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Sheaves of Algebras over Boolean Spaces

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To the memory of
ALFRED L. FOSTER,
who set me to work
representing algebras.

Preface

My involvement in the line of research leading to this book began in 1963 when I was a graduate student working under the direction of Alfred Foster, and was first learning about representing algebras as subdirect products. In particular, for a starter I learned that Stone's representation theorem was valid not just for Boolean algebras but for any class of algebras satisfying the identities of a primal algebra. Foster perceived in these algebras a Boolean part whose representation theory could be levered into representing many other kinds of algebras.

The broad motivation was to break up a complicated algebra into simpler pieces; if the pieces could be understood, then hopefully so could the whole algebra. The obvious decomposition to try first is a direct product. The advantage of direct products is the simplicity of their construction. The overwhelming disadvantage is that most algebras are indecomposable in this sense, and even when decomposable there may be no ultimate refinement. Subdirect products overcome both of these liabilities, as first demonstrated by Garrett Birkhoff.

The main drawback to subdirect products is that, while factors may be commonplace and well understood, the transfer of an argument from the components to the whole algebra may fail because one may not know in sufficient detail how the components fit together to form the original algebra. Thus one grafts topological spaces onto subdirect products to form significantly superior sheaves. Elements of the subdirect product become continuous functions, and are easier to recognize. Boolean spaces are often used since they arise naturally in representing Boolean algebras and have been the key to many other representation theorems. However, the topological spaces of algebra are intuitively quite different from the more traditional topological spaces such as manifolds with a local Euclidean topology. They may be totally disconnected or not even Hausdorff.

The question we address then is, how far can one go in representing arbitrary algebras by sheaves over general topologies, and in particular Boolean spaces? The overall structure of a given algebra should come from a systematic synthesis of the components, that is, the stalks of the sheaf. Many questions about any algebra in such a class should be answerable by analyzing locally what is happening in the components, rather than working globally with formulas over the whole algebra.

My first exposure to sheaves over Boolean spaces was in a seminar run by Joseph Kist in the spring of 1972, in which he presented the seminal paper of Stephen Comer. Here I learned of the rich and productive world of ring spaces as expounded by Richard Pierce in his memoir.

It was in this seminar that I discovered factor elements, which generalize central idempotents in rings, and how they correspond to factor congruences. Later, factor bands, ideals, and sesquimorphisms were added. The goal was to extend the classical representation of regular commutative rings as subdirect products of fields.

Although general tools are developed, applicable to all algebras, the best efforts come from settling on those that I dub ‘shells’, which assert the existence of a zero and a one for a multiplicative operation and perhaps an addition that otherwise need not satisfy any of the usual identities such as commutativity and associativity. In this context, one can generalize well beyond ring theory a number of classical results on biregularity, strong regularity, and lack of nilpotents.

This monograph adapts the intuitive idea of a metric space to universal algebra, leading to the useful device of a complex. Then a sheaf is constructed directly from a complex.

The core of this book does not look at all congruences of an algebra, but at only some of them comprising a Boolean subsemilattice of congruences, and more typically, at others splitting the algebra into a product of complementary factors. Thus there are no restrictions on the whole lattice of congruences, but only on parts of it. This is one of the themes of this monograph.

Over the course of time, terms and notations tend to grow like Topsy. In synthesizing disparate fields and even extending them, inconsistencies across them pose a dilemma for an author. Should he completely streamline the terminology, thereby shutting out the casual reader who is merely browsing but already knows something of the traditional notation? Or, should he leave every term as it has originally arisen, thereby making it difficult for the serious reader to correlate similar ideas? I have taken a middle course, respecting most terms and notations already in the literature, but occasionally changing some to better reflect the overall picture. For example, congruences that *permute* elsewhere *commute* here since other internal factor objects, such as idempotent endomorphisms, always commute when creating a product. But I left unchanged directly *indecomposable* and subdirectly *irreducible*, although one ought to have a common root word for the many kinds of algebraic atoms. The definitions of the rather general algebras, *shells* and *half-shells*, have broadened over time as weaker and weaker conditions were observed to create sheaves that would accomplish most of the same ends. *Nullity* is used for an element annihilating a binary operation as a zero does in ring theory. And *unity* is the term used where others might use 'unit element' or 'identity;' it even means 'object' in categories. Likewise, the adjective *unital* adds a unity to a ring or shell.

Many exercises and problems have been included. The distinction between them is as follows. On the one hand, the exercises come from notes I wrote to myself while trying to understand the relationships between new concepts. There was no attempt to create other exercises that might fill out the book; thus the density of exercises varies from section to section. The reader may enjoy more healthy exercise by filling in wherever a proof trails off with a phrase such as 'straightforward to prove', 'trivial', or 'left to the reader'. This is especially so in the categorical sections establishing adjointness and equivalence.

On the other hand, the problems are open questions that I have not resolved because I did not take the time. Thus, such problems may range from the trivial to the significant, perhaps to promising research to pursue. I have not attempted to distinguish these possibilities.

As for prerequisites, a reader should have a nodding acquaintance with universal algebra, logic, categories, topology, and Boolean algebra. By recalling useful facts about these topics, prerequisites have been kept to a minimum. All concepts beyond these are defined. However, as the goal is new theorems, and the ideas already in the literature are lightly illustrated here, the prospective reader will be well motivated if he is familiar with some of the classical results that are being generalized.

I am thankful to the participants who asked penetrating questions in algebra seminars at New Mexico State University, Tennessee Technological University, the University of Tennessee and Vanderbilt University; some of these led to additional insights and examples. Fruitful conversations with Joseph Kist have cleared up a number of murky points. Mai Gehrke pointed out non sequiturs, and shortened several long-winded proofs. Isadore Fleischer corrected several of the early chapters. Paul Cohn offered suggestions on the history of the subject, and Ross Willard pointed out a significant

extension of the concept of a shell. Diego Vaggione quickly dispatched several of the original open problems. All of these, including three anonymous reviewers, deserve warm handshakes for their many comments and thought-provoking suggestions. As for remaining faux pas that I should have caught, may the sympathetic reader forgive me for any difficulties they might cause.

