Wave Propagation in Fluids

Models and Numerical Techniques

Vincent Guinot

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Introduction

What is wave propagation?

In a kitchen or in a bathroom, the number of times we turn a tap every day is countless. So is the number of times we watch the liquid stream impacting the sink. The circular flow pattern where the fast and shallow water film diverging from the impact point changes into a deeper, bubbling flow is too familiar to deserve attention. Very few people looking at the circular, bubbling pattern – referred to as a hydraulic jump by the specialists of hydraulics – are aware that they are contemplating a shock wave.

Turning off the tap too quickly may result in a thud sound. This is the audible manifestation of the well-known water hammer phenomenon, a train of pressure waves propagating in the metal pipes as fast as hundreds to thousands of meters per second. The water hammer phenomenon is known to cause considerable damage to hydropower duct systems or water supply networks under the sudden operation of valves, pumps or turbines. The sound is heard because the vibrations of the duct system communicate with the ambient atmosphere, and from there with the operator’s ears.

Everyone has once thrown stones into the water in a pond, watching the concentric ripples propagate on the surface. Less visible and much slower than the ripples is the moving groundwater that displaces a pollutant front in a journey that may last for years.

As ubiquitous and familiar as wave propagation may be, the phenomenon is often poorly understood. The reason why intuition so often fails to grasp the mechanisms of wave propagation may lie in the commonly shared, instinctive

perception that waves are made of matter. This, however, is not true. In the example of the hydraulic jump in the sink, the water molecules move across an immobile wave. In the example of the ripples propagating at the free surface of a pond, the waves travel while the water remains immobile.

Waves appear when an object or a system (e.g. the molecules in a fluid, a rigid metallic structure) reacts to a perturbation and transmits it to its neighbors. In many cases, as in the example of the water ripples, the initially perturbed system returns to its initial equilibrium state, while the waves keep propagating. In this respect, waves may be seen as information. The ripples propagating in a pond are a sign that the water molecules “inform” their neighbors that the equilibrium state has been disturbed. A sound is nothing more than information about a perturbation occurring in the atmosphere.

Numerical techniques for wave propagation simulation have been the subject of intensive research over the last 50 years. The advent of fast computers has led to the development of efficient numerical techniques. Engineers and consultants now use simulation software packages for wave propagation on a daily basis. Whether it is for the purpose of acoustics, aerodynamics, flood wave propagation or contaminant transport studies, computer-based simulation tools have become indispensable to almost all domains of engineering. Such tools, however, remain instruments operated by human beings to execute tedious, repetitive operations previously carried out by hand. They cannot, and hopefully never will, replace the expert’s judgment and experience. Human presence remains necessary to the sound assessment of the relevance and accuracy of modeling results. Such an assessment, however, is possible only provided that the very specific type of reasoning required for the correct understanding of wave propagation phenomena has been acquired.

The main purpose of this book is to contribute to a better understanding of wave propagation phenomena and the most commonly used numerical techniques for its simulation. The first three chapters deal with the physics and mathematics of wave propagation. Chapters 4 and 5 provide an insight into more theoretical notions, used in specific numerical techniques. Chapters 6 and 7 are devoted to finite difference and finite volume techniques respectively. Basic notions of linear algebra and numerical methods are presented in Appendices A to C. The various formulae used in the present book are summarized in Appendix D.

What is the intended readership for this book?

This book is intended for the students of professional and research master programs and those engaged in doctoral studies, the curriculum of which contains hydraulics and/or fluid mechanics-related subjects. Engineers and developers in the field of fluid mechanics and hydraulics are also a potential target group. This book was written with the following objectives:

1)To introduce the physics of wave propagation, the governing assumptions and the derivation of the governing equations (in other words, the modeling process) in various domains of fluid mechanics. The application fields are as diverse as contaminant transport, open channel and free surface hydraulics, or aerodynamics.

2)To explain how the behavior of the physical systems can be analyzed using very simple mathematical techniques, thus allowing practical problems to be solved.

3)To introduce the main families of numerical techniques used in most simulation software packages. As today’s practising engineers cannot afford not to master modeling

packages, a basic knowledge of the existing numerical techniques appears as an indispensable engineering skill.

How should this book be read?

The chapters are divided into three parts:

– The first part is devoted to the theoretical notions applied in the remainder of the chapter.

– The second part deals with the application of these theoretical notions to various equations of hydraulics and fluid mechanics.

– The third part provides a summary of the key points developed in the chapter, as well as suggestions of application exercises.

The main purpose of the application exercises is to test the reader's ability to reuse the notions developed in the chapter and apply them to practical problems. The solution principle of the exercises may be accessed from the following URL: http://vincentguinot.free.fr/waves/exercises.htm.

