i-SMOOTH ANALYSIS

THEORY AND APPLICATIONS

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To sweet memory of my brother Vassilii

Contents

Preface

i-*S*mooth analysis is the branch of functional analysis, that considers the theory and applications of the invariant derivatives of functions and functionals.

The present book includes two parts of the *i*-smooth analysis theory. The first part presents the theory of the invariant derivatives of functionals.

The second part of the *i*-smooth analysis is the theory of the invariant derivatives of functions.

Until now, *i*-smooth analysis has been developed mainly to apply to the theory of functional differential equations. The corresponding results are summarized in the books [17], [18], [19] and [22].

This edition is an attempt to present *i*-smooth analysis as a branch of the functional analysis.

There are two classic notions of generalized derivatives in mathematics:

a) The Sobolev generalized derivative of functions [34], [35];
b) The generalized derivative of the distribution theory [32].

In works [17], [18], [19] and [25] the notion of the invariant derivative (*i*-derivative) of nonlinear functionals was introduced, developed the corresponding *i*-smooth calculus of functionals and showed that for linear continuous functionals the invariant derivative coincides with the generalized derivative of the distribution theory.

The theory is based on the notion and constructions of the invariant derivatives of functionals that was introduced around 1980.

Beginning with the first relevant publication in this direction there arose two questions:

Question A1: *Is it possible to introduce a notion of the invariant derivative of functions?*

Question A2: *Is the invariant derivative of functions concerned with the Sobolev generalized derivative?*

This book arose as a result of searching for answers to the *questions **A1*** & ***A2***: there were found positive answers on both these questions and the corresponding theory is presented in the second part.

Another question that initiated the idea for this EDITION was the following

Question B: *Does anything besides a terminological and mathematical relation between the Sobolev generalized derivative of functions and the generalized derivative of distributions?*

At first glance the question looks incorrect, because the first derivative concerns the finite dimensional functions whereas the second one applies to functional objects (distributions – linear continuous functionals).

Nevertheless as it is shown in the second part the answer to the *question* ***B*** came out positive: the mathematical relation between both generalized derivatives can be established by means of the invariant derivative.

One of the main goals writing this book was to clarify the nature of the invariant derivatives and their status in the present system of known derivatives. By this reason we do not pay much attention to applications of the invariant derivatives and concentrate on developing the theory.

The edition is not a textbook and is assigned for specialists, so statements and constructions regarding standard mathematical courses are used without justification or additional comments. Proofs of some new propositions contain only key moments if rest of the details are obvious.

Though the edition is not a textbook, the material is appropriate for graduate students of mathematical departments and be interesting for engineers and physicists. Throughout the book generally accepted notation of the functional analysis is used and new notation is used only for the latest notions.

Acknowledgements. At the initial stage of developing the invariant derivative theory the support of the professor V. K. Ivanov had been very important for me: during personal discussions and at his department seminars of various aspects of the theory were discussed. Because of his recommendations and submissions my first works on the matter were published.

At the end of the 1970-s and the beginning of the 1980-s, many questions were cleared up in discussions with my friends and colleagues: PhD-students Alexander Babenko, Alexander Zaslavskii and Alexander Ustyuzhanin. Theory of numerical methods for solving functional differential equations (FDE) based on *i*-smooth analysis was developed in

cooperation with Dr. V. Pimenov[1].

The attention and support of professor A. D. Myshkis was important to me during a critical stage of *i*-smooth analysis development.

I am thankful to Dr. U. A. Alekseeva, professor V. V. Arestov, professor A. G. Babenko, professor Neville J. Ford for their familiarization with the preliminary versions of the books and useful comments and recommendations.

The author is thankful to a book editor for the great work on book improvement and to A. V. Ivanov for preparation of the printing version of the book.

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1 The author developed a general approach to elaborating numerical methods for FDE and Dr. V. Pimenov developed complete theory, presented in the second chapter of this book

Part I

INVARIANT DERIVATIVES OF FUNCTIONALS AND NUMERICAL METHODS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

Chapter 1

The invariant derivative of functionals

In this chapter we consider the basic constructions of nonlinear i-smooth calculus of nonlinear on $C[a,b]$ functionals.

