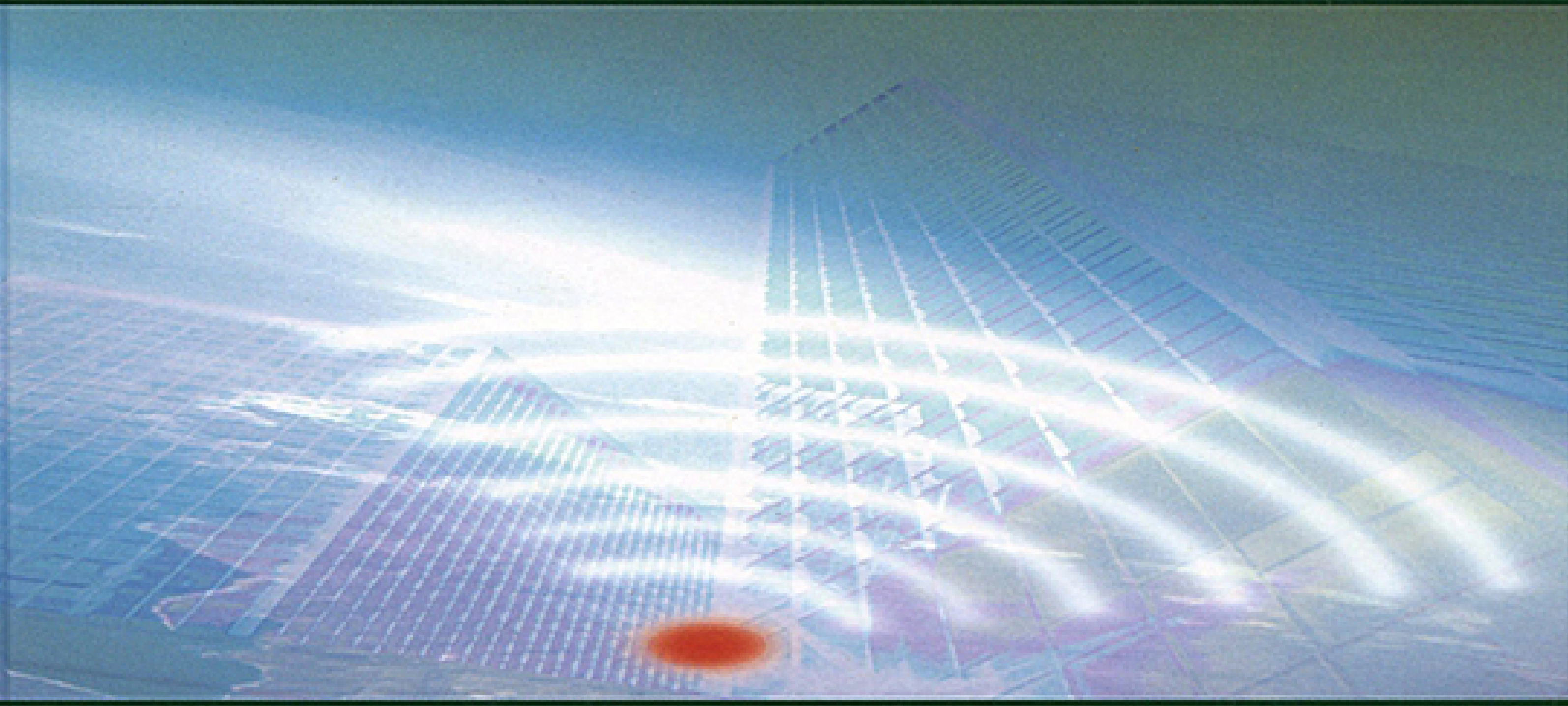

Vibration in Continuous Media

Jean-Louis Guyader

ISTE

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Preface

This book, which deals with vibration in continuous media, originated from the material of lectures given to engineering students of the National Institute of Applied Sciences in Lyon and to students preparing for their Master’s degree in acoustics.

The book is addressed to students of mechanical and acoustic formations (engineering students or academics), PhD students and engineers wanting to specialize in the area of dynamic vibrations and, more specifically, towards medium and high frequency problems that are of interest in structural acoustics. Thus, the modal expansion technique used for solving medium frequency problems and the wave decomposition approach that provide solutions at high frequency are presented.