Try to resist the temptation to read the solution immediately. Solving the exercise by yourself should be the primary objective. The solution text is provided only as a help, in case you cannot find a way to start and for you to check the validity of your reasoning after completing the exercise.

Chapter 1

Scalar Hyperbolic Conservation Laws in One Dimension of Space

1.1. Definitions

1.1.1. *Hyperbolic scalar conservation laws*

A one-dimensional hyperbolic scalar conservation law is a Partial Differential Equation (PDE) that can be written in the form

[1.1] $$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S$$

where t and x are respectively the time- and space-coordinates, U is the so-called conserved variable, F is the flux and S is the source term. Equation [1.1] is said to be the conservation form of the conservation law. The following definitions are used:

– The flux F is the amount of U that passes at the abscissa x per unit time due to the fact that U (also called the transported variable) is being displaced.

– The source term S is the amount of U that appears per unit time and per unit volume, irrespective of the amount transported via the flux F. If U represents the concentration in a given chemical substance, the source term may express degradation phenomena, or radioactive decay. S is positive

when the conserved variable appears in the domain, and is negative if U disappears from the domain.

– The conservation law is said to be scalar because it deals with only one dependent variable. When several equations in the form [1.1] are satisfied simultaneously, the term "system of conservation laws" is used. Systems of conservation laws are dealt with in Chapter 2.

Only hyperbolic conservation laws are dealt with in what follows. The conservation law is said to be hyperbolic if the flux F is a function of U (and none of its derivatives) and, possibly, of x and t. Such a dependence is expressed in the form

$$\left.\begin{aligned} F &= F(U,x,t) \\ S &= S(U,x,t) \end{aligned}\right\} \quad [1.2]$$

The function $F(U, x, t)$ is called the "flux function".

NOTE.– The expression $F(U, x, t)$ in equation [1.2] indicates that F depends on U at the abscissa x at the time t and does not depend on such quantities as derivatives of U with respect to time or space. For instance, the following expression

$$F = aU \quad [1.3]$$

is a permissible expression [1.2] for F, while the following diffusion flux,

$$F = -D\frac{\partial U}{\partial x} \quad [1.4]$$

where D is the diffusion coefficient, does not yield a hyperbolic conservation law because the flux F is a function of the first-order derivative of U with respect to space.

In the case of a zero source term, equation [1.1] becomes

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \quad [1.5]$$

In such a case (see section 1.1.2), U is neither created nor destroyed over the domain. The total amount of U over the domain varies only due to the difference between the

incoming and outgoing fluxes at the boundaries of the domain.

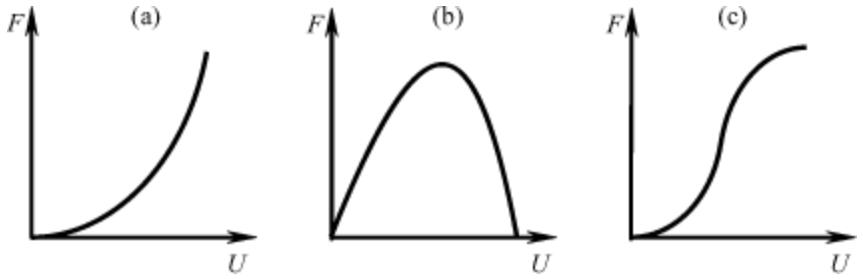
Depending on the expression of the flux function, the conservation law is said to be convex, concave or non-convex (Figure 1.1):

– The law is convex when the second-order derivative $\partial^2 F/\partial U^2$ of the flux function with respect to U is positive for all U.

– The law is concave when the second-order derivative $\partial^2 F/\partial U^2$ of the flux function with respect to U is negative for all U.

– The law is said to be non-convex when the sign of the second-order derivative $\partial^2 F/\partial U^2$ of the flux function with respect to U changes with U.

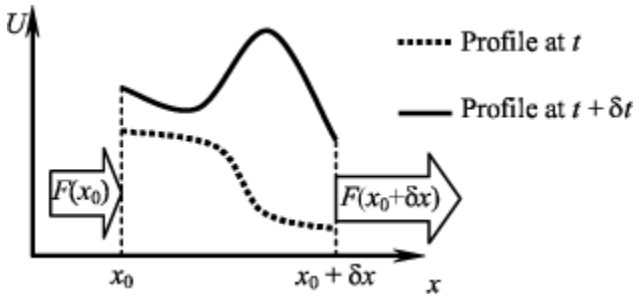
Figure 1.1. *Typical examples of flux functions: convex (a), concave (b), non-convex (c)*



1.1.2. *Derivation from general conservation principles*

The conservation form [1.1] is derived from a balance over a control volume of unit section defined between x_0 and $x_0 + \delta x$ (Figure 1.2). The balance is carried out over the control volume between two times t_0 and $t_0 + \delta t$. The variation in the total amount of U contained in the control volume is then related to the derivatives $\partial U/\partial t$ and $\partial F/\partial x$ in the limit of vanishing δt and δx.