A study of nonlinear mappings can be realized by local approximations of nonlinear operators by linear operators. Corresponding linear approximations are called derivatives of nonlinear mappings. Depending on the form of linear approximations of various types of derivatives can be introduced. Further

$C[a,b]$ is the space of continuous functions $\phi(\cdot) : [a,b] \to \mathbf{R}$ with the norm $\|\phi(\cdot)\|_C = \max\limits_{a\leq x\leq b} \|\phi(x)\|$;

$C^k[a,b]$ is the space of k–times continuous differentiable functions $\phi(\cdot) : [a,b] \to \mathbf{R}$;

$C^\infty[a,b]$ is the space of infinitely differentiable functions $\phi(\cdot) : [a,b] \to \mathbf{R}$.

1 Functional derivatives

In the general case *a derivative* of a mapping

$$f : X \to Y \tag{1.1}$$

(X & Y are topological vector spaces) at a point $x_0 \in X$ is a linear mapping

$$f'(x_0) : X \to Y \tag{1.2}$$

which approximates in an appropriate sense the difference

$$f(x_0 + h) - f(x_0) \tag{1.3}$$

by h. Subject to the specific form of difference approximation difference(1.3) one can obtain various notions of derivatives.

Consider the classic derivatives of functionals [1]

$$V[\cdot] : C[a,b] \to \mathbf{R}\,. \tag{1.4}$$

1.1 The Frechet derivative

The Frechet derivative (strong derivative) of the functional (1.4) at a point $\phi(\cdot) \in C[a,b]$ is a *linear* continuous functional

$$L_\phi[\cdot] : C[a,b] \to \mathbf{R}\,, \tag{1.5}$$

satisfying the condition

$$V[\phi + \psi] = V[\phi] + L_\phi[\psi] + o(\psi)\,, \tag{1.6}$$

where $\lim\limits_{\|\psi\|_C \to 0} \dfrac{\|o(\psi)\|_C}{\|\psi\|_C} = 0\,.$

If there exists a functional L, satisfying the above conditions, then it is denoted by $\psi \to V'[\phi]\,\psi$ and is called the *Frechet differential.*

1.2 The Gateaux derivative

The *Gateaux derivative* of the functional (1.4) at a point $\phi \in C[a,b]$ is a linear mapping $V'_\Gamma[\phi] : C[a,b] \to \mathbf{R}$, satisfying the condition

$$V[\phi + \psi] - V[\phi] = V'_\Gamma[\phi]\,\psi + \varepsilon(\psi)\,, \tag{1.7}$$

where $\lim\limits_{t \to 0} \dfrac{\varepsilon(t\,\psi)}{t} = 0\,.$

[1] Complete theory see for example in [16], [28].

2 Classification of functionals on $C[a,b]$

Many specific classes of functionals have integral forms. The investigation of such integral functionals formed the basis of the general functional analysis theory.

Along with integral functionals (which are called *regular functionals*) beginning with the works of Dirac and Schwartz mathematicians we generally use *singular functionals* among which the first and most well known is the δ–function.

Further as a rule integrals are understandood in the Riemann sense [2].

2.1 Regular functionals

Analysing the structure of specific functionals one can single out basic (elementary) types of *integral* functionals on $C[a,b]$:

$$V[\phi] = \int_a^b \alpha[\phi(x)]\, dx\,, \tag{2.1}$$

$$V[\phi] = \int_a^b \beta[x, \phi(x)]\, dx\,, \tag{2.2}$$

$$V[\phi] = \int_a^b \beta[x, \int_x^b \phi(\xi)\, d\xi\,]\, dx\,, \tag{2.3}$$

$$V[\phi] = \int_a^b \int_a^b \omega[\phi(x), \phi(\xi)]\, dx\, d\xi\,, \tag{2.4}$$

$\alpha : \mathbf{R} \to \mathbf{R}$, $\beta : [a,b] \times \mathbf{R} \to \mathbf{R}$, $\omega : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ are continuous functions.

