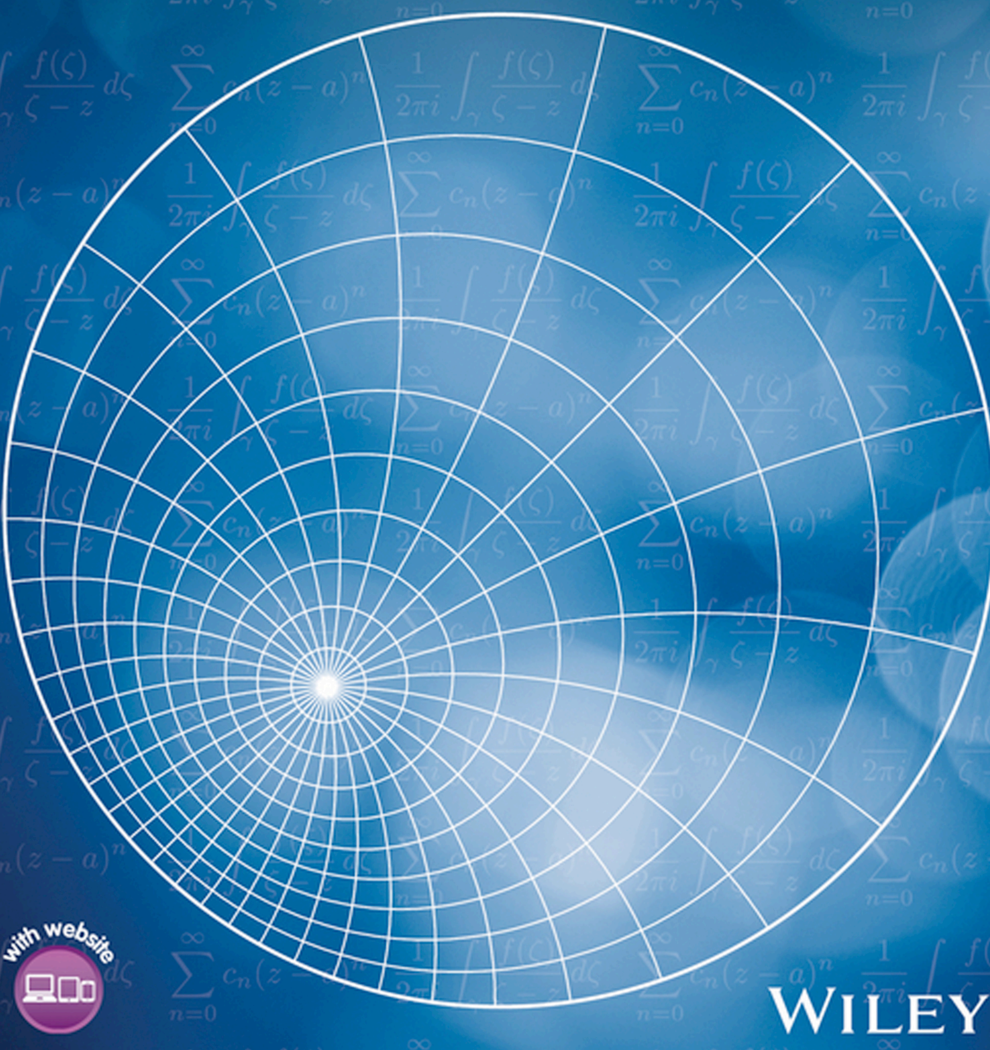
COMPLEX ANALYSIS

A MODERN FIRST COURSE IN FUNCTION THEORY

JERRY R. MUIR, JR.