Figure 1.2. *Definition sketch for the balance over a control volume*



The total amount $M(t_0)$ of U contained in the control volume at $t = t_0$ is defined as

$$M(t_0) = \int_{x_0}^{x_0+\delta x} U(x,t_0)\,\mathrm{d}x \quad [1.6]$$

At $t = t_0 + \delta t$, the total amount of U contained in the control volume is

$$M(t_0+\delta t) = \int_{x_0}^{x_0+\delta x} U(x,t_0+\delta t)\,\mathrm{d}x \quad [1.7]$$

The variation δS in the amount of U induced by the source term S over the domain between t_0 and $t_0 + \delta t$ is given by

$$\delta S = \int_{t_0}^{t_0+\delta t}\int_{x_0}^{x_0+\delta x} S(U,x,t)\,\mathrm{d}x\,\mathrm{d}t \quad [1.8]$$

The amount $\delta F(x_0)$ of U brought by the flux F across the left-hand side boundary of the control volume between t_0 and $t_0 + \delta t$ is given by

$$\delta F(x_0) = \int_{t}^{t+\delta t} F(x_0,t)\,\mathrm{d}t \quad [1.9]$$

A quantity $\delta F(x_0 + \delta x)$ leaves the domain across the right-hand side boundary

$$\delta F(x_0+\delta x) = \int_{t}^{t+\delta t} F(x_0+\delta x,t)\,\mathrm{d}t \quad [1.10]$$

Stating the conservation of U over the control volume $[x_0, x_0 + \delta x]$ between t_0 and $t_0 + \delta t$, the following equality is obtained

$$M(t_0+\delta t) = M(t_0) + \delta F(x_0) - \delta F(x_0+\delta x) + \delta S \quad [1.11]$$

Substituting equations [1.6–1.10] into equation [1.11] leads to

$$\int_{x_0}^{x_0+\delta x} [U(x, t_0+\delta t) - U(x, t_0)]\mathrm{d}x = \int_{t_0}^{t_0+\delta t} [F(x_0, t) - F(x_0+\delta x, t)]\ \mathrm{d}t + \int_{t_0}^{t_0+\delta t}\int_{x_0}^{x_0+\delta x} S(x,t)\ \mathrm{d}x\ \mathrm{d}t \quad [1.12]$$

A first-order Taylor series expansion around (x_0, t_0) gives

$$\left.\begin{aligned} U(x_0, t_0+\delta t) - U(x_0, t_0) &= \delta t \frac{\partial U}{\partial t} + O(\delta t^2) \\ F(x_0, t_0) - F(x_0+\delta x, t_0) &= -\delta x \frac{\partial F}{\partial x} + O(\delta x^2) \end{aligned}\right\} \quad [1.13]$$

where the quantities $O(\delta t^2)$ and $O(\delta x^2)$ are second- or higher-order polynomials with respect to δt and δx respectively. These polynomials contain the second- and higherorder derivatives of U and F with respect to t and x. When δt and δx tend to zero, the polynomial $O(\delta t^2)$ becomes negligible compared to the quantity $\delta t\ \partial U/\partial t$ because δt^2 decreases faster than δt. The polynomial $O(\delta x^2)$ becomes negligible compared to $\delta x \partial F/\partial x$ for the same reason. The relationships [1.13] thus become

$$\left.\begin{aligned} U(x_0, t_0+\delta t) - U(x_0, t_0) &\underset{\delta t\to 0}{\approx} \delta t \frac{\partial U}{\partial t} \\ F(x_0, t_0) - F(x_0+\delta x, t_0) &\underset{\delta x\to 0}{\approx} -\delta x \frac{\partial F}{\partial x} \end{aligned}\right\} \quad [1.14]$$

A similar reasoning leads to the following equivalence

$$\int_{t_0}^{t_0+\delta t}\int_{x_0}^{x_0+\delta x} S(x,t)\ \mathrm{d}x\ \mathrm{d}t \underset{\substack{\delta t\to 0 \\ \delta x\to 0}}{\approx} \delta t\ \delta x\ S \quad [1.15]$$

Substituting equations [1.14] and [1.15] into equation [1.12] leads to

$$\delta t \frac{\partial U}{\partial t} \delta x = -\delta x \frac{\partial F}{\partial x} \delta t + \delta t \delta x\ S \quad [1.16]$$

Dividing equation [1.16] by $\delta t\ \delta x$ yields the conservation form [1.1], recalled here

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S$$

The following comments can be made:

– The PDE [1.1] is a particular case of the more general, integral equation [1.12]. Equation [1.1] is obtained from equation [1.12] using the assumption that δt and δx tend to zero. Equation [1.12] is the so-called weak form of equation [1.1] (see Chapter 3 for more details).