[2] In some examples we use the Stilties integral.

Special classes of regular functionals can be constructed using the Stilties integral:

$$W[\phi] = \int_a^b \beta[x, \phi(x)]\, d\lambda(x)\,,$$

$$W[\phi] = \int_a^b \alpha(\nu, \int_\nu^b \beta[x, \phi(x)]\, dx)\, d\lambda(\nu)\,,$$

where $\beta : [-\tau, 0] \times \mathbf{R}^n \to \mathbf{R}$ are continuous functions, $\lambda : [-\tau, 0] \to \mathbf{R}$ is a function with bounded variation.

2.2 Singular functionals

Functionals of the form

$$V[\phi] = P(\phi(\zeta)), \quad \phi \in C[a, b], \tag{2.5}$$

$\Big(P(\cdot) : \mathbf{R} \to \mathbf{R}, \zeta \in (a, b)\Big)$ are called the *singular functionals*. For example:

$$V[\phi] = \phi\left(\frac{a+b}{2}\right) \tag{2.6}$$

is the singular functional.

Another example of the singular functional is the following

$$V[\phi] = \max_{a \le x \le b} \|\phi(x)\|\,. \tag{2.7}$$

3 Calculation of a functional along a line

3.1 Shift operators

Consider the procedure of calculation of the functional (1.4) along a continuous curve (function) [3] $\gamma(y)$, $y \in \mathbf{R}$.

[3]Without loss of generality we assume that the curve has infinite length (i.e. is defined on $\mathbf{R}$).

We have to consider the segment

$$\{\gamma(x+y)\,,\ a\le x\le b\}$$

of the function as the element of the space $C[a,b]$. Such representation is called the *segmentation principle* and allows one switch from the line $\gamma(y)$ to the set of continuous functions

$$T_y\gamma \equiv \{\gamma(x+y)\,,\ a\le x\le b\}\,. \tag{3.1}$$

The corresponding operator T_y is called the *shift operator*. For the shift operator we also use the notation $\gamma_y = T_y\gamma$.

Remark 3.1 One can introduce two types of shift operators:
a) r-shift operator (right shift operator)

$$T_y^+\gamma \equiv \{\gamma(x+y)\,,\ a\le x\le b\}\,; \tag{3.2}$$

b) l-shift operator (left shift operator)

$$T_y^-\gamma \equiv \{\gamma(x-y)\,,\ a\le x\le b\}\,. \tag{3.3}$$

□

Remark 3.2 We usually use the *right* shift operator T_y^+. So if there can be no misunderstanding we, as a rule, omit the sign "+", and write T_y instead of T_y^+. □

3.2 Superposition of a functional and a function

The function (1.4) calculated along the curve $\gamma(y)$ is the function

$$v(y) = V[T_y\gamma]\,,\ y\in\mathbf{R}\,, \tag{3.4}$$

i.e. the superposition of the functional V and the shift operator $T_y\gamma$.

3.3 Dini derivatives

For investigating properties of the functional (1.4) one can use *Dini derivatives* of the functional (1.4) along the function $\gamma(y)$:

$$D^{+}V[T_y\gamma] \equiv \limsup_{\Delta y \to 0} \frac{V[T_{y+\Delta y}\gamma] - V[T_y\gamma]}{\Delta y}\,, \qquad (3.5)$$

$$D^{-}V[T_y\gamma] \equiv \liminf_{\Delta y \to 0} \frac{V[T_{y+\Delta y}\gamma] - V[T_y\gamma]}{\Delta y}\,. \qquad (3.6)$$