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A Modern First Course in Function Theory

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To Stacey,
proofreader, patron,
and partner

CONTENTS

PREFACE

> This unfortunate name, which seems to imply that there is something unreal about these numbers and that they only lead a precarious existence in some people's imagination, has contributed much toward making the whole subject of complex numbers suspect in the eyes of generations of high school students.
>
> – Zeev Nehari [21] on the use of the term *imaginary number*

In the centuries prior to the movement of the 1800s to ensure that mathematical analysis was on solid logical footing, complex numbers, those numbers algebraically generated by adding $\sqrt{-1}$ to the real field, were utilized with increasing frequency as an ever-growing number of mathematicians and physicists saw them as useful tools for solving problems of the time. The 19th century saw the birth of complex analysis, commonly referred to as function theory, as a field of study, and it has since grown into a beautiful and powerful subject.

The functions referred to in the name "function theory" are primarily analytic functions, and a first course in complex analysis boils down to the study of the complex plane and the unique and often surprising properties of analytic functions. Familiar concepts from calculus – the derivative, the integral, sequences and series – are ubiquitous in complex analysis, but their manifestations and interrelationships are novel in this setting. It is therefore possible, and arguably preferable, to see these topics addressed in a manner that helps stress these differences, rather than following the same ordering seen in calculus.

Complex Analysis: A Modern First Course in Function Theory, First Edition. Jerry R. Muir, Jr.

This text grew from course notes I developed and tested on many unsuspecting students over several iterations of teaching undergraduate complex analysis at Rose-Hulman Institute of Technology and The University of Scranton. The following characteristics, rooted in my personal biases of how best to think of function theory, are worthy of mention.

- Complex analysis should never be underestimated as simply being calculus with complex numbers in place of real numbers and is distinguished from being so at every possible opportunity.

- Series are placed front and center and are a constant presence in a number of proofs and definitions. Analyticity is defined using power series to emphasize the difference between analytic functions and the differentiable functions studied in calculus. There is an intuitive symmetry between analyzing zeros using power series and singularities using Laurent series. The early introduction of power series allows the complex exponential and trigonometric functions to be defined as natural extensions of their real counterparts.

- Many properties of analytic functions seem counterintuitive (perhaps unbelievable) to students recently removed from calculus, and seeing these as early as possible emphasizes the distinctive nature of complex analysis. In service of this, Liouville's theorem, factorization using zeros, the open mapping theorem, and the maximum principle are considered prior to the more-involved Cauchy integral theory.

- Analytic function theory is built upon the trinity of power series, the complex derivative, and contour integrals. Consequently, the Cauchy–Riemann equations, an alternative expression of analyticity tied to differentiability in two real variables, are naturally partnered with the conformal mapping theorem at the end of the line of properties of analytic functions. Harmonic functions, also strongly reliant on this multivariable calculus topic, are the subject of the final chapter, allowing their study to benefit from the full theory of analytic functions.

- The geometric mapping properties of planar functions give intuition that was easily provided by the graphs of functions in calculus and help to tie geometry to function theory. In particular, linear fractional (Möbius) transformations are developed in service to this principle, prior to the introduction of analyticity or conformal mapping.

- The study of any flavor of analysis requires a box of tools containing basic geometric and topological facts and the related properties of sequences. These topics, in the planar setting, are addressed up front for easy reference, so as not to interrupt the subsequent presentation of function theory.

When faced with the choice of glossing over some details to more quickly "get to the good stuff" or ensuring that the development of topics is logically complete and consistent, I opted for the latter, leaving the reader the freedom to decide how to approach the text. This was done (with one caveat[1]) subject to the constraint that all material encountered in the typical undergraduate sequence of calculus courses is assumed without proof. This includes results that may not be proved in those courses but whose proofs are part of a standard course in real analysis. This helps to streamline the presentation while reducing overlap with other courses. For example, complex sequences are shown to converge if and only if their real and imaginary components converge. Then the assumed algebraic rules for convergent real sequences imply the same rules for complex sequences. A similar decomposition into real and imaginary parts readily provides familiar rules for derivatives and integrals of complex-valued functions of a real variable. It is important to clarify that material proved in a real analysis course that is not considered in calculus, such as aspects of topology or convergence of sequences of functions, is dealt with here.