Albuquerque, New Mexico Arthur Knoebel

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I

Introduction

This chapter has two sections. The first is a history of the ideas and previous theorems upon which this monograph is based. The second is a survey of the principal results presented in this book.

1. History

To set the stage, we take a short historical jaunt. This will not be a literal, detailed history, but a genetic reconstruction of key events that have come to play a role in this book. There are three areas, as befits its title: sheaves, algebras, and Boolean algebras. We begin with the last.

The attempt to decompose an involved problem into workable parts is an old one – it is called the reductionist philosophy. A good starting point for examining attempts at symbolic decomposition is the work of Gottfried von Leibniz [**Leib66**] [**Mido65**]. While his efforts did not lead directly to the analysis of algebras, the motivations and flawed solutions shed light on our work in this book. Leibniz's dream of a universal calculus of logic for deriving facts mechanically by combining together basic concepts is, in a sense, a precursor of Boolean algebra.

This view that algebra could carry the burden of logical manipulation is already seen in George Peacock's definition of algebra as "the science of general reasoning by symbolical language". [**Peac30**, p. 1] A part of this dream was realized independently by Boole [**Bool47**] and Augustus

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De Morgan [**DeMo47**] [**MacH85**, pp. 68–71]. Boole's book developed an algebra of logic, which bears his name, although 'Boolean algebras' today are not what he described [**Burr00**].

We have mentioned Boolean algebras at the outset since they will subsequently provide a calculus for decomposing algebras by sheaves. At a higher and more powerful level of logic, the successful application of the first-order predicate calculus to mathematics came later; but unfortunately this does not solve problems in the generality envisioned by Leibniz. Kurt Gödel and Jacques Herbrand showed how limited automatic problem solving could be [**Mend64**].

Having discussed an algebra of logic, we now move on to the discoveries in linear algebra, such as quaternions, vectors and exterior algebras, which paved the way to modern algebra. Hermann Grassmann [**Gras44**], William Hamilton [**Hami44**], and later Benjamin Peirce [**Peir70**], J. Willard Gibbs, [**Gibb81**] and Oliver Heaviside [**Heav93**] invented and studied many different kinds of linear algebras, thereby opening a path to the study of noncommutative and nonassociative systems. Peirce introduced what is now called the right Peirce decomposition of a linear algebra: $A = iA + (1-i)A$, for an idempotent i, which need not be central. Also important as another example of a noncommutative system is Arthur Cayley's [**Cayl54**] attempted axiomatization of abstract groups, which arose from the study of the permutation of roots of a polynomial equation.

In another direction, Richard Dedekind [**Dede97**] first recognized the notion of a lattice in the context of number theory. Lattice theory is significant for our history in providing us with laws similar to but not identical with those of arithmetic, and in generalizing Boolean algebras to nonlogical examples. The history of algebra in the nineteenth century is rich and varied; there is much we could mention that would lead into our research, but, to keep this part of the book short, we refer the interested reader to the fine histories of Luboš Nový [**Nový73**] and B. L. van der Waerden [**vdWa85**].

The twentieth century saw the flowering of six fields that have influenced our work and provided examples: rings, lattices, universal algebra, algebraic geometry, sheaf theory, and functional analysis. The oldest of these fields is commutative ring theory.[1] We mention two papers as samples of the influence of David Hilbert and Emmy Noether on the development of modern algebra with its distinctive perspective and abstract axiomatics. Hilbert's work [**Hilb96**] on invariants cut through the Gordian knot of case-by-case construction of bases for polynomial invariants by indirect, qualitative methods – non-constructive, if you like.

[1] Rings are assumed, in this section, to have a unity, since historically they always had one; otherwise, throughout the rest of this monograph, they will not, unless designated 'unital'.

A most significant event in the history of modern algebra occurred with the publication of Noether's [1921] *Idealtheorie in Ringbereichen.* Noether's mathematical philosophy [**Noet21**] was to replace arguments that manipulated elements by structural proofs using ideals, thereby creating a powerful theory predicated only on the ascending chain condition on ideals. Here, for the first time, an ideal in an abstract ring is decomposed as a product of primary ideals. This notion was systematized by Wolfgang Krull into a principle: whether a ring $\boldsymbol{R}$ is indecomposable in a certain sense is equivalent to determining whether the intersection of a certain class of ideals in $\boldsymbol{R}$ is the null ideal [**Krull35**]. This idea was exploited by Garrett Birkhoff as the construction of subdirect products in the context of universal algebra [**Birk44**].

Another perspective on looking for representations is to specify the kind of rings we want to draw components from and the candidates in the way of ideals and congruences that are initially proposed to obtain these building blocks. A classic example is the class of semisimple rings. The factors must be quotients by maximal ideals whose intersection is the trivial ideal. But in general the intersection of all maximal ideals is not the zero ideal – witness local rings – so the representation is not faithful. Historically, to overcome this, one appropriately restricts the class of rings, for example, to those that are Artinian (that is, they satisfy a descending chain condition on ideals) and contain no nilpotent ideals other than the null ideal. The Wedderburn-Artin theorem then concludes that such a ring is a direct sum of a finite number of ideals each of which is isomorphic to the ring of all linear transformations of a finite-dimensional vector space over a division ring [**Jaco80**, vol. 2, p. 203].

In 1929, Krull was one of the first to attain theorems without chain conditions; these had the advantage of giving representations with an infinite number of factors [**Krull29**]. Gottfried Köthe defines the notion of a transcendent reducible ring and proves that each transcendent reducible commutative ring is a direct product of fields [**Köthe30**, p. 548]. Here, we see the first theorem in which the quotients have no divisors of zero, which will be a recurring theme later in this book. John von Neumann defined regularity of rings and discovered the isomorphism of the lattice of factors with the lattice of central idempotents [**vonN36**].