– The conservation form [1.1] is based on the implicit assumption that F is differentiable with respect to x and U is differentiable with respect to t. Consequently, [1.1] is meaningful only when U is continuous in space and time. In contrast, equation [1.12] is meaningful even when U is discontinuous in space and/or time. This has consequences on the calculation of discontinuous solutions, as shown in Chapter 3.

1.1.3. *Non-conservation form*

Equation [1.1] can be rewritten in the so-called non-conservation form, which involves only derivatives of U. The non-conservation form of equation [1.1] is

$$\frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = S' \quad [1.17]$$

where λ is called the wave celerity, or wave propagation speed, and S' is a source term that may be identical (although not necessarily) to the source term S in equation [1.1]. Equation [1.17] is obtained from equation [1.1] by rewriting the derivative $\partial F/\partial x$ as

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial U}\frac{\partial U}{\partial x} + F' \quad [1.18]$$

where the term $F' = (\partial F/\partial x)_{U=\text{Const}}$ contains all the derivatives of F other than the derivative with respect to U. The expression of F being known, $\partial F/\partial U$ and F' are easily determined. Substituting equation [1.18] into equation [1.1] yields

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial U}\frac{\partial U}{\partial x} + F' = S \quad [1.19]$$

i.e.

$$\frac{\partial U}{\partial t}+\frac{\partial F}{\partial U}\frac{\partial U}{\partial x}=S-F' \quad [1.20]$$

Comparing equation [1.20] to equation [1.17] leads to the following definitions for λ and S'

$$\left.\begin{aligned}\lambda &= \frac{\partial F}{\partial U}\\ S' &= S-\left(\frac{\partial F}{\partial x}\right)_{U=\text{Const}}\end{aligned}\right\} \quad [1.21]$$

As the expressions of F and S are known, λ and S' can be calculated at any point in time and space if U is known. From definition [1.21], in the case where the variations in F are due to variations in U only, $F' = 0$ and S' is identical to S.

Example: assume that the flux function F is defined as in equation [1.3], recalled here

$$F = aU$$

where a is a function of x and t. Equation [1.18] thus becomes

$$\frac{\partial F}{\partial x}=\frac{\partial}{\partial x}[a(x,t)U]=a\frac{\partial U}{\partial x}+U\frac{\partial a}{\partial x} \quad [1.22]$$

and λ and F' are given by

$$\left.\begin{aligned}\lambda &= a\\ F' &= U\frac{\partial a}{\partial x}\end{aligned}\right\} \quad [1.23]$$

If a does not depend on x, $F' = 0$ because $\partial a/\partial x = 0$.

1.1.4. *Characteristic form – Riemann invariants*

Writing a conservation law in non-conservation form leads to the notions of characteristic form and Riemann invariant. Such notions are essential to the understanding of hyperbolic conservation laws. A very convenient way of determining the behavior of the solutions of hyperbolic conservation laws consists of identifying invariant quantities

(i.e., quantities that do not change) along certain trajectories, also called “characteristic curves” (or more simply “characteristics”).

The solution is calculated by “following” the invariants along the characteristics, which allows the value of U to be determined at any point. To do this, non-conservation form [1.17] is used

$$\frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = S'$$

The purpose is to derive the expression of the variation δU in U observed by an observer traveling at a given speed v. A small time interval δt is considered, over which the traveler moves by a distance $\delta x = v\ \delta t$. The variation δU “seen” by the observer is given by

[1.24] $$\delta U = \frac{\partial U}{\partial t}\delta t + \frac{\partial U}{\partial x}\delta x = \left(\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial x}\right)\delta t$$

Note that from the observer’s point of view, U is a function of time only, because the observer’s location $x(t)$ is defined by $\mathrm{d}x/\mathrm{d}t = v$. When δt tends towards zero, the ratio $\delta U / \delta t$ tends to the so-called total derivative $\mathrm{d}U/\mathrm{d}t$. Therefore, equation [1.24] becomes

[1.25] $$\frac{\delta U}{\delta t} \underset{\delta t \to 0}{\simeq} \frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\partial U}{\partial t} + v\frac{\partial U}{\partial x} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = v$$

In the particular case of an observer moving at the celerity λ, equation [1.25] becomes

[1.26] $$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\partial U}{\partial t} + \lambda\frac{\partial U}{\partial x} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda$$

Comparing equations [1.26] and [1.17] leads to

[1.27] $$\frac{\mathrm{d}U}{\mathrm{d}t} = S' \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda$$

Equation [1.27] is the so-called characteristic form of equation [1.1]. The trajectory, the equation of which is $\mathrm{d}x/\mathrm{d}t = \lambda$, is called a characteristic. λ is called the celerity, or wave propagation speed.