It is my hope that this text provides a clear, concise exposition of function theory that allows the reader to observe the development of the theory without "losing the forest through the trees." To wit, connections between complex analysis and the sciences and engineering are noted but not explored. Although interesting and important, presenting applications to areas with which the reader is not assumed familiar while maintaining the previously mentioned commitment to logical completeness would have significantly lengthened the text and interrupted the flow of ideas.

The apex of a first complex analysis course is the residue theorem and its application to the evaluation of real integrals (Sections 5.3 and 5.4), and a number of pedagogical options are available for reaching or surpassing that point depending on the background of the students, the goals of the instructor, and the amount of rigor desired. The minimal route to get to Section 5.4 without skipping material used at some point along the way is: Sections 1.2–1.7, 2.1–2.4, 2.7–2.9, 3.1–3.4, 4.1–4.3, 5.1–5.4. It has been my experience that Section 5.4 can be reached in a fourteen-week-semester course taught to students with a multivariable calculus background with the possible omission of only one or two of the latter sections of Chapter 3, if necessary. The completeness of the text leaves the decision in the instructor's hands of what results could be accepted without proof or covered lightly. For instance, one willing to assert that properties of complex sequences or continuous functions behave like their real counterparts, assume that integrals may be exchanged with series as needed eschewing the details of uniform convergence, or postpone the presentation of geometric mapping properties and linear fractional transformations should find the text suitably flexible, as should an instructor whose students already have comfort with the material covered in the first chapter or so. An instructor venturing into the last chapter on harmonic functions will want to have covered Sections 3.5 and 3.6 at its start and Section 2.6 later on.

Each section of the text concludes with remarks, under the heading "Summary and Notes," that review the main points of the section, note who discovered them and when, place them in context either within the whole of the text or within the broader mathematical world, and/or note interesting connections to the sciences. No

[1]We assume the Jordan curve theorem, that every simple closed planar curve has an inside and outside. See Remark 4.1.4 for discussion.

one will ever accuse me of being well versed in the history of mathematics, but I have always found the topic interesting. I hope the tidbits provided within these notes whet the reader's appetite to learn more about it than I know and find the history-related references in the bibliography to be as educational as I did.

Exercises of both a computational and theoretical nature are given. Many of the problems asking students to "prove" or "show" something require little more than a calculation, while others have more depth. Those marked with the symbol "▷" contain results that are referenced later in the text. Of note, there are several exercises that either provide an alternative definition of, or point of view on, a concept or introduce a complex analysis topic that is not included in the body of the text. Examples include the definition of compactness using open covers, the definition of contour integrals using limits of Riemann sums, Hadamard products of analytic functions, a proof of Cauchy's theorem for cycles, the argument principle, the application of the residue theorem to sum certain numerical series, the uniqueness of the inverse Laplace transform up to continuity, and Fejér's theorem for the Cesàro summability of Fourier series. These would serve as interesting projects for good students, as would a thorough reading of the epilogue on the Riemann mapping theorem.

My undergraduate professors at SUNY Potsdam were fond of saying that a mathematics book should always be read with paper and pencil in hand. I hope that any detail lost to the tug of war between brevity and lucidity can be recovered with a bit of work on the side, to the reader's benefit.

Function theory has been around for a long time, and I am grateful for the instruction I received as a student and the excellent complex analysis books listed in the bibliography. My love of, and point of view on, the subject surely germinated during the privileged time I spent as a student of Ted Suffridge at the University of Kentucky and from the books [7, 26] from which I learned and referenced during those years. I have no doubt their influence is present throughout the following pages in places even I would not expect.

I thank my students over the years, especially those who weren't shy about pointing out flaws when this work was in its infancy, and Stephen Aldrich, Michael Dorff, and Stacey Muir, who used early drafts and offered valuable feedback. Stacey's unlimited help and support deserves special recognition – for the last decade, she has been contractually obligated not to run away when I'm at my wit's end with a project.

