precise relations between analytic invariants arising in the Arakelov theory of algebraic curves and hyperbolic geometry (see [13]), proving the noncompleteness of a newly defined metric on the moduli space of algebraic curves of a fixed genus (see [14]), deriving bounds for canonical and hyperbolic Green's functions (see [15]), and obtaining bounds for Faltings's delta function with applications associated with Arakelov theory (see [16]). In this article, we expand our application of the results from [13] to analytic number theory. In brief, we first generalize the identity (1) to general noncompact, finite volume hyperbolic Riemann surfaces without elliptic fixed points; this relation is stated in equation (2) below. We then compute the Rankin–Selberg convolution with respect to (2), and show that the result yields a new relation involving Rankin–Selberg L-functions of cusp forms of weight two and Maass forms, as well as the scattering matrix of the nonholomorphic Eisenstein series of weight zero.

1.2 The Basic Identity

Let X denote a compact hyperbolic Riemann surface, necessarily of genus $g \geq 2$. Let $\{f_j\}$ be a basis of the g-dimensional space of cusp forms of weight two, which we assume to be orthonormal with respect to the Petersson inner product. Then we set

$$\mu_{\mathrm{can}}(z) = \frac{1}{g} \cdot \frac{i}{2} \sum_{j=1}^{g} |f_j(z)|^2 \mathrm{d}z \wedge \mathrm{d}\bar{z}$$

for any point $z \in X$. Let Δ_{hyp} denote the hyperbolic Laplacian acting on the space of smooth functions on X, and $K(t; z, w)$ the corresponding heat kernel; set $K(t; z) = K(t; z, z)$. We use μ_{shyp} to denote the $(1, 1)$-form of the constant negative curvature metric on X such that X has volume one, and μ_{hyp} to denote the $(1, 1)$-form of the metric on X with constant negative curvature equal to -1. With this notation, the key identity of [13] states

$$\mu_{\mathrm{can}}(z) = \mu_{\mathrm{shyp}}(z) + \frac{1}{2g} \int_0^{\infty} \Delta_{\mathrm{hyp}} K(t; z) \, \mathrm{d}t \mu_{\mathrm{hyp}}(z) \qquad (z \in X). \qquad (1)$$

The first result in this paper is to generalize (1) to general noncompact, finite volume hyperbolic Riemann surfaces without elliptic fixed points. Specifically, if X is such a noncompact, finite volume hyperbolic Riemann surface of genus g with p cusps and no elliptic fixed points, then

$$\mu_{\mathrm{can}}(z) = \left(1 + \frac{p}{2g}\right)\mu_{\mathrm{shyp}}(z) + \frac{1}{2g}\int_0^\infty \Delta_{\mathrm{hyp}}K(t;z)\,\mathrm{d}t\;\mu_{\mathrm{hyp}}(z) \qquad (z \in X). \tag{2}$$

The proof of (2) we present here is to study (1) for a degenerating family of hyperbolic Riemann surfaces and to use known results for the asymptotic behavior of the canonical metric form μ_{can} (see [12]), the hyperbolic heat kernel (see [18]), and small eigenvalues and eigenfunctions of the Laplacian (see [21]).

In [2], the author extends the identity (2) to general finite volume quotients of the hyperbolic upper half-plane, allowing for the presence of elliptic elements. The proof does not employ degeneration techniques, as in this paper, but rather follows the original method of proof given in [13] and [15]. The article [2] is part of the Ph.D. dissertation completed under the direction of the second named author of the present article.

1.3 The Rankin–Selberg Convolution

For the remainder of this article, we assume $p > 0$. Let P denote a cusp of X and $E_{P,s}(z)$ the associated nonholomorphic Eisenstein series of weight zero. In essence, the purpose of this article is to evaluate the Rankin–Selberg convolution with respect to (2), by which we mean to multiply both sides of (2) by $E_{P,s}(z)$ and to integrate over all $z \in X$.

By means of the uniformization theorem, there is a Fuchsian group of the first kind $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{R})$ such that X is isometric to $\Gamma\backslash\mathbb{H}$. Furthermore, we can choose Γ so that the point $i\infty$ in the boundary of $\mathbb{H}$ projects to the cusp P, which we assume to have width b. Writing $z = x + iy$, well-known elementary considerations then show that the expression

$$\int_X E_{P,s}(z)\mu_{\mathrm{can}}(z)$$
$$= \int_X E_{P,s}(z)\left(\left(1 + \frac{p}{2g}\right)\mu_{\mathrm{shyp}}(z) + \frac{1}{2g}\int_0^\infty \Delta_{\mathrm{hyp}}K(t;z)\,\mathrm{d}t\;\mu_{\mathrm{hyp}}(z)\right)$$

is equivalent to

$$\int_{y=0}^\infty \int_{x=0}^b y^s \mu_{\mathrm{can}}(z)$$
$$= \int_{y=0}^\infty \int_{x=0}^b y^s \left(\left(1 + \frac{p}{2g}\right)\mu_{\mathrm{shyp}}(z) + \frac{1}{2g}\int_0^\infty \Delta_{\mathrm{hyp}}K(t;z)\,\mathrm{d}t\;\mu_{\mathrm{hyp}}(z)\right). \tag{3}$$