S' being a function of U, x and t, its value may be calculated at any point (x, t) if the value of U is known. The

first-order ODE [1.27] is applicable along the characteristic.

One very important, specific case is where the source term S is zero, equation [1.17] becomes

[1.28] $$\frac{\partial U}{\partial t} + \lambda \frac{\partial U}{\partial x} = 0$$

and equation [1.27] becomes

[1.29] $$\frac{\mathrm{d}U}{\mathrm{d}t} = 0 \quad \text{for} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda$$

Equation [1.29] can also be written as

[1.30] $$U = \text{Const} \qquad \text{for} \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \lambda$$

Consequently, quantity U is invariant to an observer moving at speed λ. U is called a Riemann invariant.

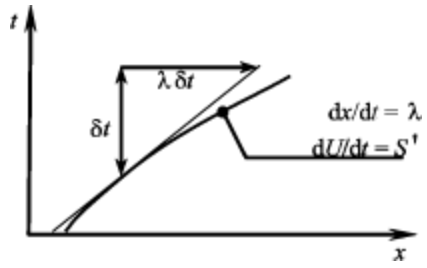
The physical meaning of the celerity, or wave propagation speed, is the following. The celerity is the speed at which the variations in U (and not U itself) propagate. A perturbation appearing in the profile of U at a given time propagates at speed λ. The celerity can be viewed as the speed at which “information”, or “signals” created by variations in U, propagates in space.

1.2. Determination of the solution

1.2.1. *Representation in the phase space*

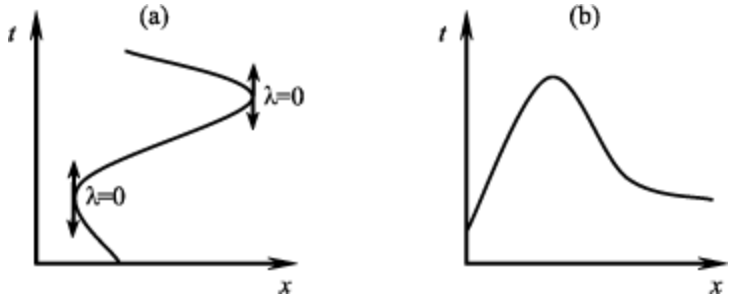
The phase space is a very useful tool in the determination of the behavior of the solutions of hyperbolic conservation laws. The term “phase space” indicates the (x, t) plane formed by space coordinate x and time coordinate t (Figure 1.3).

Figure 1.3. *Representation of characteristic curves in the phase space*



The trajectory $dx/dt = \lambda$ is represented by a curve in the phase space. The distance δx covered by the characteristic over a time interval δt is given by $\delta x = \lambda\ \delta t$, therefore the slope of the line is $\delta t/\delta x = 1/\lambda$. Note that the sign of λ may change with time depending on the variations in U and the expressions of λ and S. When λ becomes zero the tangent to the characteristic curve is vertical in the phase space (Figure 1.4a). In contrast, an extremum with respect to time is not physically permissible (Figure 1.4b) because "traveling backwards in time" is not possible.

Figure 1.4. *Physically permissible (a) and non-permissible (b) characteristics*



The representation in the phase space may be used to determine the behavior of the solutions of conservation law [1.1] using the so-called "characteristics method". The following simple case is considered:

– Source term S in equation [1.1] is zero.

– The flux depends only on U, therefore $F' = 0$ in equations [1.18–1.20].

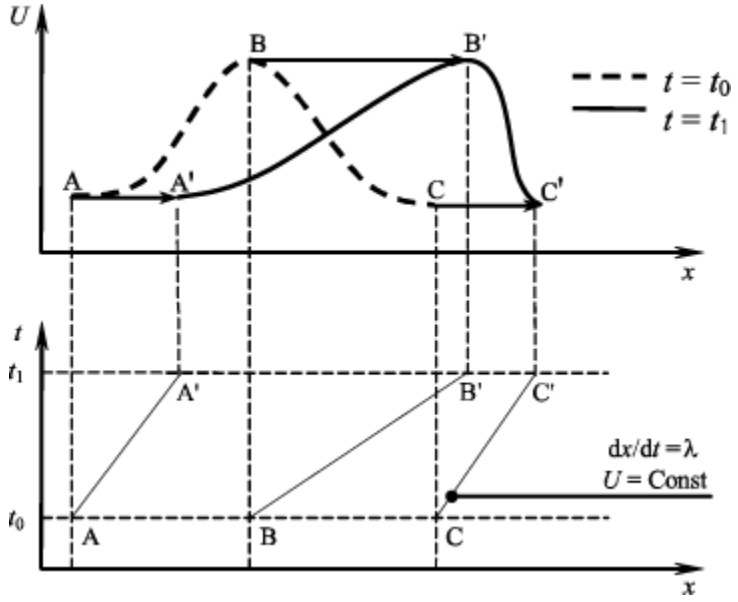
Characteristic form [1.27] then reduces to equation [1.30], recalled here