Finally, I could not be more appreciative of the team at Wiley for their able support during the preparation of this text, for allowing me the freedom to write the book I imagined, and for believing in the first place that the course notes of a crackpot professor could become a reasonable book. That said, my fingers did the typing, and, alas, any misteaks are mine.

JERRY R. MUIR, JR.

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CHAPTER 1

THE COMPLEX NUMBERS

In this chapter, we introduce the complex numbers and their interpretation as points in a number plane, an analog to the real number line. We develop the algebraic, geometric, and topological properties of the set of complex numbers, many of which mirror those of the real numbers. These properties, especially the topological ones, are connected to sequences, and thus we conclude the chapter by studying the basic nature of sequences and series. At the conclusion of the chapter, we will possess the tools necessary to begin the study of functions of a complex variable.

1.1 Why?

Our work in this text can best be understated as follows: Let's throw $\sqrt{-1}$ into the mix and see what happens to the calculus. The result is a completely different flavor of analysis, a separate field distinguished from its real-variable sibling in some striking ways.

The use of $\sqrt{-1}$ as an intermediate step in finding solutions to real-variable problems goes back centuries. In the Renaissance, Italian mathematicians used complex numbers as a tool to find *real* roots of cubic equations. The algebraic use of complex

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numbers became much more mainstream due to the work of Leonhard Euler in the 18^{th} century and later, Carl Friedrich Gauss. Euler and Jean le Rond d'Alembert are generally credited with the first serious considerations of functions of a complex variable – the former considered such functions as an intermediate step in the calculation of certain *real* integrals, while the latter saw these functions as useful in his study of fluid mechanics.

Introducing complex numbers as a stepping stone to solve real problems is a common historical theme, and it is worth recalling how other familiar systems of numbers can be viewed to solve particular algebraic and analytic problems. The natural numbers, integers, rational numbers, and real numbers satisfy the set containments $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, but each subsequent set has characteristics not present in its predecessor. Where the natural numbers $\mathbb{N} = \{1, 2, \dots\}$ are closed under addition and multiplication, extending to the integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ provides an additive identity and inverses. The set of rational numbers $\mathbb{Q}$, consisting of all fractions of integers, has multiplicative inverses of its nonzero elements and hence is an algebraic *field* under addition and multiplication.

The move from $\mathbb{Q}$ to the real numbers $\mathbb{R}$ is more analytic than algebraic. Although $\mathbb{Q}$ is a field, it is not *complete*, meaning there are "holes" that need to be filled. For instance, consider the equation $x^2 = 2$. Since $1^2 = 1$ and $2^2 = 4$, it seems that a solution to the equation should exist and lie somewhere in between. Further analysis reveals that a solution should lie between $5/4$ and $3/2$. Successive subdivisions may be used to target where a solution should lie, but that point is not in $\mathbb{Q}$. Beyond unsolvable algebraic equations lies the number π, the ratio of a circle's circumference to its diameter, which can also be shown not to lie in $\mathbb{Q}$. The alleviation of these problems comes by allowing the set $\mathbb{R}$ of real numbers to be the *completion* of $\mathbb{Q}$. In satisfyingly imprecise terms, $\mathbb{R}$ is equal to $\mathbb{Q}$ with the "holes filled in." This is done so that the *axiom of completeness* (i.e. the least upper bound property) holds. See Appendix B for more detail. The result is that $\mathbb{R}$ is a *complete ordered field*.

The upgrade from the real numbers to the complex numbers has both algebraic and analytic motivation. The real numbers are not *algebraically* complete, meaning there are polynomial equations such as $x^2 = -1$ with no solutions. The incorporation of $\sqrt{-1}$ mentioned earlier is a direct response to this. But the work of Euler and d'Alembert shows how moving outside $\mathbb{R}$ facilitates analytic methods as well. While their work did much for bringing credibility to the use of complex numbers, it was during the 19^{th} century, in the movement to deliver rigor to mathematical analysis, that complex function theory gained its footing as a separate subject of mathematical study, due largely to the work of Augustin-Louis Cauchy, Bernhard Riemann, and Karl Weierstrass.