The majority of the computations carried out in this article are related to the evaluation of (3). To be precise, for technical reasons we consider the integrals in (3) multiplied by the factor $2gb^{-1}\pi^{-s}\Gamma(s)\zeta(2s)$, where $\Gamma(s)$ is the Γ-function and $\zeta(s)$ is the Riemann ζ-function.

1.4 The Main Result

Having posed the problem under consideration, we can now state the main result of this article after establishing some additional notation.

The cusp forms f_j, being invariant under the map $z \mapsto z + b$, allow a Fourier expansion of the form

$$f_j(z) = \sum_{n=1}^{\infty} a_{j,n} \mathrm{e}^{2\pi i n z/b}.$$

Following notations and conventions in [4], we let

$$\widetilde{L}(s, f_j \otimes \overline{f}_j) = G_\infty(s) \cdot L(s, f_j \otimes \overline{f}_j), \tag{4}$$

where

$$G_\infty(s) = (2\pi)^{-2s-1}\Gamma(s)\Gamma(s+1)\zeta(2s),$$

$$L(s, f_j \otimes \overline{f}_j) = \sum_{n=1}^{\infty} \frac{|a_{j,n}|^2}{(n/b)^{s+1}}.$$

As shown in [4], the Rankin–Selberg L-function $\widetilde{L}(s, f_j \otimes \overline{f}_j)$ is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, admits a meromorphic continuation to all $s \in \mathbb{C}$, and is symmetric under $s \mapsto 1 - s$.

Let φ_j be a nonholomorphic weight zero form which is an eigenfunction of Δ_{hyp} with eigenvalue $\lambda_j = s_j(1 - s_j)$, hence $s_j = 1/2 + ir_j$. From [11], we recall the expansion

$$\varphi_j(z) = \alpha_{j,0}(y) + \sum_{n \neq 0} \alpha_{j,n} W_{s_j}(nz/b),$$

where

$$\alpha_{j,0}(y) = \alpha_{j,0} y^{1-s_j},$$

$$W_{s_j}(w) = 2\sqrt{\cosh(\pi r_j)}\sqrt{|\mathrm{Im}(w)|}K_{ir_j}(2\pi|\mathrm{Im}(w)|)\mathrm{e}^{2\pi i \mathrm{Re}(w)} \quad (w \in \mathbb{C}),$$

and $K_{\cdot}(\cdot)$ denotes the classical K-Bessel function. Again, following notations and conventions in [4], we let

$$\widetilde{L}(s, \varphi_j \otimes \overline{\varphi}_j) = G_{r_j}(s) \cdot L(s, \varphi_j \otimes \overline{\varphi}_j),$$

where

$$G_{r_j}(s) = s(1-s)\pi^{-2s}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+ir_j\right)\Gamma\left(\frac{s}{2}-ir_j\right)\zeta(2s),$$
$$L(s, \varphi_j \otimes \overline{\varphi}_j) = \sum_{n\neq 0} \frac{|\alpha_{j,n}|^2}{(n/b)^{s-1}}.$$

As shown in [4], the Rankin–Selberg L-function $\widetilde{L}(s, \varphi_j \otimes \overline{\varphi}_j)$ is holomorphic for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, admits a meromorphic continuation to all $s \in \mathbb{C}$, and is symmetric under $s \mapsto 1-s$. Observe that our completed L-function $\widetilde{L}(s, \varphi_j \otimes \overline{\varphi}_j)$ differs from the L-function defined in [4] because of the appearance of the multiplicative factor $s(1-s)$ in the definition of $G_{r_j}(s)$.

Similarly, one can define completed Rankin–Selberg L-functions associated with the nonholomorphic Eisenstein series $E_{P,s}(z)$ for any cusp P on X having a Fourier expansion of the form

$$E_{P,s}(z) = \delta_{P,\infty} y^s + \phi_{P,\infty}(s) y^{1-s} + \sum_{n\neq 0} \alpha_{P,s,n} W_s(nz/b)$$

with $\phi_{P,\infty}(s)$ denoting the (P,∞)-th entry of the scattering matrix.