In 1936, Marshall Stone found his far-reaching representation theorem for Boolean algebras and rings: every such algebra is a subdirect power of a two-element algebra [**Stone36**]. Thus, the kernel into which all Boolean algebras decompose is the two-element Boolean algebra (or ring) of traditional truth values. Stone's paper is the spur and inspiration for much of the work that leads to the work explored in this monograph. Compare the length of Stone's original paper to Birkhoff's much shorter proof of the same result [**Birk44**]; this is a considerable distillation to take place within a decade. In Stone's paper of the next year, he explored the duality between

Boolean algebras and Boolean spaces [**Stone37**]. The power of this theory to tackle problems about Boolean algebras by going to their corresponding Boolean spaces was subsequently illustrated by Paul Halmos [**Halm63**, chap. 28], when he used it to demonstrate William Hanf's result that Boolean algebras need not have unique roots, in the sense that $\boldsymbol{A}^2 \cong \boldsymbol{B}^2$ need not imply $\boldsymbol{A} \cong \boldsymbol{B}$ [**Hanf57**].

Representing Boolean algebras both as rings of sets and topological spaces stimulated a number of mathematicians. Over a two-year period, 1937–1938, several papers appeared, apparently independently of each other, giving subdirect representations of special classes of commutative regular rings in terms of fields, without assuming either the ascending or descending chain conditions on ideals. The first result of this kind was the theorem of Neal McCoy and Deane Montgomery, who proved that any p-ring (commutative, $px = 0$ and $x^p = x$) is a subdirect product of prime fields $\mathbb{Z}_p$ [**McCMo37**]. A theorem of this type without regard to characteristic is due to McCoy for commutative rings: any commutative, von Neumann regular ring is a subdirect product of fields [**McCoy38**]. More generally, McCoy, using a lemma of [**Krull29**], showed that any commutative ring without nilpotent elements is a subdirect product of integral domains. Since each integral domain is embeddable in a field, it follows that any such ring is a subring of a direct product of fields. Birkhoff systematized the presentation of such results by proving a lemma suggested by McCoy: a subdirectly irreducible commutative ring without nilpotents is a field [**Birk44**]. Shortly thereafter, Alexandra Forsythe and McCoy extended this result to the noncommutative case: any regular ring without nilpotents is a subdirect product of division rings [**ForMc46**].

Since we will be talking about variants of regular rings shortly, we should mention the relationship between regularity and nilpotents in the case of a commutative ring $\boldsymbol{R}$. On the one hand, it is easy to show that if $\boldsymbol{R}$ is regular, then it has no nilpotent elements. On the other hand, by the result of McCoy above, if $\boldsymbol{R}$ has no nilpotent elements, then it is a subring of a product of fields, which are always regular. Hence, if $\boldsymbol{R}$ has no nilpotent elements, then $\boldsymbol{R}$ is embeddable in a regular ring, still having no nilpotents.

Richard Arens and Irving Kaplansky give examples showing that, in the noncommutative case, biregular rings and regular rings are independent notions [**AreKa48**]. They prove in their theorem 6.2 that, if $\boldsymbol{A}$ is an algebra over the field $\boldsymbol{GF}(p)$ in which every element a of A satisfies the equation $a^{p^n} = a$ for a fixed n, then there is a locally compact, zero-dimensional space $\boldsymbol{X}$ with a homeomorphism σ for which $\sigma^n = 1$, such that $\boldsymbol{A}$ is isomorphic to the algebra of all continuous functions f from $\boldsymbol{X}$ to $\boldsymbol{GF}(p^n)$ that vanish outside a compact set and respect σ: $f(\sigma x) = [f(x)]^p$. The authors go on to concoct a counterexample built around $\boldsymbol{GF}(4)$ showing that the equation, $a^4 = a$, is necessary.

Reinhold Baer studied the condition that each annihilator is generated by a projection, which will be significant in some of our applications [**Baer52**]. Somewhat earlier, the application of topological methods to non-Boolean rings began with Israel Gel'fand and George Šilov generalizing the Stone topology of prime ideals in Boolean rings to commutative normed rings [**Gel'Ši41**]. Shortly thereafter, Nathan Jacobson generalizes the Stone topology by adapting it to the set of primitive ideals in an arbitrary ring, not necessarily with a unity [**Jaco45**]. It is no longer Hausdorff; but if there is a unity, it will still be compact.

The writing of a history of representing general algebras is complicated by the history not being linear. Instead, the history may be thought of as more like a braided stream with many strands and rivulets, some of them running in parallel, some bifurcating, and others merging back together. Certainly, to judge from the references absent in published papers, there must have been considerable independent effort. How to identify these strands and how many to pay attention to are matters of opinion. In any case, a strictly chronological account would be confusing and misleading. So we must often follow one strand for a while, then back up in time to pursue another. We now go back to look at the origins of universal algebra.

Alfred North Whitehead, in writing his book on universal algebra, also had a lofty but less sweeping goal than Leibniz: he wished to create a theory of algebra capable of unifying and comparing the many linear algebras that had been proposed in the nineteenth century [**Whit98, Fear82**].

We follow the current view that the concept of universal algebra as it is recognized today, despite Whitehead's treatise by the same name, began with the two seminal papers of Garrett Birkhoff, who showed that there were significant theorems simultaneously covering groups, rings, fields and vector spaces as well as lattices and Boolean algebras. Birkhoff formulated the concept of a general algebraic system as we know it today [**Birk35**]. In his next paper on the subject [**Birk44**], Birkhoff presented the theorem that every algebra is a subdirect product of subdirectly irreducible algebras. This theorem is fundamental to our purposes and illustrates how the finitary nature of algebra makes for a good theory with many applications.

Another stream flowed into universal algebra from logic via Emil Post's generalization of classical two-valued logic [**Post21**]. Post algebras, as defined by Paul Rosenbloom, are to Boolean algebras as n-valued logic is to two-valued logic [**Rosen42**]. Following close upon [**Birk44**], L. I. Wade established that any Post algebra is a subdirect power of a primal Post algebra [**Wade45**], Rosenbloom having proven this first for finite Post algebras.

The work of Alfred Foster, influenced by that of Wade and McCoy, was a watershed in the way we view representations of algebras [**Fost53**].

Foster realized that a significant property for an algebra to be the kernel in subdirect representations is primality: all operations on the carrier can be composed from the fundamental operations. This gave primal algebras of all finite cardinalities; each was the sole subdirectly irreducible algebra of the equational class generated by it. Foster identified a Boolean part in these primal algebras that could be extended into representing the other algebras of the class. This is achieved by creating Boolean partitions over a primal algebra and convolving these partitions to define operations.