Function theory is the study of the calculus of complex-valued functions of a complex variable. The analysis of functions on this new domain will quickly distinguish itself from real-variable calculus. As the reader will soon see, by combining the algebra and geometry inherent in this new setting, we will be able to perform a great deal of analysis that is not available on the real domain. Such analysis will include some intuition-bending results and techniques that solve problems from calculus that are not easily accessible otherwise.

Before setting out on our study of complex analysis, we must agree on a starting point. We assume that the reader is familiar with the fundamentals of differential, integral, and multivariable calculus. The language of sets and functions is freely used; the unfamiliar reader should examine Appendix A.

Lastly, to whet our appetites for what is to come, here are a handful of exercises appearing later in the text the statements of which are understandable from calculus, but whose solutions are either made possible or much simpler by the introduction of complex numbers.

Forthcoming Exercises

1. Derive triple angle identities for $\sin 3\theta$ and $\cos 3\theta$. [Section 2.4, Exercise 6]
2. Find a continuous one-to-one planar transformation that maps the region lying inside the circles $(x-1)^2+y^2=4$ and $(x+1)^2+y^2=4$ onto the upper half-plane $y \geq 0$. [Section 2.6, Exercise 13]
3. Find the radius of convergence of the Taylor series expansion of the function

$$f(x) = \frac{\sin x}{1+x^4}$$

   about $a=2$. [Section 3.2, Exercise 3]
4. Verify the summation identity, where $c \in \mathbb{R}$ is a constant.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+c^2} = \begin{cases} \dfrac{\pi^2}{6} & \text{if } c = 0, \\ \dfrac{\pi}{2c}\coth \pi c - \dfrac{1}{2c^2} & \text{if } c \neq 0 \end{cases}$$

   [Section 5.3, Exercise 11]
5. Evaluate the following integrals, where $n \in \mathbb{N}$ and $a, b > 0$.

   (a) $\displaystyle\int_0^{2\pi} \cos^n t\,dt,$ [Section 2.8, Exercise 4]

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin ax}{x(x^2+b^2)}\,dx,$ [Section 5.4, Exercise 7]

   (c) $\displaystyle\int_0^{\infty} \frac{\sqrt[n]{x}}{x^2+a^2}\,dx,$ [Section 5.4, Exercise 10]
6. Find a real-valued function u of two variables that satisfies

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

   inside the unit circle and continuously extends to equal 1 for points on the circle with $y > 0$ and 0 for points on the circle with $y < 0$. [Section 6.2, Exercise 3]

1.2 The Algebra of Complex Numbers

As alluded to in Section 1.1, we desire to expand from the set of real numbers in a way that provides solutions to polynomial equations such as $x^2 = -1$. One may be tempted to simply define a number that solves this equation. The drawback to doing so is that the negative of this number would also be a solution, and this could cause some ambiguity in the definition. We therefore choose a different method.

1.2.1 Definition. A *complex number* is an ordered pair of real numbers. The set of complex numbers is denoted by $\mathbb{C}$.

By definition, any $z \in \mathbb{C}$ has the form $z = (x, y)$ for numbers $x, y \in \mathbb{R}$. What distinguishes complex numbers from their counterparts, the two-dimensional vectors in $\mathbb{R}^2$, is their algebra – specifically, their multiplication.

1.2.2 Definition. If (a, b) and (c, d) are complex numbers, then we define the algebraic operations of *addition* and *multiplication* by

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, ad + bc).$$

Clearly, $\mathbb{C}$ is closed under both of these operations. (Adding or multiplying two complex numbers results in another complex number.)