With all this, the main result of this article is the following theorem. For any $\varepsilon > 0$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, define the Θ-function

$$\Theta_\varepsilon(s) = \sum_{\lambda_j > 0} \frac{\cosh(\pi r_j) \mathrm{e}^{-\lambda_j \varepsilon}}{2\lambda_j} \widetilde{L}(s, \varphi_j \otimes \overline{\varphi}_j)$$
$$+ \frac{1}{8\pi} \sum_{P \text{ cusp}} \int_{-\infty}^{\infty} \frac{\cosh(\pi r) \mathrm{e}^{-(r^2+1/4)\varepsilon}}{r^2 + 1/4} \widetilde{L}(s, E_{P,1/2+ir} \otimes \overline{E}_{P,1/2+ir})\, \mathrm{d}r$$

and the universal function

$$F_\varepsilon(s) = \frac{\zeta(s) b^{s-1}}{2\pi^2} \int_0^\infty \frac{r \sinh(\pi r) \mathrm{e}^{-(r^2+1/4)\varepsilon}}{r^2+1/4} G_r(s)\, \mathrm{d}r.$$

Then the L-function relation involving Rankin–Selberg L-functions of cusp forms and Maass forms

$$\lim_{\varepsilon\to 0}\big(\Theta_\varepsilon(s) - F_\varepsilon(s)\big)$$
$$= \sum_{j=1}^{g} \widetilde{L}(s, f_j \otimes \overline{f}_j) - 4\pi\zeta(s)b^{s-1}G_\infty(s) - \pi^{-s}\frac{2s}{s+1}\Gamma(s)\zeta(2s)\phi_{\infty,\infty}\left(\frac{s+1}{2}\right) \tag{5}$$

holds true. By taking $\varepsilon > 0$ in (5), one has an error term which is $o(1)$ as ε approaches zero. This error term is explicit and given in terms of integrals involving the hyperbolic heat kernel.

A natural question to ask is to what extent the relation of L-functions (5) implies relations between the Fourier coefficients of the holomorphic weight two forms and the Fourier coefficients of the Maass forms under consideration. In general, extracting such information from a limiting relationship such as (5) could be very difficult. However, as stated, our analysis yields an explicit expression for the error term by rewriting (5) for a fixed $\varepsilon > 0$, which allows for additional considerations. The problem of using (5) to study possible relations among the Fourier coefficients is currently under investigation.

1.5 General Comments

If X is the Riemann surface associated with a congruence subgroup, then the series $\phi_{\infty,\infty}(s)$ can be expressed in terms of Dirichlet L-functions associated with even characters with conductors dividing the level (see [8] or [10]). With these computations, one can rewrite (5) further so that one obtains an expression involving Rankin–Selberg L-functions associated with cusp forms of weight two, Maass forms, nonholomorphic Eisenstein series, and classical zeta functions. However, the relation stated in (5) holds for any finite volume hyperbolic Riemann surface without elliptic fixed points. In order to eliminate the restriction that X has no elliptic fixed points, one needs to revisit the proof of (2), and possibly (1), in order to allow for elliptic fixed points. As stated above, this project currently is under investigation in [2]; however, we choose to focus in this paper on deriving (5) with the simplifying assumption that X has no elliptic fixed points in order to draw attention to the presence of an L-function relation coming from the basic identity (2). We will leave for future work the generalization of (2) to arbitrary finite volume hyperbolic Riemann surfaces, which may have elliptic fixed points, and derive the relation analogous to (5).

From Riemannian geometry, theta functions naturally appear as the trace of a heat kernel, and the small time expansion of the heat kernel has a first-order term which is somewhat universal and a second-order term which involves integrals of a curvature of the Riemannian metric. In this regard, (5) suggests that the sum of Rankin–Selberg L-functions

$$\sum_{j=1}^{g} \widetilde{L}(s, f_j \otimes \overline{f}_j)$$

represents some type of curvature integral relative to the theta function $\Theta_\varepsilon(s)$. Further investigation of this heuristic observation is warranted.

1.6 Outline of the Paper

In Sect. 2, we recall necessary background material and establish additional notation. In Sect. 3, we prove (2) and further develop the identity (2) using the spectral expansion of the heat kernel $K(t; z, w)$. In Sect. 4, we evaluate the integrals in (3) using the revised analytic expressions of (2), and in Sect. 5, we gather the computations from Sect. 4 and prove (5).