Primality was seminal and a central strand in the evolution of decomposition theorems through a sequence of papers of Stone, Foster, and his students, leading from Boolean algebras through primal algebras to a diversity of generalizations, such as semiprimal and hemiprimal algebras, which would produce analogous constructions in the varieties generated by them. The kernels of such representations no longer need to have all operations derivable from the fundamental operations, but only those preserving some prescribed structure, such as subalgebras or congruences. (For a history of these variations, see the surveys of Robert Quackenbush [**Quac79**] and Alden Pixley [**Pixl96**].) The class of algebras being represented need not look, upon first glance, at all like the traditional classes of rings, groups or lattices, either in the type of operations or in the identities they satisfy. Even when the underlying primal algebra only two elements, this kernel may look superficially very different from the two-element Boolean algebra; for example, the Sheffer stroke, a functionally complete binary operation, satisfies many unusual and unexpected identities; hence, so does the equational class it generates. Foster's work opened up new vistas, beckoning us to try to find structural clues independent of the usual operations in which rings and lattices are defined.

Tah-Kai Hu put Foster's work into a categorical setting by extending Stone's duality between Boolean algebras and Boolean spaces to a natural equivalence between the category of all algebras satisfying the identities of a given primal algebra and the dual category of Boolean spaces [**Hu69**]. Actually, this was done in the more general setting of locally primal algebras, which is a generalization of primality to infinite algebras. Joachim Lambek and Basil Rattray gave a categorical proof by means of adjoint functors [**LamRa78**].

Another strand in the unfolding of universal algebra also comes from logic, both in the more general setting of relational structures, as well as in the desire to represent logics as algebras. New algebraic systems discovered outside universal algebra gave impetus to solving problems within it; very important among these were the cylindric algebras of Alfred Tarski, designed to provide an algebraic model of first-order calculus, the next step after modeling the sentential calculus by Boolean algebra [**HenkMT71**]. Once a logic is captured algebraically, we want to know how good the match is. A natural answer to seek is an analog to Stone's theorem: every algebra in the class should be a subdirect product of 'primitive' algebras defined directly from the logic. Examples are the multi-valued algebras invented by Chen-Chung Chang [**Chang58**], and the already mentioned Post algebras studied by Rosenbloom [**Rosen42**]. Helena Rasiowa's book describes many such logics turned into algebra [**Rasi74**].

A Johnny-come-lately in our history of algebra is the theory of sheaves. What is missing in the construction of a subdirect product is a criterion for determining whether an element of the full product belongs to the subdirect product or not. So we implant topologies into subdirect products, wherever possible, creating a sheaf space, and incorporate the Boolean part of an algebra into the topology of the index set, making a base space. Any element of the subdirect product must be, among other things, a continuous function from the base space to the sheaf space; this adds a coherence to subdirect products otherwise lacking.

But the roots of sheaf theory itself are deeper and earlier – they may be found in the works of Henri Cartan [**Cart49**], Jean Leray [**Leray50**], Jean-Pierre Serre [**Serre55**], Roger Godement [**Gode58**], Alexander Grothendieck and Jean Dieudonné [**GroDi60**], and Armand Borel [**Borel64**]. These pioneers used sheaves in algebraic topology (Poincaré duality), complex analysis (De Rham's theorem), algebraic geometry (Riemann–Roch theorem), and differential equations (distributions). The history of sheaves is sketched by John Gray [**Gray79**], Christian Houzel [**Houz98**], and Concepción Romo Santos [**Romo94**].

These early papers and books provided the impetus for other workers to solve problems in algebra by means of sheaves. For example, John Dauns and Karl Hofmann [**DauHo66**], to get around the counterexample of Arens and Kaplansky [**AreKa48**] built out of a single finite field, introduced sheaves and obtained the following theorem. Every biregular ring, not necessarily unital, is isomorphic to the ring of global sections with compact supports in a sheaf of simple unital rings; the base space is locally compact, totally disconnected, and Hausdorff. Further, if the original ring has a unity, then the base space is also compact and hence Boolean. Most importantly, any number of different rings may appear as stalks in the same sheaf.

The memoir of Richard Pierce has many worthwhile results [**Pier67**]. In particular, his theorem 6.6 gives a categorical equivalence between the categories of rings (the homomorphisms must preserve central idempotents) and their reduced sheaves. Pierce also gives in his lemma 4.2 a sufficient condition for a sheaf to be reduced: when the stalks are directly indecomposable; if the rings are commutative this condition is also necessary.

Joseph Kist proves for a commutative ring $\boldsymbol{R}$ that, if the space $\boldsymbol{X}$ of minimal prime ideals is compact and $\boldsymbol{R}$ has no nilpotents, then $\boldsymbol{R}$ is isomorphic to a subring of the ring $\boldsymbol{\Gamma}(\boldsymbol{\mathcal{A}})$ of all global sections of a sheaf $\boldsymbol{\mathcal{A}}$ over $\boldsymbol{X}$ in which the stalks are integral domains [**Kist69**]. Further, when $\boldsymbol{R}$ is a Baer ring, then this isomorphism is onto $\boldsymbol{\Gamma}(\boldsymbol{\mathcal{A}})$. Carl Ledbetter, by considerably different methods, shows that this last result is still true even when $\boldsymbol{R}$ is noncommutative [**Ledb77**]. Hofmann surveys much that is known about sheaves of rings [**Hofm72**]. To generalize such results beyond rings, new techniques are needed, which are discussed next.

Comer realized that the construction of the Pierce sheaf for rings could be extended to a rather broad class of algebras [**Comer71**]. These are algebras whose factor congruences form a Boolean algebra. Besides rings, there are lattices, semilattices, and the shells of this book. This came out of Comer's investigation into the question of the decidability of the theory of cylindric algebras [**Comer72**]. Stanley Burris and Ralph McKenzie make some unique comments on this portion of our history [**BurMc81**, pp. 15–20, 67–70].