$$U = \text{Const} \qquad \text{for} \frac{dx}{dt} = \lambda$$

F being a function of U only, $\lambda$is also a function of U only. Consequently, if U is constant along a characteristic line, $\lambda$is also constant and the characteristic is a straight line in the phase space (Figure 1.5). Assume that the profile $U(x, t_0)$ is known for all x at the time t_0. The purpose is to determine the profile $U(x, t_1)$ for all x at the time $t_1 > t_0$. Consider point A, the abscissa of the which is denoted by x_A, at which the value of U at (x_A, t_0) is denoted by U_A. Since the celerity λ depends on U only, the characteristic passing at A is a straight line. Its (constant) celerity is $\lambda_A = \partial F/\partial U$ (U_A). At the time t_1, the characteristic has moved to point A′, the abscissa $x_{A'}$ of which is given by

[1.31] $$x_{A'} = x_A + (t_1 - t_0)\lambda_A$$

Figure 1.5. *Representation of the characteristics in the phase space (bottom) and behavior of the physical profile (top) in the specific case $F' = S = 0$*



From the property of invariance of U along the characteristic, U remains unchanged between A and A′

[1.32] $U_{A'} = U_A$

Extending the reasoning above to any value of x, the following relationship is obtained

[1.33] $U(x + \lambda\Delta t, t + \Delta t) = U(x, t)$

where Δt represents the quantity ($t_1 - t_0$) and $\lambda$is estimated at (x, t).

Figure 1.5 shows how the characteristics method can be used to determine the evolution of a given profile [ABC]. The figure is drawn assuming that $\lambda$is an increasing function of U. Therefore, point B moves faster than points A and C because U_B is larger than U_A and U_C. Consequently, region [AB] tends to spread in time, while region [BC] becomes narrower. After a certain amount of time point B catches up point C and the solution becomes discontinuous at point B′ = C′. The derivatives $\partial U/\partial t$ and $\partial U/\partial x$ are no longer defined and a specific treatment must be applied to determine the solution at later times. Such a treatment is detailed in Chapter 3.

In general, S and F are non-zero. Thus, relationship [1.33] cannot be used because

– U is not invariant along a characteristic line;

– the characteristics are therefore curved lines, the slope of which depends on the local value of x and U.

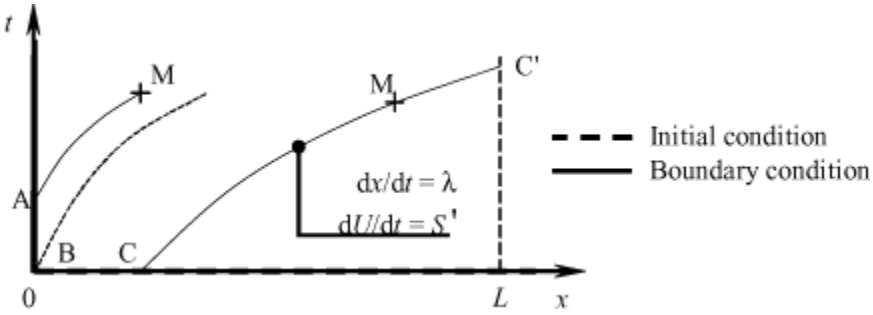
Therefore, no simple relationship can be derived between the initial profile at $t = t_0$ and the final profile at $t = t_1$. In most cases, the solution must be calculated approximately using numerical methods. Such methods are dealt with in Chapters 6 and 7.

1.2.2. *Initial conditions, boundary conditions*

In practical applications, the solution of equation [1.1] is sought over a domain of finite length. A key issue is the

amount of information needed for the calculation of U at point M(x, t) in the domain. This question is best answered using the phase space (Figure 1.6). The solution domain is assumed to extend from $x = 0$ to $x = L$.