The aim of this work is to facilitate the comprehension of the physical phenomena and prediction methods; moreover, it offers a synthesis of the reference results on the vibrations of beams and plates. We are going to develop three aspects: the derivation of simplified models like beams and plates, the description of the phenomena and the calculation methods for solving vibration problems. An important aim of the book is to help the reader understand the limits hidden behind every simplified model. In order to do that, we propose simple examples comparing different simplified models of the same physical problem (for example, in the study of the transverse vibrations of beams).

The first few chapters are devoted to the general presentation of continuous media vibration and energy method for building simplified models. The vibrations of continuous three-dimensional media are presented in Chapter 1 and the equations which describe their behavior are established thanks to the conservation laws which govern the mechanical media. Chapter 2 presents the problem in terms of variational formulation. This approach is fundamental in order to obtain, in a systematic way, the equations of the simplified models (also called condensed media), such as beams, plates or shells. These simplified continuous media are often sufficient models to describe the vibrational behavior of the objects encountered in practice. However, their importance is also linked to the richness of the information which is accessible thanks to the analytical solutions of the equations which characterize them. Nevertheless, since these models are obtained through *a priori* restriction of possible three-dimensional movements and stresses, it is necessary to master the underlying hypothesis well, in order to use them advisedly. The aim of Chapters 3 and 4 is to provide these hypotheses in the case of beams and plates. The derivation of equations is done thanks to the variational formulations based on Reissner and Hamilton's functionals. The latter is the one which is traditionally used, but we have largely employed the former, as the limits of the simplified models obtained in this way are established more easily. This approach is given comprehensive coverage in this book, unlike others books on vibrations, which dedicate very little space to the establishment of simplified models of elastic solids.

Chapters 5, 6 and 7 deal with the different aspects of the behavior of beams and plates in free vibrations. The vibrations modes and the modal decomposition of the response to initial conditions are described, together with the wave approach and the definition of image source linked to the reflections on the limits. We must also insist on the influence of the “secondary effects”, such as shearing, in the problems of bending plates. From a general point of view, the discussion of the phenomena is done on two levels: that of the mechanic in terms of modes and that of the acoustician in terms of wave's propagation. The notions of phase speed and group velocity will also be exposed.

We will provide the main analytical results of the vibrations modes of the beams and rectangular or circular plates. For the rectangular plates, even quite simple boundary conditions often do not allow analytical calculations. In this case, we will describe the edge effect method which gives a good approximation for high order modes.

Chapter 8 is dedicated to the introduction of damping. We are going to show that the localized source of damping results in the notion of complex modes and in a difficulty of resolution which is much greater than the one encountered in the case of distributed damping, where the traditional notion of vibrations modes still remains.

The calculation of the forced vibratory response is at the center of two chapters. We will start by discussing the modal decomposition of the response (Chapter 9), where we are going to introduce the classical notions of generalized mass, stiffness and force. Then we will continue with the decomposition in forced waves (Chapter 10) which offers an alternative to the previous method and is very effective for the resolution of beam problems.

For the modal decomposition, the response calculations are conducted in the frequency domain and time domain. The same instances are treated in a manner which aims to highlight the specificities of these two calculation techniques. Finally we will study the convergence of modal series and the way to accelerate it.

In the case of forced wave decomposition, we will show how to treat the case of distributed and non-harmonic excitations, starting from the solution for a localized, harmonic excitation. This will lead us to the notion of integral equation and its key idea: using the solution of a simple case to treat a complicated one.

Chapters 11 and 12 deal with the problem of approximating the solutions of vibration problems, using the Rayleigh-Ritz method. This method employs directly the variational equations of the problems. The classical approach, based on Hamilton's functional, is used and the convergence of the solutions studied is illustrated through some examples. The Rayleigh-Ritz quotient – which stems directly from this approach – is also introduced.

A second approach is proposed, based on the Reissner’s functional. This is a method which has not been at the center of accounts in books on vibrations; however, it presents certain advantages, which will be discussed in some examples.

Chapter 1

Vibrations of Continuous Elastic Solid Media

1.1. Objective of the chapter

This work is addressed to students with a certain grasp of continuous media mechanics, in particular, of the theory of elasticity. Nevertheless, it seems useful to recall in this chapter the essential points of these domains and to emphasize in particular the most interesting aspects in relation to the discussion that follows.

After a brief description of the movements of the continuous media, the laws of conservation of mass, momentum and energy are given in integral and differential form. We are thus led to the basic relations describing the movements of continuous media.

The case of small movements of continuous elastic solid media around a point of static stable equilibrium is then considered; we will obtain, by linearization, the equations of vibrations of elastic solids which will be of interest to us in the continuation of this work.