2 Notations and Preliminaries

2.1 Hyperbolic and Canonical Metrics

Let Γ be a Fuchsian subgroup of the first kind of $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations on the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \,|\, z = x + iy,\ y > 0\}$. We let X be the quotient space $\Gamma\backslash\mathbb{H}$ and denote by g the genus of X. We assume that Γ has no elliptic elements and that X has $p \geq 1$ cusps. We identify X locally with its universal cover $\mathbb{H}$.

In the sequel μ denotes a (smooth) metric on X, i.e., μ is a positive $(1,1)$-form on X. In particular, we let $\mu = \mu_{\mathrm{hyp}}$ denote the hyperbolic metric on X, which is compatible with the complex structure of X, and has constant negative curvature equal to -1. Locally, we have

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{\mathrm{d}z \wedge \mathrm{d}\overline{z}}{y^2}.$$

We write $\mathrm{vol}_{\mathrm{hyp}}(X)$ for the hyperbolic volume of X; recall that $\mathrm{vol}_{\mathrm{hyp}}(X)$ is given by $2\pi(2g - 2 + p)$. The scaled hyperbolic metric $\mu = \mu_{\mathrm{shyp}}$ is simply the rescaled hyperbolic metric $\mu_{\mathrm{hyp}}/\mathrm{vol}_{\mathrm{hyp}}(X)$, which measures the volume of X to be one.

Let $S_k(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k with respect to Γ equipped with the Petersson inner product

$$\langle f, g\rangle = \frac{i}{2}\int_X f(z)\,\overline{g(z)}\,y^k\,\frac{\mathrm{d}z\wedge \mathrm{d}\overline{z}}{y^2} \qquad \big(f, g\in S_k(\Gamma)\big).$$

By choosing an orthonormal basis $\{f_1, ..., f_g\}$ of $S_2(\Gamma)$ with respect to the Petersson inner product, the canonical metric $\mu = \mu_{\mathrm{can}}$ of X is given by

$$\mu_{\mathrm{can}}(z) = \frac{1}{g}\cdot\frac{i}{2}\sum_{j=1}^{g}|f_j(z)|^2\,\mathrm{d}z\wedge\mathrm{d}\overline{z}.$$

We denote the hyperbolic Laplacian on X by Δ_{hyp}; locally, we have

$$\Delta_{\mathrm{hyp}} = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right). \tag{6}$$

The discrete spectrum of Δ_{hyp} is given by the increasing sequence of eigenvalues

$$0=\lambda_0<\lambda_1\leq\lambda_2\leq\ldots$$

2.2 Modular Forms, Maass Forms, and Eisenstein Series

Throughout we assume, as before, that the cusp width of the cusp $i\infty$ equals b. In Sect. 1.4, we established the notation for holomorphic cusp forms of weight two and Maass forms with respect to Γ, as well as the corresponding Rankin–Selberg L-functions, so we do not repeat the discussion here.

The eigenfunctions for the continuous spectrum of Δ_{hyp} are provided by the Eisenstein series $E_{P,s'}$ (associated with each cusp P of X) with eigenvalue $\lambda = s'(1-s')$, hence $s' = 1/2 + ir$ ($r\in\mathbb{R}$). They have Fourier expansions of the form

$$E_{P,s'}(z) = \alpha_{P,s',0}(y) + \sum_{n\neq 0}\alpha_{P,s',n}W_{s'}(nz/b),$$

where

$$\alpha_{P,s',0}(y) = \delta_{P,\infty}y^{s'} + \phi_{P,\infty}(s')y^{1-s'},$$
$$W_{s'}(w) = 2\sqrt{\cosh(\pi r)}\sqrt{|\mathrm{Im}(w)|}K_{ir}(2\pi|\mathrm{Im}(w)|)\mathrm{e}^{2\pi i\mathrm{Re}(w)} \quad (w\in\mathbb{C});$$

here $\delta_{P,\infty}$ is the Kronecker delta and $\phi_{P,\infty}(s')$ is the (P,∞)-th entry of the scattering matrix (see [11]). For example, the function $\phi_{\infty,\infty}(s')$ is given by a Dirichlet series of the form

$$\phi_{\infty,\infty}(s') = \sqrt{\pi}\,\frac{\Gamma(s'-1/2)}{\Gamma(s')}\sum_{n=1}^{\infty}\frac{a_n}{\mu_n^{2s'}}, \tag{7}$$

where the quantities a_n and μ_n are explicitly given in [11], p. 60.