Brian Davey pushed the work of Comer further by realizing that all we need in order to obtain a sheaf over a Boolean space is a Boolean sublattice of congruences [**Davey73**]. In this very general set-up, there is the Gel'fand morphism, named for his work [**Gel'f41**], that takes the original algebra to the algebra of all global sections of the sheaf; it is injective, but not necessarily surjective. Davey notes further, however, that if we start with a Boolean sublattice of *factor* congruences, then the Gel'fand morphism is indeed surjective. Since most of the remaining contributors to the unfolding of sheaves in general algebra in the 1970s will be discussed more fully in later chapters, we only list them here: Klaus Keimel [**Keim70**], Maddana Swamy [**Swam74**], Albrecht Wolf [**Wolf74**], William Cornish [**Corn77**], and Peter Krauss and David Clark [**KraCl79**]. For a survey see [**Keim74**].

Another area with many examples of interest is this problem: for which equational classes can one express each algebra as a Boolean product of a finite number of finite algebras, depending only on the original class? Unfortunately, as shown by the work of Burris and McKenzie, this traditional situation in both classical and universal algebra – a Hausdorff sheaf over a Boolean space with a finite number of finite stalks – is limited by the generators having to be simple Abelian or quasiprimal [**BurMc81**].

Sheaves have proved their worth in model theory by establishing theorems in decidability, elementary equivalence and embedding, preservation and transfer properties, and model completeness. See Sect. XII.3 for definitions of these concepts and for some sample theorems.

Categories are another general concept useful to us; they help to systematize equivalences among diverse classes of mathematical objects. Samuel Eilenberg and Saunders MacLane [**EilMa45**] formally presented this concept, although many examples were already known informally before then. What will be historically significant for us are three such categorical equivalences: Stone's duality between Boolean algebras and what are now called Boolean spaces [**Stone37**], Pierce's equivalence of the category of rings with conformal homomorphisms and the category of their sheaves [**Pier67**], and Hu's equivalence of primal varieties, mentioned earlier [**Hu69**]. See [**MacL65**] for early developments and applications of this rapidly growing subject.

Modern analysis, and indirectly general topology, has been a source of inspiration for the ideas in this monograph. Early on we have John von Neumann's paper on rings of operators [**vonN36**]. A later influence on our efforts came from functional analysis, where Melvin Henrikson and Meyer Jerison [**HenrJe65**] and Kist [**Kist69**] systematically exploited the spectrum of minimal prime ideals. Kist had moved from functional analysis to commutative ring theory.

It is easy to appreciate the significance of functional analysis for the type of theorems we are heading toward. The ring $\boldsymbol{C}(\boldsymbol{X})$ of all continuous functions from a topological space $\boldsymbol{X}$ to the real numbers $\mathbb{R}$ has the appearance of a sheaf space $\boldsymbol{X} \times \mathbb{R}$, with the obvious projection $\pi : \boldsymbol{X} \times \mathbb{R} \to \boldsymbol{X}$, where $\boldsymbol{X} \times \mathbb{R}$ is given the product topology. Thus, it is already decomposed into a subdirect product whose stalks $\mathbb{R}$ have no divisors of zero. But its similarity to a sheaf is only that, for the technical condition that π be a local homeomorphism fails; related to this failure is the fact that each stalk $\mathbb{R}$ is not discrete. (Sometimes we speak of $\boldsymbol{X} \times \mathbb{R}$ as being merely a 'bundle'.) Thus, we might seek a different factoring of $\boldsymbol{C}(\boldsymbol{X})$. Marshall Stone has some intriguing comments in [**Stone70**, p. 240] about how this interest in $\boldsymbol{C}(\boldsymbol{X})$ shifted to concerns with representing algebras by sets of functions subject to certain constraints. See also the selection of essays [**Aull85**], edited by Charles Aull, for some current views on $\boldsymbol{C}(\boldsymbol{X})$ as an algebraic object. Hofmann has additional comments in [**Hofm72**, pp. 295–296] on the influence of functional analysis on the evolution of sheaf theory towards ring theory.

As a beginning to the book, this tour through the evolution of ideas leading to sheaves is over. Now we are ready to delve into what this book covers – both known theorems and new ones.

2. Survey of Results

With the historical background of Sect. 1 in mind, we now briefly describe the principal results of this monograph, chapter by chapter, omitting minor caveats. The flow of this book as a whole is first to develop tools for constructing sheaves that are helpful in understanding the structure of general algebras, specializing as needed, with applications in the middle chapters, and finally to close with a backward glance at how some of the earlier theorems might be extended.

Chapter II lays out the traditional background needed from general algebra. In Sect. 1, there is one novelty: 'sesquimorphisms' as a substitute for congruences. The three isomorphism theorems are presented both conventionally and in terms of sesquimorphisms.

Section 2 introduces direct and categorical products, and then studies the five kinds of factor objects that may identify products internally: bands, congruences, sesquimorphisms, ideals, and elements; the last four coming in complementary pairs (Theorems II.2.5, II.2.12, II.2.19).

Chapter III outlines the concepts and theorems needed from several disciplines: equational logic, categories, topology, and Boolean algebras, including Stone's representation of these, the grandfather of many of the theorems in this book and an essential tool for proving them.

Chapter IV sets the stage for the book proper by introducing the notion of a complex and showing that it always gives a sheaf of algebras. Crucial to decomposing an algebra as a subdirect product of quotient algebras is a measure of how close or far apart the elements of the algebra are. Complexes originated in the theory of rings and modules, and they are the algebraic analog of metric spaces. A metric now becomes a binary operation from the carrier of an algebra, not necessarily going to the real numbers, but taking as values open sets in some topological space. This binary operation satisfies axioms similar to both those for a metric space and those for a congruence preserving the operations of an algebra. Complexes are an intermediate step on the way to sheaves.

Sheaf constructions next illustrate how a well-developed topological tool can shed light on a principally algebraic device. Out of each complex, one constructs a sheaf whose algebra of all global sections contains a subalgebra isomorphic to the original (Theorem IV.2.1); arguments common to this construction can now be made once and for all in the context of complexes. Sheaves have proven their value in many situations, and here they will also do so.