At the end of the chapter, a brief exposition of the equations of linear vibrations of viscoelastic solids is outlined. The equations in the temporal domain are given as well as those in the frequency domain, which are obtained by Fourier transformation. We then note a formal analogy of elastic solids equations with those of the viscoelastic solids, known as principle of correspondence.

Generally, the presentation of these reminders will be brief; the reader will find more detailed presentations in the references provided at the end of the book.

1.2. Equations of motion and boundary conditions of continuous media

1.2.1. *Description of the movement of continuous media*

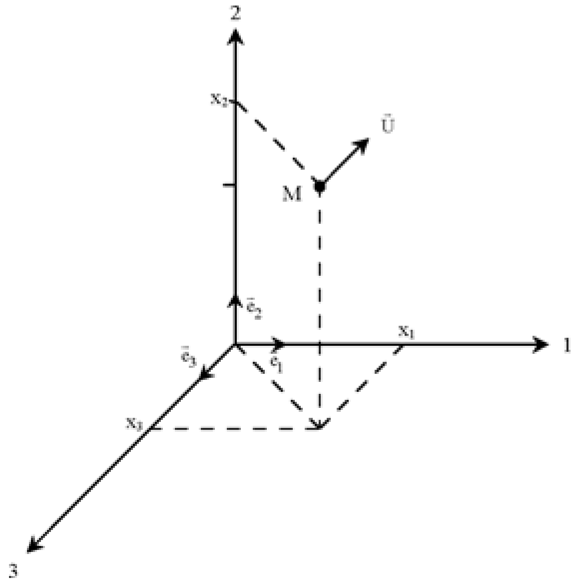
To observe the movement of the continuous medium, we introduce a Galilean reference mark, defined by an origin O and an orthonormal base $\vec{e}_1, \vec{e}_2, \vec{e}_3$. In this reference frame, a point M, at a fixed moment T, has the co-ordinates (x_1 , x_2 , x_3).

The Euler description of movement is carried out on the basis of the four variables (x_1 , x_2 , x_3 , t); the Euler unknowns are the three components of the speed $\vec{U}$ of the particle which is at the point M at the moment t.

[1.1] $\vec{U} = U_i(x_1, x_2, x_3, t)$

Derivation with respect to time of quantities expressed with Euler variables is particular; it must take into account the variation with time of the co-ordinates x_i of the point M.

Figure 1.1. *Location of the continuous medium*



For example, for each acceleration component γ_i of the particle located at the point M, we obtain by using the chain rule of derivation:

$$\gamma_i = \frac{dU_i}{dt} = \frac{\partial U_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial U_i}{\partial x_j}\frac{\partial x_j}{\partial t},$$

and noting that:

$$U_j = \frac{\partial x_j}{\partial t},$$

we obtain the expression of the acceleration as the total derivative of the velocity:

$$\gamma_i = \frac{dU_i}{dt} = \frac{\partial U_i}{\partial t} + \sum_{j=1}^{3} \frac{\partial U_i}{\partial x_j} U_j \,;$$

or in index notation:

[1.2] $$\gamma_i = \frac{dU_i}{dt} = \frac{\partial U_i}{\partial t} + U_{i,j} U_j \,.$$

In the continuation of this work we shall make constant use of the index notation, which provides the results in a compact form. We shall briefly point out the equivalences in the traditional notation:

– partial derivation is noted by a comma:

$$\frac{\partial U_i}{\partial x_j} = U_{i,j} \,;$$

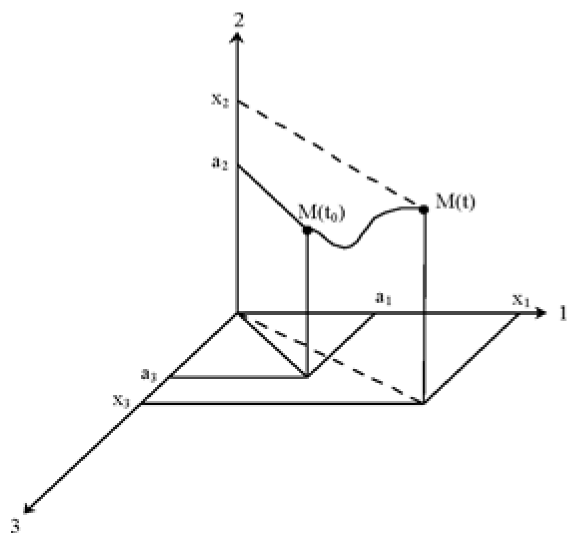
– an index repeated in a monomial indicates a summation:

$$\sum_{j=1}^{3} U_{i,j} U_j = U_{i,j} U_j .$$

The Lagrangian description is an alternative to the Euler description of the movement of continuous media. It consists of introducing Lagrange variables (a_1 , a_2 , a_3 , t), where (a_1 , a_2 , a_3) are the co-ordinates of the point where the particle is located at the moment of reference t_0. The Lagrange unknowns are the coordinates x_i of the point M where the particle is located at the moment t:

[1.3] $x_i = \phi_i (a_1 , a_2 , a_3 , t)$.

Figure 1.2. *Initial a_i and instantaneous x_i co-ordinates*



a_j being independent of time, the speed or the acceleration of the particle M with co-ordinates x_i is deduced from it by partial derivation:

[1.4] $U_i (a_j , t) = \frac{\partial \phi_i}{\partial t} (a_j , t) \qquad \gamma_i (a_j , t) = \frac{\partial^2}{\partial t^2} \phi_i (a_j , t)$.

The Lagrangian description is direct: it identifies the particle; the Euler description is indirect: it uses variables with instantaneous significance, which eventually proves to be interesting for the motion study of continuous media; it is the reason for the frequent use of Euler’s description. The two descriptions are, of course, equivalent; the demonstration thereof can be found in the titles on the mechanics of continuous media provided in the references section.

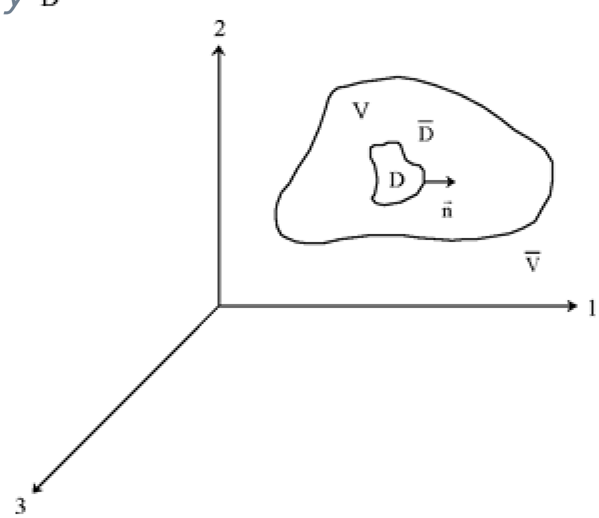
1.2.2. *Law of conservation*

Laws that govern the evolution of continuous media over time are the laws of conservation: conservation of mass, conservation of momentum and conservation of energy. These laws can be expressed in an integral form [1.5] or in a differential form [1.6] with the boundary condition [1.7].

The general form of the conservation law is provided in this section; it will be detailed in the next sections with the conservation of mass, momentum and energy.

Let us consider a part D of the continuous medium whose movement is being observed. Let us also introduce its boundary $\overline{D}$ and nj the direction cosines of the exterior normal $\vec{n}$, which is supposed to exist in all the points of $\overline{D}$. V is the volume of the continuous medium and $\overline{V}$ is the surface delimiting it. These quantities are defined in Figure 1.3.

Figure 1.3. *Continuous medium* V *with boundary* $\overline{V}$ *and part* D *with boundary* $\overline{D}$



The integral form of a conservation equation, in a very general case, is given by the following equation:

[1.5] $\frac{d}{dt}\int_D A_i + \int_D \alpha_{ij} n_j = \int_D B_i$.

$\frac{d}{dt}$ indicates the total derivative, i.e. the derivative with respect to time when the derived quantity is followed in its movement. A_i and B_i are vector quantities, in the general case of dimension 3, but may also be scalar values, in the particular case of dimension 1.

From a physical point of view:

$\frac{d}{dt}\int_D A_i$ represents the fluctuation over time of a physical value, attached to the part D of the continuous medium, whose movement is being followed.

$\int_D \alpha_{ij} n_j$ represents the action of the exterior surface on D.

$\int_D B_i$ represents the action of the exterior volume on D.

The law of conservation [1.5] thus translates the fact that the fluctuation over time of a quantity attached to the part D, followed in its movement, results from the actions of surface and volume affecting the part D of the considered continuous medium from the outside.