Figure 1.6. *Initial and boundary conditions in the phase space*



For the sake of clarity, the celerity λ is assumed to be positive over the entire domain (the case where the sign of the celerity changes is examined at the end of the section). Two possibilities arise:

– If point M is located on the right-hand side of the characteristic that passes at point B ($x = 0$, $t = 0$), there exists a point C on the line ($t = 0$) such that the characteristic passing at C passes at M. Point C is called the foot of the characteristic at $t = 0$. If the value of U is known at point C, U can be calculated along the characteristic line by solving characteristic form [1.27] using any analytical or numerical method. Therefore, the value of U can be calculated at any point M located on the right-hand side of the characteristic that passes at B (0, 0), provided that $U(x, 0)$ is known for all x between 0 and L. The function that describes profile $U(x, 0)$ is called the initial condition. It is expressed as follows

[1.34] $U_0(x) = U(x, t = 0), \qquad x \in [0, L]$

– If point M is located on the left-hand side of the characteristic passing at B, the value of U at M cannot be calculated from the initial condition and the knowledge of the value of U at all A points along the line ($x = 0$) is

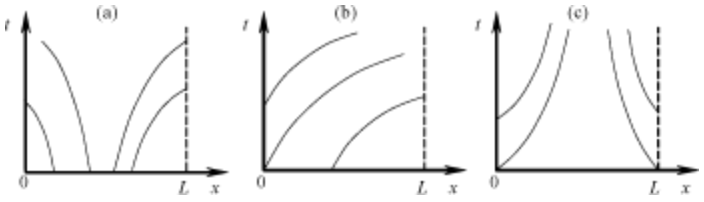
necessary. The function that describes the profile $U(0, t)$ is called a boundary condition. In the case of a positive λ, the characteristics enter the domain on the left-hand side and the left boundary condition must be used. This is expressed as follows

[1.35] $$U_b(t) = U(x = 0, t), \qquad t > 0$$

Note that a boundary condition can be prescribed only if the characteristics enter the domain. In the situation illustrated by Figure 1.6, prescribing a boundary condition at the point C′ would be meaningless because the value of U at C′ is entirely determined by the initial condition at C via the characteristic form [1.27] and cannot be prescribed independently of it. Depending on the variations of λ with U, x and t, the number of boundary conditions needed to determine U uniquely over the domain $[0, L]$ may be 0, 1 or 2 (see Figure 1.7).

In configuration (a), the characteristics leave the domain at both boundaries ($x = 0$) and ($x = L$). The value of U at both boundaries is determined entirely by the initial condition $U(x, 0)$. In configuration (b), the left-hand boundary condition is needed because the characteristics enter the domain at $x = 0$ and the value of U at this location cannot be determined from the values inside the domain. In contrast, the knowledge of U at the right-hand boundary is not required because U is determined uniquely from the value of U inside the domain. In configuration (c), the characteristics enter the domain at both $x = 0$ and $x = L$. Consequently, two boundary conditions are needed, one at each end of the domain, because $U(0, t)$ and $U(L, t)$ cannot be determined from inside the domain and must therefore be specified independently in the form of boundary conditions.

Figure 1.7. *Number of boundary conditions needed depending on the variations of the wave celerity: none (a) one (b) two (c)*



1.3. A linear law: the advection equation

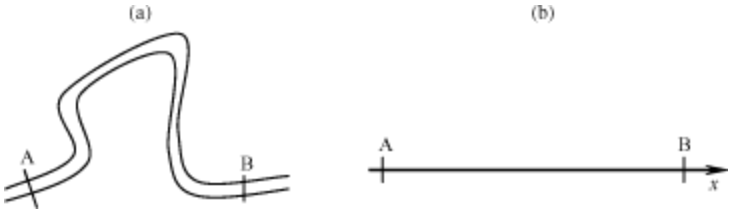
1.3.1. *Physical context – conservation form*

The linear advection equation is the simplest possible hyperbolic conservation law. It is found in many domains of fluid mechanics because it expresses a widespread phenomenon, the transport of a given quantity in a moving fluid. The transported variable may be the temperature of the fluid, the concentration in a given chemical, etc. The expression “advection” is often understood as the advection of a passive scalar, i.e., a quantity that does not influence the behavior of the flow by which it is transported. In a number of cases however, the transported quantity influences the velocity field, a phenomenon known as coupling. This is the case of the inviscid Burgers equation dealt with in section 1.4.

In the present section, a passive scalar is considered. The example of a chemical substance dissolved in water with a concentration variable in space and time is used. The water is assumed to flow in a channel, the transverse dimensions of which are assumed to be negligible compared to the longitudinal dimension. The channel may be an open channel (a river, a canal) or a closed channel (a conduit) with a cross-sectional area variable in space and time. The assumption of negligible transverse dimensions for the

channel allows the assumption of a one-dimensional, longitudinal flow and transport process to be used. The channel is represented as a one-dimensional object. The space coordinate is the curvilinear abscissa (Figure 1.8).