Notice that if $a, b \in \mathbb{R}$, then $(a, 0) + (b, 0) = (a + b, 0)$ and $(a, 0)(b, 0) = (ab, 0)$. Therefore $a \mapsto (a, 0)$ is a natural algebraic embedding of $\mathbb{R}$ into $\mathbb{C}$. Accordingly, it is natural to write a for the complex number $(a, 0)$, and in this way, we consider $\mathbb{R} \subseteq \mathbb{C}$.

For any complex number $z = (x, y)$,

$$z = (x, 0) + (0, 1)(y, 0) = x + (0, 1)y.$$

In other words, each complex number can be written uniquely in terms of its two real components and the complex number $(0, 1)$. This special complex number gets its own symbol.

1.2.3 Definition. The *imaginary unit* is the complex number $i = (0, 1)$. A complex number z expressed as

$$z = x + iy \tag{1.2.1}$$

is said to be in *rectangular form*.

Because every complex number can be written uniquely as above, we (usually) refrain from using the ordered pair notation in favor of using the rectangular form. Notice that i is a solution to the equation $z^2 = -1$.

It is left as an exercise to verify that 0 is the additive identity and 1 is the multiplicative identity, every member of $\mathbb{C}$ has an additive inverse, both operations are associative and commutative, multiplication distributes over addition, and if $z \neq 0$ is written as in (1.2.1), then it has the multiplicative inverse

$$z^{-1} = \frac{1}{z} = \frac{x}{x^2 + y^2} + i\left(\frac{-y}{x^2 + y^2}\right) \tag{1.2.2}$$

in $\mathbb{C}$. This shows that $\mathbb{C}$ is a *field*.

1.2.4 Definition. For a complex number z written as in (1.2.1), we call the real numbers x and y the *real part* and *imaginary part* of z, respectively, and use the symbols $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$. If $\operatorname{Re} z = 0$, then z is called *imaginary* (or *purely imaginary*). The *conjugate* of z is the complex number $\overline{z} = x - iy$. The *modulus* (or *absolute value*) of z is the nonnegative real number $|z| = \sqrt{x^2 + y^2}$.

It is a direct calculation to verify the relationship

$$|z|^2 = z\overline{z} \tag{1.2.3}$$

for all $z \in \mathbb{C}$. Other useful identities involving moduli and conjugates of complex numbers are left to the exercises.

1.2.5 Example. The identity (1.2.3) is useful for finding the rectangular form of a complex number. For instance, consider the quotient

$$z = \frac{1 + 2i}{2 + i}.$$

To find the expressions from Definition 1.2.4, we multiply by the conjugate of the denominator over itself,

$$z = \frac{1 + 2i}{2 + i}\,\frac{2 - i}{2 - i} = \frac{4 + 3i}{5},$$

to get a positive denominator. We see that $\operatorname{Re} z = 4/5$, $\operatorname{Im} z = 3/5$, $|z| = 1$, and $\overline{z} = (4 - 3i)/5$.

Summary and Notes for Section 1.2.

The set of complex numbers $\mathbb{C}$ consists of ordered pairs of real numbers. The real numbers are the those complex numbers of the form $(x, 0)$ for $x \in \mathbb{R}$, and $i = (0, 1)$ is the imaginary unit. Algebraic operations are defined to make $\mathbb{C}$ a field and so that $i^2 = -1$. We write the complex number $z = (x, y)$ in the rectangular form $z = x + iy$, and the conjugate of z is $\overline{z} = x - iy$.

In their attempts to find real solutions to cubic equations, Italian mathematicians found it necessary to manipulate complex numbers. Perhaps the first to consider them was Gerolamo Cardano in the 16^{th} century, who named them "fictitious numbers." Rafael Bombelli introduced the algebra of complex numbers shortly thereafter. At that time, the square roots of negative numbers were just manipulated as a means to an end. The ordered pair definition can be traced to William Rowan Hamilton almost three centuries later.

Exercises for Section 1.2.

1. For the following complex numbers z, calculate $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, and $\overline{z}$.