We may associate a differential form to the integral form of the conservation equation.

The differential form of the conservation law:

[1.6] $\frac{\partial A_i}{\partial t} + (A_i U_j + \alpha_{ij})_{,j} = B_i \quad \text{in } V,$

[1.7] $\alpha_{ij}\, n_j = C_i \quad \text{on } \bar{V}.$

Equation [1.6] supposes that A_i, α_{ij}, B_i and C_i are continuously derivable in any point of V. This assumption, which we adopt, excludes the existence of discontinuity surfaces in volume V. For a detailed account of discontinuity surfaces we refer the reader to specialized works on continuous media mechanics.

The boundary condition [1.7] translates the equality of the projection of the tensor a_{ij} following the external normal to an external action of surface contact C_i. This action of contact will generally be a given in a problem; we shall see, however, that sometimes it will be preferable to modify the boundary condition, in order to more easily introduce the action of the exterior upon the continuous medium.

1.2.3. *Conservation of mass*

This law of conservation postulates that the mass of a part D of the continuous medium, whose movement is followed, remains constant over time.

To give the integral form of this conservation law, let us introduce the density ρ(M, t); under these conditions the law of conservation of mass is written:

[1.8] $$\frac{d}{dt}\int_D \rho(M,t) = 0 .$$

Equation [1.8] is a particular case of the general form [1.5]. The associated differential form is deduced from it:

[1.9] $$\frac{d}{dt}\rho + (\rho U_j)_{,j} = 0 .$$

Equation [1.9] is called continuity relation.

1.2.4. *Conservation of momentum*

A fundamental law of mechanics is introduced. To apply this law to every part D of the continuous medium, it is necessary to define the external efforts applied to D. These are of two kinds:

– efforts exerted on D by systems external to the continuous medium, which are remote actions or forces of volume written $f_i(M, t)$;

– efforts exerted on D through surface actions on $\bar{D}$; these are actions of local contact verifying the two following conditions:

a) at each point M of the boundary $\bar{D}$ and at every moment t, these efforts are represented by a density of force T_i,

b) the vector T_i at the moment t depends only on the point M and the unitary vector normal to $\bar{D}$ in M.

Let us state [1.10], where σ_{ij} is a second-order tensor, called a stress tensor:

[1.10] $T_i = \sigma_{ij} n_j$.

Note: in [1.10], T_i is the i^{th} component of the resulting stress for the vector $\vec{n}$; σij is the ij^{th} component of the stress tensor. Somewhat abusing the language, the σ_{ij} will also be called stresses.

Let us write the fundamental law of the dynamics applied to a part D of the continuous medium. Equality of the dynamic torque and the torque of the external efforts applied to D led to the two relations [1.11] and [1.12]; O is a point related to the point of reference, which we take as the origin without restricting the general case:

[1.11] $\frac{d}{dt}\int_D \rho U_i = \int_{\bar{D}} \sigma_{ij} n_j + \int_D f_i$,

[1.12] $\frac{d}{dt}\int_D (x_l \rho U_k - x_k \rho U_l) = \int_{\bar{D}} (x_l \sigma_{kj} - x_k \sigma_{lj})\ n_j + \int_D (x_l f_k - x_k f_l)$

with (1,k) = {(1,2), (2,3), (3,1)}.

Relations [1.11] and [1.12] express the conservation of momentum. Their expressions can also be given in vectorial notation:

$$\frac{d}{dt}\int_D \rho\vec{U} = \int_{\bar{D}} \vec{T} + \int_D \vec{f} ,$$

$$\frac{d}{dt}\int_D \overrightarrow{OM} \wedge \rho\vec{U} = \int_{\bar{D}} \overrightarrow{OM} \wedge \vec{T} + \int_D \overrightarrow{OM} \wedge \vec{f}.$$

The associated partial derivative equation [1.11] is:

[1.13] $$\frac{d}{dt}(\rho U_i) + (\rho U_j U_i)_{,j} = \sigma_{ij,j} + f_i \quad \text{in } V.$$

By using the continuity equation [1.9] in [1.13] and after appropriate grouping, we obtain:

[1.14] $$\rho\left(\frac{d}{dt}U_i + U_j\ U_{i,j}\right) = \sigma_{ij,j} + f_i \quad \text{in } V.$$

The first member of [1.14] represents $\rho\gamma_i$ where γ_i is the acceleration of the particle located at the point M, which we calculated in [1.2]. Equation [1.14] thus appears as a generalization of the point mechanics. It bears the name of the equation of motion.