If (3.4) is the differentiable function then

$$D^{+}V[T_y\gamma] = D^{-}V[T_y\gamma] = \dot{v}(y)\,.$$

4 Discussion of two examples

4.1 Derivative of a function along a curve

Consider a differentiable function $g(x) : \mathbf{R} \to \mathbf{R}$ and a smooth curve[4] $x = \gamma(y)$, $y \in \mathbf{R}$. The derivative of the superposition

$$w(y) = g(\gamma(y)),\ y \in \mathbf{R}\,, \qquad (4.1)$$

by the rule of differentiation of a superposition can be calculated as

$$\dot{w}(y) = \left.\frac{dg(x)}{dx}\right|_{x=\gamma(y)} \dot{\gamma}(y)\,, \quad y \in \mathbf{R}. \qquad (4.2)$$

In applications and concrete examples as a rule *elementary* functions or their combinations are usually used. The derivatives of these functions can be calculated *apriori*. In this case for calculating derivative of a function $g(x)$ *along the curve* $\gamma(y)$ one can formally substitute $\gamma(y)$ into beforehand calculated function (derivative) $\frac{dg(x)}{dx}$, and multiply the obtained expression by $\dot{\gamma}(y)$.

[4]which is also a smooth function.

4.2 Derivative of a functional along a curve

Calculation of the Dini derivative by the definition (see. paragraph 3.3) requires calculating the functional (1.4) along the curve γ, while for calculating derivative of the differentiable function $g(x)$ along the curve one can use the formula (4.2) which does not require calculating the function along the curve.

One can state the following *question*: can we calculate the derivative of the functional (1.4) along the curve γ by the analogy with the elementary function g? In other words consider the following

Question C: Is it possible define *apriori* a derivative ∂V of the functional (1.4) and obtain the derivative of the functional V along the function γ substituting γ into ∂V? $\square$

Example 4.1 Let us calculate the derivative of the functional

$$V[\phi] = \int\limits_a^b \alpha[\phi(x)]\,dx, \quad \phi \in C[a,b]\,, \tag{4.3}$$

along a curve $x = \gamma(y)$, $y \in \mathbf{R}$ (here $\alpha : \mathbf{R} \to \mathbf{R}$ is a continuous function). The functional (4.3) calculated along the curve γ is the function

$$v(y) \equiv V[T_y\gamma] = \int\limits_a^b \alpha[\gamma(y+x)]\,dx\,, \tag{4.4}$$

The function (4.4) is differentiable and its derivative is

$$\frac{dV[T_y\gamma]}{dy} = \frac{d}{dy}\int\limits_a^b \alpha[\gamma(y+x)]\,dx =$$

$$= \frac{d}{dy} \int\limits_{y+a}^{y+b} \alpha[\gamma(x)]\, dx = \alpha[\gamma(y+b)] - \alpha[\gamma(y+a)] \,. \quad (4.5)$$

Let us consider the functional

$$\partial V[\phi] = \alpha[\phi(b)] - \alpha[\phi(a)] \ , \ \phi \in C[a,b] \ , \qquad (4.6)$$

Then obviously

$$\dot{v}(y) = \partial V[T_y \gamma] \ . \qquad (4.7)$$

Hence the derivative of the functional (4.3) along the curve γ coincides with the functional (4.6) calculated along the curve γ. □

Further the functional (4.7) is called the *invariant derivative* of the functional (4.3).

Example 4.2 Calculating the singular functional (2.6) along the curve γ. We obtain the superposition

$$v(y) \equiv V[T_y \gamma] = \gamma \left(y + \frac{a+b}{2} \right) \ . \qquad (4.8)$$

The function $v(y)$ is differentiable if $\gamma(y)$ is the smooth (differentiable) curve. In this case

$$\frac{dV[T_y \gamma]}{dy} = \dot{v}(y) = \dot{\gamma} \left(y + \frac{a+b}{2} \right) \ . \qquad (4.9)$$

Consider the functional

$$\partial V[\phi] = \dot{\phi} \left(\frac{a+b}{2} \right) \ , \ \ \phi \in C[a,b] \ , \qquad (4.10)$$

Then

$$\dot{v}(y) = \partial V[T_y \gamma] \ . \qquad (4.11)$$

Hence the derivative of the functional (2.6) along the curve γ coincide with the functional (4.10) calculated along γ. □

5 The invariant derivative

5.1 The invariant derivative

In this section we give rigorous definition of the invariant derivative of the functional (1.4).