   (a) $z = 3 + 2i$

   (b) $z = \dfrac{1 + i}{i}$

(c) $z = \dfrac{2-i}{1+i} + i$

(d) $z = (4+2i)\overline{(3+i)}$

2. Verify the following algebraic properties of $\mathbb{C}$.

   (a) The complex numbers 0 and 1 are the additive and multiplicative identities of $\mathbb{C}$, respectively.

   (b) Each $z \in \mathbb{C}$ has an additive inverse.

   (c) Addition and multiplication of complex numbers is associative. In other words, $z + (w + v) = (z + w) + v$ and $z(wv) = (zw)v$ for all $z, w, v \in \mathbb{C}$.

   (d) Addition and multiplication of complex numbers is commutative. That is, $z + w = w + z$ and $zw = wz$ for all $z, w \in \mathbb{C}$.

   (e) Multiplication of complex numbers distributes over addition. That is, $a(z + w) = az + aw$ for all $a, z, w \in \mathbb{C}$.

   (f) If $z \in \mathbb{C}$ is nonzero, then its multiplicative inverse is as given in (1.2.2).

3. $\triangleright$ Verify the following identities involving the conjugate.

   (a) For each $z \in \mathbb{C}$, $\overline{\overline{z}} = z$.

   (b) For each $z \in \mathbb{C}$,
$$\operatorname{Re} z = \frac{z + \overline{z}}{2}, \qquad \operatorname{Im} z = \frac{z - \overline{z}}{2i}.$$

   (c) For all $z, w \in \mathbb{C}$,
$$\overline{z + w} = \overline{z} + \overline{w}, \qquad \overline{zw} = \overline{z}\,\overline{w}.$$

4. $\triangleright$ Verify the following identities involving the modulus. For each, let $z, w \in \mathbb{C}$.

   (a) $|zw| = |z||w|$

   (b) $|z/w| = |z|/|w|$ if $w \neq 0$

   (c) $|\overline{z}| = |z|$

   (d) $-|z| \leq \operatorname{Re} z \leq |z|$

5. $\triangleright$ Prove that for all $z, w \in \mathbb{C}$,
$$|z + w|^2 = |z|^2 + 2\operatorname{Re} z\overline{w} + |w|^2.$$

6. Let $p\colon \mathbb{C} \to \mathbb{C}$ be a polynomial. That is, $p(z) = \sum_{k=0}^{n} a_k z^k$ for $a_0, \ldots, a_n \in \mathbb{C}$ and $a_n \neq 0$. A *root* (or *zero*) of p is a number $r \in \mathbb{C}$ such that $p(r) = 0$. Show that if $a_0, \ldots, a_n \in \mathbb{R}$, then $\overline{r}$ is a root of p whenever r is a root of p.

7. Let $a, b \in \mathbb{R}$. Consider the function $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by
$$T(x, y) = (\operatorname{Re}[(a + ib)(x + iy)], \operatorname{Im}[(a + ib)(x + iy)]).$$
   Prove that T is a linear transformation on $\mathbb{R}^2$, and determine a 2×2 matrix form for T. What does T represent?

8. For ordered pairs of real numbers (a, b) and (c, d), what drawbacks are there to defining multiplication of complex numbers by $(a, b)(c, d) = (ac, bd)$?

1.3 The Geometry of the Complex Plane

The real number line is the geometric realization of the set of real numbers and accordingly is a useful tool for conceptualization. Since complex numbers are defined to be ordered pairs of real numbers, it is only natural to visualize the set of complex numbers as the points in the Cartesian coordinate plane $\mathbb{R}^2$. This geometric interpretation is essential to the analysis of complex functions.

1.3.1 Definition. When its points are considered to be complex numbers, the Cartesian coordinate plane is referred to as the *complex plane* $\mathbb{C}$. The x- and y-axes in the plane are called the *real and imaginary axes*, respectively, in $\mathbb{C}$.