Let us now exploit the law of conservation [1.12], by writing the associated partial derivative equation:

[1.15] $$\frac{d}{dt}(x_l\,\rho U_k - x_k\,\rho U_l) + \left[(x_l\,\rho U_k - x_k\,\rho U_l)\,U_j - (x_l\,\sigma_{kj} - x_k\,\sigma_{lj})\right]_{,j} = x_l\,f_k - x_k\,f_l$$

with (1,k) = {(1,2), (2,3), (3,1)}.

Let us take the example of the couple (1,k) = (1,2) and develop the derivations. After rearranging the terms we obtain:

$$x_1\left(\frac{d}{dt}(\rho U_2) + (\rho U_2 U_j)_{,j} - \sigma_{2j,j} - f_2\right) - x_2\left(\frac{d}{dt}(\rho U_1) + (\rho U_1 U_j)_{,j} - \sigma_{1j,j} - f_1\right) = \sigma_{21} - \sigma_{12}.$$

Taking into account the relation [1.13] the first member is nil; it is thus noted that:

$$\sigma_{12} = \sigma_{21}.$$

Proceeding in an identical manner for couples (2,3) and (3,1), we obtain the general relation of reciprocity of stresses:

[1.16] $$\sigma_{ij} = \sigma_{ji}.$$

The conservation of momentum involves the symmetry of the stress tensor.

1.2.5. *Conservation of energy*

At every moment the total derivative of the energy E (D) of a part D of the continuous medium is the sum of the power of the external efforts exerted on D and the rate of heat received by D.

Energy E (D) is the sum of kinetic and potential energy, i.e.:

[1.17] $$E(D) = \int_D \rho \left(e + \frac{1}{2} U_i^2 \right)$$

with e as the specific potential energy.

The integral form of the law of conservation of energy is given by [1.18], where q_j is the heat flow vector. The minus sign is related to taking into account the external normal, thus $q_j n_j$ represents the heat flow emitted by the continuous medium.

[1.18] $$\frac{d}{dt} \int_D \rho \left(e + \frac{1}{2} U_i^2 \right) = \int_D \sigma_{ij} n_j U_i - q_j n_j + \int_D f_i U_i .$$

The differential form of the law of conservation of energy results from [1.18]; we obtain all the calculations done:

[1.19] $$\frac{d}{dt} \left(\rho \left(e + \frac{1}{2} U_i^2 \right) \right) + \left(\rho U_j \left(e + \frac{1}{2} U_i^2 \right) - U_i \sigma_{ij} + q_i \right)_{,j} = f_i U_i \quad \text{in } V .$$

It follows from transforming [1.19] using relations [1.9] and [1.14]:

[1.20] $$\rho \left(\frac{\partial}{\partial t} e + U_i \, e_{,i} \right) = \sigma_{ij} \, U_{i,j} - q_{j,j} \qquad \text{in } V .$$

This partial derivative equation has a simple physical interpretation, since the total derivative of specific potential energy appears in the term between the brackets (on the left-hand side of the equation). Thus the variation of specific potential energy results from the power of interior efforts ($\sigma_{ij} U_{i,j}$) and from a contribution of heat ($-q_{j,j}$).

1.2.6. *Boundary conditions*

The boundary conditions represent the natural prolongation of the conservation equations, over the surface $\overline{V}$ of the continuous medium. They are obtained through the relation [1.7] given in the general case of a conservation law, which will have to be further specified by the conservation of mass, momentum and energy.

Let us note first of all that the conservation of mass [1.8] does not involve a boundary condition because the term α_{ij} does not appear in [1.8].

Equation [1.11] of the conservation of momentum involves the boundary condition:

[1.21] $\sigma_{ij} n_j = F_i \quad \text{on } \overline{V}.$

F_i represent the components of the external surface forces applied to the continuous medium.

Equation [1.12] of the conservation of momentum involves the boundary condition:

[1.22] $x_l \sigma_{kj} n_j - x_k \sigma_{lj} n_j = x_l F_k - x_k F_l \quad \text{on } \overline{V},$

with (1,k) = {(1,2), (2,3), (3,1)}.

The second member represents the moment of external surface forces applied to V. The verification of the boundary condition [1.21] involves the verification of [1.22] which, therefore, does not bring any additional information.