For $\phi \in C[a,b]$ and $\kappa > 0$ denote by

$$E_\kappa[\phi] \equiv \left\{ \Phi \in C[a-\kappa, b+\kappa] \;\middle|\; \Phi(x) = \phi(x),\, x \in [a,b] \right\}$$

the set of continuous extensions of the function ϕ, and by

$$[T_y\Phi](x) = \Phi(x+y)\,,\ a \le x \le b \in C[a,b]\,,$$

the *shift operator* of a function $\Phi \in E_\kappa[\phi]$ (i.e. contraction of the function Φ on the interval $[a+y, b+y]$). At that $T_\Phi \in C[a,b], y \in [-\kappa, \kappa]$.

Definition 5.1 *A functional $V : C[a,b] \to \mathbf{R}$ has at a point $\phi \in C[a,b]$ the invariant derivative $\partial V[\phi]$, if for any $\Phi \in E_\kappa[\phi]$ the corresponding function*

$$v_\Phi(y) = V[T_y\Phi]\,,\ y \in [-\kappa, \kappa], \tag{5.1}$$

has at the zero finite derivative $\dot{v}_\Phi(0)$ invariant with respect to functions $\Phi \in E_\kappa[\phi]$, i.e. the value $\dot{v}_\Phi(0)$ of this derivative is the same for all $\Phi \in E_\kappa[\phi]$. In this case

$$\partial V[\phi] = \dot{v}_\Phi(0). \tag{5.2}$$

□

Note that the invariant derivative does not depend on the value $\kappa > 0$ because it is defined as the limit.

If a functional V has at a point $\phi \in C[a,b]$ the invariant derivative, then we say that V is *invariantly differentiable* (*i-differentiable*) at the point ϕ. Operation of calculating the invariant derivative is called *i-differentiation* (invariant differentiation).

Example 5.1 Consider the linear on $C[a,b]$ functional

$$V[\phi] = \int_a^b \phi(x)\, dx\,, \tag{5.3}$$

and fix arbitrary $\phi \in C[a,b]$ and $\Phi \in E_\kappa[\phi]$. The corresponding function $v_\Phi(y) = V[T_y\Phi]$ has the form

$$v_\Phi(y) \equiv \int_a^b \Phi(y+x)\, dx = \int_{y+a}^{y+b} \Phi(x)\, dx\,,$$

Its derivative at the point $y = 0$ has the form

$$\left.\frac{dv_\Phi(y)}{dy}\right|_{y=0} = \left(\frac{d}{dy}\int_{y+a}^{y+b} \Phi(x)\, dx\right)_{y=0} =$$

$$= \Phi(b) - \Phi(a) = \phi(b) - \phi(a)$$

and is invariant with respect to $\Phi \in E_\kappa[\phi]$. Thus the invariant derivative of the functional (5.3) is

$$\partial V[\phi] = \phi(b) - \phi(a)\,. \tag{5.4}$$

□

This example shows that the invariant derivative differs from the Frechet derivative (the Gateaux derivative) because the Frechet derivative of a linear continuous functional coincides with the functional.

5.2 The invariant derivative in the class $B[a,b]$

Continuity of a function $\phi \in C[a,b]$ and as the consequence of continuity of $\Phi \in E_\kappa[\phi]$ can be insufficient for differentiability of the function $v_\Phi(y)$ (for instance in the example 4.2, if γ is not the smooth curve). Therefore in this section we introduce the notion of the *invariant derivative in* *a class of functions* $B[a,b]$, where $B[a,b]$ is a space of sufficiently smooth functions, for example $C^1[a,b]$, $C^\infty[a,b]$, etc.