We also look at systems of congruences from which one obtains a subdirect product. This is proven equivalent to the notion of a complex whenever the underlying topology is T_0 and the equalizers of global sections form a subbasis (Theorem IV.2.5).

Another concept is the 'Hausdorff sheaf', where the sheaf space is T_2. A sheaf being Hausdorff is equivalent to equalizers being clopen and the base space being Hausdorff (Proposition IV.2.9). When the base space is also a Boolean space, we have the well-studied notion of a 'Boolean product'.

The constructions of this chapter are set into an adjoint situation between the categories $\mathfrak{Complex}$ and $\mathfrak{Sheaf}$ for a given algebraic type (Theorems IV.3.15 and IV.3.18). The functors and natural transformations entering into this adjoint situation will be successively specialized in subsequent chapters.

Another way to capture the separation of two elements of an algebra is through a congruence by which they are not related. The typical situation introduced in Chap. V, which will occur repeatedly throughout this monograph, is where a subset of congruences separating all elements is singled out for special attention. We want to pick a set of congruences that is appropriate to the algebra at hand and to the aspects of it we wish to study. It is noteworthy that usable sets of congruences need not be sublattices of the lattice of all congruences. Such sets need only be closed to intersection and a complementation respecting the partial ordering of inclusion on congruences. This fragment, to be called a 'Boolean subsemilattice', will be a complemented distributive lattice, in which the join operation may be greater than that in the complete lattice of all congruences.

To set things up for later ideas covered in the book, we prove Theorem V.2.1, which states that any Boolean subsemilattice of an algebra determines a complex over a Boolean space, which in turn determines a sheaf of algebras over the same space. The original algebra will be a subalgebra of the algebra of all global sections of a sheaf of quotient algebras. In general, the larger the subsemilattice, the more numerous are the stalks of the sheaf; and the larger the congruences themselves, the smaller the quotients. But the quotients do not come directly from the congruences of the Boolean subsemilattice. Instead, the essential construction behind this theorem is to look at the Boolean space of prime ideals in this selected Boolean algebra of congruences. Each prime ideal has a supremum that is again a congruence in the given algebra, although not usually in the Boolean subsemilattice. These suprema are the points of the Boolean space over which floats the sheaf space of stalks, which are the quotient algebras by the suprema. The continuous cuts through the sheaf space are the global sections. The mapping of elements of the original algebra into them is called the Gel'fand morphism.

But the converse is also true: we prove in Theorem V.2.9 that every representation of an algebra by a sheaf of algebras over a Boolean space must arise by the previous construction from some Boolean subsemilattice of congruences. As one is free to choose the Boolean subsemilattice, so one is also free to choose the nature of the quotient algebras, and thus to tailor the extent of their indecomposability. For example, discovering the right congruences will factor out divisors of zero in shells.

The patchwork, partition, and interpolation properties associated with sheaves over Boolean spaces make the global sections of such sheaves especially malleable. Even easier to work with are the more specialized Boolean products, which have been used to good advantage in universal algebra, and which are briefly looked at for the sake of comparison. Also included in Sect. V.3 are Boolean powers, Boolean extensions, and Hausdorff sheaves.

Introduced at the end of Chap. V is the category $\mathfrak{BooleBraceRed}$ of reduced Boolean braces – they consist of an algebra and a selected Boolean subsemilattice of congruences. This category forms an adjunction with the category $\mathfrak{CompBooleRed}$ of reduced complexes over Boolean spaces

(Theorem V.4.14). In turn, this last category is a full subcategory of the category of all complexes over arbitrary topological spaces, and thus, this adjunction, when composed with the adjunction of the last chapter, forms an adjunction of $\mathfrak{BooleBraceRed}$ with the category $\mathfrak{SheafBooleRed}$ of reduced sheaves over Boolean spaces, which is a full subcategory of $\mathfrak{Sheaf}$ (Theorem V.4.17). Reduction limits the number of trivial stalks in a sheaf and its related structures.

As the set of all congruences of an algebra is a lattice, it is natural to consider sublattices. Of special interest in Chap. VI are those congruences θ having a complementary partner θ' in the sense of forming a factorization: $\boldsymbol{A} \cong (\boldsymbol{A}/\theta) \times (\boldsymbol{A}/\theta')$. Davey [**Davey73**] considered a Boolean sublattice of commuting (= permuting) factor congruences – this sublattice together with its algebra we call a 'factorial brace'. As Boolean lattices are equivalent to Boolean algebras, one has a Boolean subsemilattice, the previous situation. Thus, one obtains a sheaf over a Boolean space. But now we have an isomorphism: Theorem VI.1.8 states that the algebra of global sections of this sheaf is isomorphic to the original algebra, that is, the Gel'fand morphism is also surjective, not just injective, as in Chap. V.

This set-up is important enough to warrant a section devoted to characterizing Boolean algebras of factor congruences alternatively by factor bands and sesquimorphisms.

Comer postulated in his paper [**Comer71**] that all the factor congruences form a distributive sublattice, that is, a Boolean sublattice of the lattice of all congruences, which is described as the algebra having 'Boolean factor congruences' (BFC). For many algebras occurring in practice, such a condition is easy to check. Section 3 characterizes their sheaves in Theorem VI.3.15 as those that are 'reduced' and 'factor-transparent'. This is called the 'canonical' sheaf representation of an algebra with BFC. Historically these results of Comer came before those of Chaps. IV and V, but it is now easier to present them as a special case of those earlier chapters. But not all algebras have BFC. Theorems VI.3.2 and VI.3.9 give many conditions equivalent to an algebra having Boolean factor congruences.

As was done in the previous two chapters, the final section of this chapter recasts its achievements in categorical terms. If a Boolean brace is taken into $\mathfrak{Sheaf}$ and then back again to the algebra of global sections, then the new Boolean brace has additional properties. The new Boolean subsemilattice now has a distributive sublattice of commuting factor congruences, creating a factorial brace as examined in the previous paragraph. All reduced factorial braces constitute the category $\mathfrak{FactorBraceRed}$, which is isomorphic to a full subcategory of $\mathfrak{BooleBraceRed}$. Most importantly, $\mathfrak{FactorBraceRed}$ is categorically equivalent to $\mathfrak{SheafBooleRed}$, by Theorem VI.4.2; thus we extend Stone's representation theorem for Boolean algebras.