The conservation of energy involves the boundary condition:

[1.23] $q_i n_i + \sigma_{ij} n_j U_i = \Pi + F_i U_i \quad \text{on } \overline{V}.$

Π is the amount of heat introduced into the continuous medium, by action of contact at its boundary surface. $F_i U_i$ is the power introduced by the surface forces applied to $\overline{V}$.

By using the relation [1.21] in [1.23], we obtain:

[1.24] $q_i n_i = \Pi \quad \text{on } \overline{V}.$

The formulation of a problem of continuous media mechanics is summarized to finding the density ρ(M, t), speed U_i(M, t) , stress σ_{ij}(M, t) and the specific energy density e(M, t), knowing the forces exiting the volume f_i(M, t) and the surface F_i(M, t) as well as the quantity of heat input Π (M, t). All these quantities are related by the 4 partial derivative equations [1.9], [1.14], [1.16], [1.20] to be verified in the volume V and the two boundary conditions [1.21], [1.24] to be verified over the surface $\overline{V}$.

1.3. Study of the vibrations: small movements around a position of static, stable equilibrium

1.3.1. *Linearization around a configuration of reference*

Linearized equations that we are going to establish only reflect a physical reality if the continuous medium keeps the positions close to those, which it occupies in the configuration of reference, during its movement. We choose a Lagrange position of reference, and the displacement of the particle M is expressed by the formula:

[1.25] $x_i = a_i + W_i(a_j, t)$.

x_i is the i^{th} co-ordinate of particle M whose movement is being followed (Euler's variable). a_i is the i^{th} co-ordinate of particle M in the configuration of reference (Lagrange's variable). $W_i(a_j, t)$ is the i^{th} co-ordinate of the displacement of point M around its position in the situation of reference. We suppose that this displacement as well as its derivatives are small:

[1.26] $\left|\frac{dW_i}{dt}\right| << 1$ and $\left|\frac{dW_i}{dx_j}\right| << 1$.

We will examine the consequences of the assumption [1.26]:

a) Let us at first consider a regular function $f(x_i, t)$, and let us express its value in the vicinity of the position of reference. The components x_i of the position of the point M are close to the co-ordinates a_i, of the same point M that had occupied it in the position of reference; consequently, a first approximation of the value of the function may be obtained by considering the first terms of its development in a Taylor series in the vicinity of a_i:

$$f(x_i, t) = f(a_i, t) + \sum_{j=1}^{3} (x_j - a_j) \frac{\partial f}{\partial x_j}(a_i, t),$$

that is, taking into account the decomposition of movement [1.25]:

[1.27] $$f(x_i, t) = f(a_i, t) + \sum_{j=1}^{3} (x_j - a_j) \frac{\partial f}{\partial x_j}(a_i, t),$$

Taking into account the regularity of $f(x_i, t)$, the partial derivative $\frac{\partial f}{\partial x_j}(a_i, t)$ is bounded. From [1.26] and [1.27] we deduce that in the first approximation:

[1.28] $f(x_i, t) = f(a_i, t)$.

b) Let us now take the derivative $\frac{\partial f}{\partial a_j}$; by using the chain derivation formula it follows:

$$\frac{\partial f}{\partial a_j} = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial a_j}.$$

Introducing the form [1.25] of the movement x_i, we shall obtain:

$$\frac{\partial f}{\partial a_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \frac{\partial W_i}{\partial a_j}.$$

The second term of the right-hand side member being infinitely small, it can be deduced that in the first approximation:

[1.29] $$\frac{\partial f}{\partial a_j} = \frac{\partial f}{\partial x_j}.$$

c) Let us calculate the total derivative of a regular function G (x_i , t) :

$$\frac{dG}{dt} = \frac{\partial G}{\partial t}(x_i\ ,t) + \sum_{j=1}^{3} \frac{\partial G}{\partial x_j}\ (x_i\ ,t)\ \ U_j\ (x_i\ ,t),$$

that is, taking into account the decomposition of movement [1.25]:

$$\frac{dG}{dt} = \frac{\partial G}{\partial t}(x_i\ ,t) + \sum_{j=1}^{3} \frac{\partial G}{\partial x_j}\ (x_i\ ,t)\ \ \frac{\partial W_j}{\partial t}\ (a_i\ ,t).$$

The function G (x_i , t) being regular, $\frac{\partial G}{\partial x_j}(x_i\ ,t)$ is bounded, the second term of the second member is infinitely small; we thus have at first approximation:

$$\frac{dG}{dt} = \frac{\partial G}{\partial t}(x_i\ ,t)\ ,$$

i.e. also taking into account [1.28]:

$$\frac{dG}{dt} = \frac{\partial G}{\partial t}(a_i\ ,t)\ .$$

To sum up, for small movements:

[1.30] $$\frac{dG}{dt} = \frac{\partial G}{\partial t}(x_i\ ,t) + \sum_{j=1}^{3} \frac{\partial G}{\partial x_j}(x_i\ ,t)\ \ U_j\ (x_i\ ,t) = \frac{\partial G}{\partial t}(a_i\ ,t)\ .$$

The distinction between the Euler and Lagrangian descriptions is no longer necessary: on the one hand the initial and current co-ordinates a_i and x_i can be assimilated and the particulate derivative can be replaced by the partial derivative with respect to time. This is true for regular functions, i.e. not for discontinuity surfaces.

Let us examine the effects of [1.28], [1.29] and [1.30] on the equations describing the behavior of the continuous medium.

The equation of conservation of mass [1.9] becomes:

$$\frac{\partial \rho}{\partial t}(a_i, t) = 0 \qquad \text{in V},$$

that is:

[1.31] $\rho(a_i, t) = \rho(a_i) \qquad \text{in V}.$

During small movements, the density of the continuous medium does not vary over time. This property is valid only at first approximation; at a higher degree of accuracy, there is an additional small term, which fluctuates with time. In linear acoustics, this small disturbance must be preserved in calculations as it intervenes in the ideal gas law of the acoustic medium. In the case of elastic solids considered here, the constant term is sufficient to describe the conservation of mass.

Equations [1.14] and [1.16], translating the conservation of momentum, become:

[1.32] $$\rho(a_i)\frac{\partial^2 W_i}{\partial t^2}(a_i, t) = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial a_j}(a_i, t) + f(a_i, t) \qquad \text{in V},$$

[1.33] $\sigma_{ij}(a_i, t) = \sigma_{ji}(a_i, t) \qquad \text{in V}.$

Equation [1.20], characterizing the conservation of energy, becomes:

[1.34] $$\rho(a_i)\frac{\partial e}{\partial t}(a_i, t) = \sum_{i=1}^{3}\sum_{j=1}^{3} \sigma_{ij}(a_i, t)\frac{\partial^2 W_i}{\partial t\, \partial a_j}(a_i, t) - \sum_{j=1}^{3}\frac{\partial q_j}{\partial a_j}(a_i, t) \qquad \text{in V}.$$

Boundary conditions:

[1.35] $$\sum_{j=1}^{3} \sigma_{ij}(a_i, t)\, n_j(a_i) = F_i(a_i, t) \qquad \text{on } \overline{V},$$

[1.36] $$\sum_{j=1}^{3} -q_j(a_i, t)\, n_j(a_i) = \Pi(a_i, t) \qquad \text{on } \overline{V}.$$

Equations [1.31] to [1.36] constitute the linearized model of general equations within the framework of small movements, around a configuration of reference, defined by the relations [1.25] and [1.26].

All quantities appearing in the linearized equations [1.31] to [1.36] are variables of the pair (a_i, t); thus, for the study of small movements, the equations and the boundary conditions are inscribed directly on the configuration of reference.

In the continuation of the course, we will often consider the case of adiabatic movements. This assumption involves $q_i(a_i, t) = 0$; there follows a modification of the equation of energy [1.34] and boundary condition [1.36] which become:

[1.37] $$\rho \frac{\partial e}{\partial t} = \sigma_{ij} \frac{\partial W_{i,j}}{\partial t} \quad \text{in } V,$$

[1.38] $$\Pi = 0 \quad \text{on } \bar{V}.$$

The boundary condition [1.38] translates the impossibility for the adiabatic medium to exchange heat.

The equation of energy [1.37] shows that the variation of specific potential energy is due only to the power of interior efforts.

We have used the index notation in [1.37], and from now we will make constant use of it.

1.3.2. *Elastic solid continuous media*

The unknowns of a problem of vibration of an elastic solid are: W_i , σ_{ij} and e. The calculation reveals 10 independent quantities (taking into account the symmetry of the stress tensor). However, the equations of continuity, movement and energy provide only 5 relations at each point. Thus, information is missing to determine the solution of the problem; that is the stress-strain relation of the continuous medium.

The stress-strain relation is characteristic of material; it connects the stress tensor to that of the strain of the continuous medium. In the case of small movements, considered here, the behavior of the continuous medium is