For $s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$, we define the completed Rankin–Selberg L-function attached to $E_{P,s'}$ by

$$\widetilde{L}(s, E_{P,s'} \otimes \overline{E}_{P,s'}) = G_r(s) \cdot L(s, E_{P,s'} \otimes \overline{E}_{P,s'}), \tag{8}$$

where

$$G_r(s) = s(1-s)\pi^{-2s}\Gamma^2\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+ir\right)\Gamma\left(\frac{s}{2}-ir\right)\zeta(2s),$$

$$L(s, E_{P,s'} \otimes \overline{E}_{P,s'}) = \sum_{n\neq 0}\frac{|\alpha_{P,s',n}|^2}{(n/b)^{s-1}}.$$

2.3 Hyperbolic Heat Kernel and Variants

The hyperbolic heat kernel $K_{\mathbb{H}}(t;z,w)$ ($t \in \mathbb{R}_{>0}$; $z,w \in \mathbb{H}$) on $\mathbb{H}$ is given by the formula

$$K_{\mathbb{H}}(t;z,w) = K_{\mathbb{H}}(t;\rho) = \frac{\sqrt{2}\mathrm{e}^{-t/4}}{(4\pi t)^{3/2}}\int_{\rho}^{\infty}\frac{r\mathrm{e}^{-r^2/(4t)}}{\sqrt{\cosh(r)-\cosh(\rho)}}\,\mathrm{d}r,$$

where $\rho = d_{\mathrm{hyp}}(z,w)$ denotes the hyperbolic distance from z to w. The hyperbolic heat kernel $K(t;z,w)$ ($t \in \mathbb{R}_{>0}$; $z,w \in X$) on X is obtained by averaging over the elements of Γ, namely

$$K(t;z,w) = \sum_{\gamma\in\Gamma} K_{\mathbb{H}}\big(t;z,\gamma(w)\big).$$

The heat kernel on X satisfies the equations

$$\left(\frac{\partial}{\partial t} + \Delta_{\mathrm{hyp},z}\right)K(t;z,w) = 0 \qquad (w \in X),$$

$$\lim_{t\to 0}\int_X K(t;z,w)\,f(w)\,\mu_{\mathrm{hyp}}(w) = f(z) \quad (z \in X)$$

for all C^{∞}-functions f on X. As a shorthand, we write $K(t;z) = K(t;z,z)$.

With the notations from Sect. 2.2, we introduce the modified heat kernel function

$$K^{\mathrm{cusp}}(t;z) = K(t;z) - \sum_{0\leq\lambda_j<1/4} |\alpha_{j,0}|^2 y^{2-2s_j} \mathrm{e}^{-\lambda_j t}$$
$$- \frac{1}{4\pi} \sum_{P\ \mathrm{cusp}} \int_{-\infty}^{\infty} |\delta_{P,\infty} y^{1/2+ir} + \phi_{P,\infty}(s) y^{1/2-ir}|^2 \mathrm{e}^{-(r^2+1/4)t} \mathrm{d}r. \quad (9)$$

Denoting by Γ_∞ the stabilizer of the cusp ∞, we can define the following partial heat kernel functions

$$K_0(t;z) = \sum_{\gamma\in\Gamma\setminus\Gamma_\infty} K_{\mathbb{H}}\big(t;z,\gamma(z)\big), \quad (10)$$

$$K_\infty(t;z) = \sum_{\gamma\in\Gamma_\infty} K_{\mathbb{H}}\big(t;z,\gamma(z)\big) \quad (11)$$

giving rise to the decomposition

$$K(t;z) = K_0(t;z) + K_\infty(t;z).$$

3 The Fundamental Identity

In this section, we derive the identity (2) by studying the relation (1) for a degenerating family of compact hyperbolic Riemann surfaces. The corresponding statement is proven in Lemma 3.1. In the remainder of the section, we manipulate the terms in (2) assuming $p > 0$ in order to obtain an equivalent formulation of the relation which then will be suited for our computations in the subsequent sections. Specifically, we first express the heat kernel on the underlying Riemann surface in terms of its spectral expansion, which involves Maass forms and nonholomorphic Eisenstein series, and we remove the terms associated with the constant terms in the Fourier expansions of the Maass forms and the nonholomorphic Eisenstein series (see Proposition 3.3). We then express the heat kernel as a periodization over the uniformizing group and remove the contribution from the parabolic subgroup associated with a single cusp (see Lemma 3.8 as well as the preliminary computations and remarks). The main result of this section is Theorem 3.9.

Lemma 3.1. *With the above notations, we have*

$$\mu_{\mathrm{can}}(z) = \left(1 + \frac{p}{2g}\right) \mu_{\mathrm{shyp}}(z) + \frac{1}{2g} \int_0^\infty \Delta_{\mathrm{hyp}} K(t;z)\, \mathrm{d}t\ \mu_{\mathrm{hyp}}(z). \quad (12)$$