For $\phi \in B[a,b]$, $\kappa > 0$ denote

$$\hat{E}_\kappa[\phi] \equiv \left\{ \Phi : [a-\kappa, b+\kappa] \to \mathbf{R} \;\middle|\; \begin{array}{r} T_y\Phi \in B[a,b] \\ y \in [-\kappa,\kappa];\; T_0\Phi = \phi \end{array} \right\}.$$

Definition 5.2 *A functional*

$$V[\cdot] : B[a,b] \to \mathbf{R} \tag{5.5}$$

has at a point $\phi \in B[a,b]$ *the invariant derivative* $\partial V[\phi]$ *in the class* $B[a,b]$, *if for any* $\Phi \in \hat{E}_\kappa[\phi]$ *the function*

$$v_\Phi(y) = V[T_y\Phi],\; y \in [-\kappa,\kappa]\,, \tag{5.6}$$

has at the point $y = 0$ *the derivative* $\dot{v}_\Phi(0)$ *invariant with respect to functions* $\Phi \in \hat{E}_\kappa[\phi]$. □

The *invariant derivative in the class* $B[a,b]$ is also called B*-invariant derivative*, for example, C^1-invariant derivative in case of $B[a,b] = C^1[a,b]$.

For $B[a,b] = C^m[a,b]$ the set $\hat{E}_\kappa[\phi]$ is denoted by $E_\kappa^{(m)}[\phi]$.

5.3 Examples

Example 5.2 We already noted in the example 4.1 that the derivative of the functional (2.1) has along the curves the invariance property. Let us calculate the invariant derivative of this functional by the definition 5.1; We assume that $\alpha : [a,b] \to \mathbf{R}$ is a continuous function.

Fix arbitrary functions $\phi \in C[a,b]$ and $\Phi \in E_\kappa[\phi]$. For the functional (2.1) the corresponding function $v_\Phi(y) = V[T_y\Phi]$ has the form

$$v_\Phi(y) \equiv \int\limits_a^b \alpha[\Phi(y+x)]\,dx = \int\limits_{y+a}^{y+b} \alpha[\Phi(x)]\,dx\,,$$

and its derivative

$$\left.\frac{dv_\Phi(y)}{dy}\right|_{y=0} = \left(\frac{d}{dy}\int\limits_{y+a}^{y+b} \alpha[\Phi(x)]\,dx\right)_{y=0} =$$

$$= \alpha[\Phi(b)] - \alpha[\Phi(a)] = \alpha[\phi(b)] - \alpha[\phi(a)]$$

is invariant with respect to $\Phi \in E_\kappa[\phi]$. Therefore the functional

$$\partial V[\phi] = \alpha[\phi(b)] - \alpha[\phi(a)] \tag{5.7}$$

is the invariant derivative of the functional (2.1). □

Example 5.3 Let in the functional

$$V[\phi] = \int\limits_a^b \alpha\left[\int\limits_\xi^b \beta[\phi(x)]\,dx\right] d\xi \tag{5.8}$$

$\alpha : \mathbf{R} \to \mathbf{R}$ is a continuous differentiable function, $\beta : \mathbf{R}^n \to \mathbf{R}$ is a continuous function.

To calculate the invariant derivative of the functional V at a point $\phi \in C[a,b]$ let us fix a function $\Phi \in E_\kappa[\phi]$ and consider

$$v_\Phi(y) \equiv \int\limits_a^b \alpha\,[\int\limits_\xi^b \beta[\Phi(y+x)]\,dx\,]\,d\xi =$$

$$= \int\limits_a^b \alpha\,[\int\limits_{\xi+y}^y \beta[\Phi(x)]\,dx\,]\,d\xi\,.$$

Calculate the derivative

$$\left.\frac{dv_\Phi(y)}{dy}\right|_{y=0} = \frac{d}{dy}\Big(\int\limits_a^b \alpha\,[\int\limits_{\xi+y}^y \beta[\Phi(x)]ds\,]\;d\xi\,\Big)_{y=+0} =$$

$$= \beta[\Phi(b)]\int\limits_a^b \dot\alpha\,[\int\limits_\xi^b \beta[\Phi(x)]\,dx\,]\;d\xi -$$