Figure 1.8. *One-dimensional representation of a channel, the transverse dimension of which can be considered negligible: reality (a) and model (b)*



The governing PDE for the one-dimensional transport of a dissolved substance is derived by carrying out a balance as in section 1.1.2. The total quantity M (t) of substance (the "mass" as introduced in section 1.1.2) over a basic slice of channel of length δx (Figure 1.9) is given by

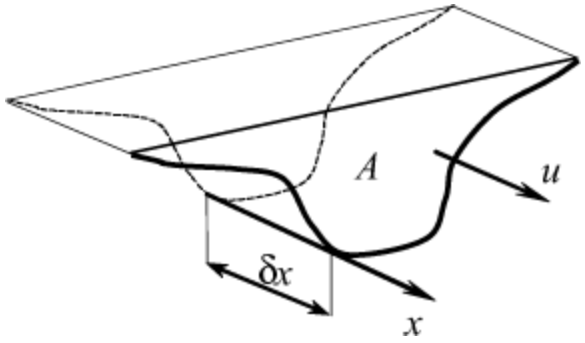
[1.36] $M = C\ \delta V$

where C is the concentration of the dissolved substance and δV is the volume of the basic channel slice, given by

[1.37] $\delta V = A\ \delta x$

where A is the cross-sectional area of the channel (Figure 1.9).

Figure 1.9. *Perspective view of a basic channel section*



Amount $\delta F(x_0)$ of dissolved chemical that passes at x_0 during a basic time interval δt is given by

[1.38] $\delta F(x_0) = AuC\ \delta t$

where u is the flow velocity. Using the same reasoning as in equations [1.11–1.16] with a zero source term, the PDE that describes the conservation of mass (also called the continuity equation) is obtained

[1.39] $$\frac{\partial}{\partial t}(AC) + \frac{\partial}{\partial x}(AuC) = 0$$

Equation [1.39] can be written in the form [1.1] by defining the conserved variable U, flux F and source term S as

[1.40] $$\left.\begin{aligned} U &= AC \\ F &= QC \\ S &= 0 \end{aligned}\right\}$$

where $Q = Au$ is the so-called liquid discharge.

1.3.2. *Characteristic form*

Several approaches may be used to rewrite equation [1.39] in characteristic form. A first approach consists of defining the conserved quantity as AC and rewriting equations [1.39–1.40] as

[1.41] $$\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}(uU) = 0$$

noting that $\partial/\partial x(uU) = u\ \partial U/\partial x + U\ \partial u/\partial x$, equation [1.41] becomes

[1.42] $$\frac{\partial U}{\partial t} + u\frac{\partial U}{\partial x} = -U\frac{\partial u}{\partial x}$$

As shown in section 1.1.4 (see equations [1.24–1.27]), equation [1.42] is equivalent to

[1.43] $$\frac{\mathrm{d}U}{\mathrm{d}t} = -U\frac{\partial u}{\partial x} \qquad \text{for } \frac{\mathrm{d}x}{\mathrm{d}t} = u$$

Equation [1.43] is of limited interest because U does not appear as an invariant quantity along a characteristic line.

In a second approach, equation [1.39] is rewritten with respect to the concentration C by developing the derivatives

[1.44] $$A\frac{\partial C}{\partial t}+C\frac{\partial A}{\partial t}+Q\frac{\partial C}{\partial x}+C\frac{\partial Q}{\partial x}=0$$

equation [1.44] is rewritten as

[1.45] $$A\frac{\partial C}{\partial t}+Q\frac{\partial C}{\partial x}=-\left(\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}\right)C$$

Noting that the continuity equation for the flow can be written as

[1.46] $$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0$$

substituting equation [1.46] into equation [1.45] yields the following equation

[1.47] $$A\frac{\partial C}{\partial t}+Q\frac{\partial C}{\partial x}=0$$

Dividing by A and noting that $Q/A = u$ leads to

[1.48] $$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=0$$

From the developments carried out in section 1.1.4 (see equations [1.24–1.27]), equation [1.48] is known to be equivalent to the following characteristic form

[1.49] $$\frac{\mathrm{d}C}{\mathrm{d}t}=0 \quad \text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t}=u$$

Equation [1.49] is equivalent to

[1.50] $$C=\text{Const} \qquad \text{for}\ \frac{\mathrm{d}x}{\mathrm{d}t}=u$$

Equation [1.50] is an interesting alternative to equation [1.43] because it allows a Riemann invariant to be derived. The Riemann invariant is the concentration of the dissolved substance. Note that the conserved quantity (the mass AC per unit length of channel) is not identical to the invariant quantity (the concentration).

1.3.3. *Example: movement of a contaminant in a river*