Additionally, if the set of *all* factor congruences is a Boolean sublattice of **Con** $\boldsymbol{A}$, then these algebras constitute the category $\mathfrak{AlgBFC}$ of algebras with Boolean factor congruences. By Theorem VI.4.5 it is isomorphic to the category of reduced and factor-transparent sheaves over Boolean spaces. Table VI.1 lists the many categories of structures and sheaves that occur in this book and the various adjunctions and equivalences that exist among them. Figure 1 summarizes in a Venn diagram the various levels of generality considered so far, as well as the shells to be discussed next.

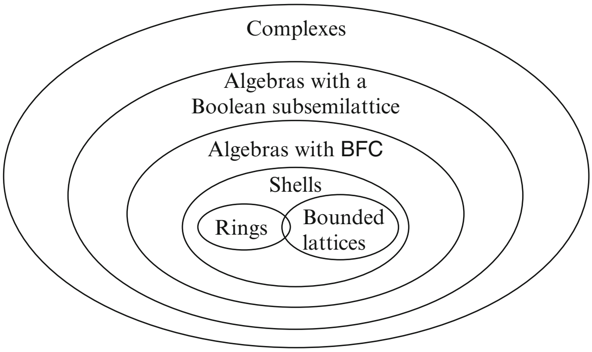

FIGURE 1. Kinds of algebras with a ready-made sheaf structure

In the heart of this monograph, Chap. VII introduces the notion of a 'unital shell', which is an algebra $\langle A; +, \times, 0, 1 \ldots, \omega, \ldots \rangle$ with two binary operations $+$ and $\times$, two constants 0 and 1, and other arbitrary operations ω as desired, in which no identities need hold other than what is expected of what we call a nullity and unity:

$$0 + a = a = a + 0, \qquad 0 \times a = 0, \qquad 1 \times a = a. \tag{2.1}$$

The remaining operations ω, if any, need not have any relationship to the first four. Clearly, a nullity is needed in order to talk about divisors of zero, which will appear in the Chap. VIII. Although addition is not always needed, a unity for multiplication appears to be needed to obtain our results on factor objects.

Examples abound: rings and linear algebras with a unity, Boolean algebras with operators, as well as bounded lattices and trellises, perhaps also with operators. The sparse identities of (2.1) provide all we need to mimic the factorization of rings by central idempotents. This definition is that of 'unital shell' in a strict sense; but we will also use 'shell' in a loose sense to refer to various weakenings of this definition. By omitting $+$, we prune it to 'unital half-shell'. Examples are bounded semilattices, and more generally, monoids with a nullity. Much of the theory still holds for even weaker shells, and this will now be explained in detail.

We start off this chapter with sesqui-elements and sesquishells, temporary concepts leading us to prove BFC in Theorem VII.1.10 with the weakest hypothesis possible in the context of shells, with the sheaf being reduced and factor-transparent. The slightly stronger unital half-shells also have BFC, and are studied in Sect. 2. Theorems tie factor elements to sesqui-elements. Within a unital half-shell, the set of its factor elements is now a Boolean algebra, anti-isomorphic to the Boolean algebra of factor congruences (Theorem VII.2.15).

Products were captured internally in Sect. II.2 by factor congruences, bands, and sesquimorphisms. In unital shells, we add to this list complementary pairs of factor ideals and complementary pairs of factor elements. In Sect. 3, the concept of unital shell is sufficiently stronger to support a characterization of factor elements solely in terms of factor identities, as given in Theorem VII.3.4. Better still, Theorems VII.3.7 and VII.3.14 characterize the inner direct product of ideals independently of the other factor objects, the latter theorem becoming the traditional definition of inner product in unital rings.

In Sect. 4, we explore the one-to-one correspondences between the five kinds of factor objects in two-sided unitary shells. (A unitary shell is 'two-sided' if, in addition to (2.1), the equations, $a \times 0 = 0$ and $a \times 1 = a$, also hold.) Each factor ideal is the 0-coset of a factor congruence, and the congruence is uniquely determined by its 0-coset. Each factor ideal is generated by a factor element, which serves as a relative unity, and conversely, each factor element generates a factor ideal. In a unital shell, the set of all factor ideals forms a Boolean algebra, and thus, so do the factor elements, and these Boolean algebras are isomorphic or anti-isomorphic to the Boolean algebras of the previously defined factor objects. Formulas are developed for these correspondences and the Boolean operations.

But not all algebras with Boolean factor congruences have factor elements. 'Separator algebras', generalizing shells, are introduced as a device for proving that any algebra $\boldsymbol{A}$ with BFC is embeddable in a ring or lattice, whose new factor elements capture the factorizations of $\boldsymbol{A}$ where there were none before (Theorem VII.5.5).

The category $\mathfrak{UnitShell}$ of unital shells of a given type is a full subcategory of $\mathfrak{AlgBFC}$ of the same type. Theorem VII.6.4 and its corollary establish that this new category is isomorphic to the category of reduced and factor-transparent sheaves of unital shells over Boolean spaces. The morphisms of $\mathfrak{UnitShell}$ are characterized as those homomorphisms, called 'conformal', that take factor elements into factor elements. Similar and equivalent categories also exist for the more general unital half-shells and their sheaves.

One of the high points of this monograph is the generalization to unital shells of Kist's theorem [**Kist69**] on the decomposition of commutative

Baer unital rings into a sheaf of integral domains over a Boolean space, which in turn is a generalization of the classical result that every von Neumann regular, commutative, unital ring is a subdirect product of fields. This application in Chap. VIII illustrates the power of sheaves over spaces constructed out of factor elements, and for which the previous chapters have prepared the reader. Here, the adjective 'Baer' originally defined for rings, becomes 'Baer-Stone' to include 'Stone' lattices: the annihilator of any element a is generated by a single factor element e, that is, the annihilator is a principal ideal:

$$a^{\perp} = \{b \in R \mid ab = 0\} = [e].$$

Theorem VIII.1.13 then states that every two-sided unital half-shell that is Baer-Stone has a canonical sheaf representation where the stalks are integral, that is, they have no divisors of zero. Here we apply the crucial fact that an ideal is integral[2] if it is associated with a congruence that is the supremum of a prime ideal of factor congruences.

The categorical interpretation of Chap. VII can be further specialized to this result: the category of Baer-Stone two-sided unital shells with conformal homomorphisms is categorically equivalent to the category of sheaves of integral shells ('integral' means no divisors of zero). This theorem can also be phrased outside of the language of sheaves. Each Baer-Stone shell is isomorphic to a subdirect product of integral shells.

The biregular rings of Arens and Kaplansky [**AreKa48**], and Dauns and Hofmann [**DauHo66**] present another situation that can be generalized; this was extended to near-rings by [**Szeto77**]. But we extend it further to unital half-shells; they are called biregular if every principal ideal is generated by a factor element. Theorem VIII.2.3 then reads essentially as it does for the classical case: a biregular unital half-shell is isomorphic to the half-shell coming from a reduced and factor-transparent sheaf over a Boolean space with simple stalks (providing certain technical conditions hold).

The sheaf representations of the last two chapters were all surjective, that is, each continuous section represents some element in the original algebra. Chapter IX relaxes this; an algebra now may be represented merely as some subalgebra of the algebra of all global sections of some sheaf. This means that the hypotheses of the last chapter, such as being Baer-Stone, may also be relaxed.

[2] This means that if a product of two elements is in the ideal, then one of the factors must be in there also. In commutative rings, this is synonymous with being a prime ideal.

In the first section of Chap. IX, to achieve any results on integrality, one must first study shells without an addition, a unity, or additional operations; I dub these new algebras ‘strict half-shells’. Also, one must expect the multiplication to satisfy certain nilpotent conditionals, too involved to state here. We then prove in Theorem IX.1.3 that any such half-shell is isomorphic to a half-shell of some of the global sections of a sheaf over a Boolean space of half-shells without divisors of zero.

In the second section, consequences of this theorem are explored. We abstract its conclusion by calling a half-shell *semi-integral* if it is a subdirect product of half-shells without divisors of zero, and give several equivalent formulations of semi-integrality, for example, Theorem IX.2.3.

In the third section, it is further assumed that the strict half-shell has a unity and consequently every factor ideal is principal, which leads to an especially transparent form for the factor elements when the half-shell is semi-integral. Then this section returns to shells where analogous results hold. But now it is necessary to make some additional assumptions: + satisfies the loop laws; + is distributive over ×; and × satisfies the nilpotence conditions mentioned above. Theorem IX.3.8 tells us that such a strict shell is isomorphic to a subshell of the Baer-Stone shell coming from a sheaf over an extremally disconnected base space whose stalks are integral. In some detail, we trace the relationship of this result to some older theorems in ring theory.

Chapter X starts off with a new proof of the classical result that any algebra in the variety generated by a primal algebra is isomorphic to the algebra of all global sections of a sheaf over a Boolean space all of whose stalks are the primal algebra. Recall that a primal algebra is a finite nontrivial algebra whose operations lead by composition to all finitary functions on the carrier. It is natural to seek other algebras close to primality whose generated varieties will have nice sheaf representations. The preprimal algebras do not disappoint us.

An algebra is *preprimal* if it is one step away from being primal, that is, if any one function not composed from its operations is added to them, then the new algebra is primal. The preprimal algebras fall naturally into seven classes, identified by relations that all their operations preserve. Most of the varieties generated by them have BFC (see Table X.1). For three of the classes we find the stalks in the sheaves of their algebras: those coming from a preprimal preserving certain permutations (Theorem X.2.1); those preserving certain Abelian groups (Theorem X.3.2); and those preserving a proper subalgebra (Theorem X.4.1).

In Chap. XI, we attempt to get away from the language of shells and half-shells, and return to arbitrary algebras, in so far as possible; there are two independent sections.

The first section iterates our sheaf construction for a shell until all quotients have become directly indecomposable. In a commutative unital ring, the stalks are always directly indecomposable. As surprising as it may seem, when noncommutative, the stalks need not be directly indecomposable. Walter Burgess and William Stephenson [**BurgSt78**] took this opportunity for unital rings to iterate the construction of Pierce sheaves on the stalks themselves until it could be pushed no further. This forces the ultimate 'factors' to be directly indecomposable; but we may no longer have a sheaf, only a subdirect product. Theorem XI.1.2 adapts this iterative construction to general algebras in varieties with Boolean factor congruences.

The second section looks at the lattice of congruences as a shell and considers how its decomposition might lead to a decomposition of the algebra itself. The crucial observation upon which Theorem XI.2.3 is based is that the factor elements of the congruence lattice form a Boolean lattice; hence, we can obtain its associated Boolean space. This, in turn, induces a sheaf, and so the Gel'fand morphism maps elements of the given algebra to some of the global sections.

As this book was being written, it became clear that there were many related topics to be pursued, and many tempting trails on which to venture. The techniques introduced in this monograph might well be extended in any number of directions. In order to draw this book to a close, rather than try to develop these ideas in detail, Chap. XII outlines additional applications, without proofs; these point to five regions ripe for research, beyond what is already known. The first application wants to extend the sheaf representations in classical ring and lattice theory to shells and beyond. The second considers algebras derived from logic. The third is about model theory: preservation of properties, decidability, and model completeness. The fourth weakens the metrics of complexes and the topology of Boolean spaces. The fifth ranges over the diverse sheaves that may exist for a particular algebra.

This introductory chapter delineates the scope of this book. It could be summarized by saying that there are two approaches to decomposing algebras by sheaves: (1) take one large algebra at a time and decompose it into smaller pieces with a sheaf; and (2) take one small algebra, or a finite collection of finite algebras, and decompose all the algebras in the variety generated by them. This book concentrates mostly on (1) and only on (2) in Chap. X about preprimals.

Outlined below is what might have been covered but was not, and a few topics that are introduced but not pursued at length since there are already excellent monographs covering these.

Relational structures are not included; most of our examples are algebras, or they can be made into them, such as lattice-ordered groups. Other structures omitted are several-sorted algebras, which have more than one carrier,