Because addition of complex numbers mirrors addition of vectors in $\mathbb{R}^2$, we use vectors to geometrically interpret addition in terms of parallelograms. Continuing this line of thought, we see that the value $|z|$, as the distance from the point $z \in \mathbb{C}$ to 0, is the length (or magnitude) of the vector z. If $z, w \in \mathbb{C}$, then $z - w$, in vector form, is the vector pointing from w to z. Therefore $|z - w|$ is equal to the distance between z and w. Lastly, we note that the operation of complex conjugation is realized geometrically as reflection in the real axis. See Figure 1.1.

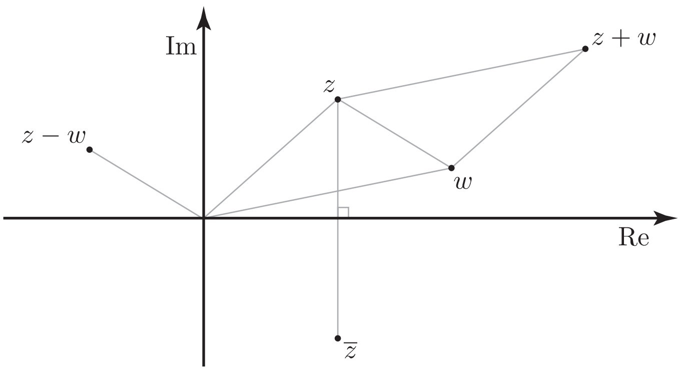

Figure 1.1 $z + w$, $z - w$, and $\overline{z}$ for some $z, w \in \mathbb{C}$

Another geometric consequence of the parallelogram interpretation is the triangle inequality, which gives that if $z, w \in \mathbb{C}$, then the distance from 0 to $z + w$ is never greater than the sum of the distances from 0 to z and 0 to w. We prove it as follows.

1.3.2 Triangle Inequality. *If $z, w \in \mathbb{C}$, then*

$$|z + w| \leq |z| + |w|. \tag{1.3.1}$$

Proof. We use Exercises 4 and 5 of Section 1.2 to calculate

$$\begin{aligned} |z + w|^2 &= |z|^2 + 2 \operatorname{Re} z\overline{w} + |w|^2 \\ &\leq |z|^2 + 2|z\overline{w}| + |w|^2 \end{aligned}$$

$$= |z|^2 + 2|z||w| + |w|^2$$
$$= (|z| + |w|)^2.$$

Taking the square root of both sides completes the proof. □

If $a \in \mathbb{C}$ and $r > 0$, then the circle in $\mathbb{C}$ centered at a of radius r is the set of all points whose distance from a is r. From our above observation, this circle can be described in set notation by $\{z \in \mathbb{C} : |z - a| = r\}$. The inside of a circle, called a *disk*, is a commonly used object and is denoted by

$$D(a; r) = \{z \in \mathbb{C} : |z - a| < r\}. \tag{1.3.2}$$

The most prominently used disk in the plane is the unit disk – the disk centered at 0 of radius 1. For this, we use the special symbol

$$\mathbb{D} = D(0; 1). \tag{1.3.3}$$

1.3.3 Example. Let us consider the geometry of the set

$$E = \{z \in \mathbb{C} : |1 + iz| < 2\}$$

in two ways. First, write $z = x + iy$ to see that the condition defining E is equivalent to $|1 - y + ix| < 2$ or $\sqrt{x^2 + (y - 1)^2} < 2$. This describes all planar points of distance less than 2 from $(0, 1) = i$. Hence $E = D(i; 2)$.

In this circumstance, there is an advantage to eschewing real and imaginary parts. Note that

$$|1 + iz| = |i(-i + z)| = |i||z - i| = |z - i|.$$

Thus $E = \{z \in \mathbb{C} : |z - i| < 2\} = D(i; 2)$.

Summary and Notes for Section 1.3.

Since the complex numbers are ordered pairs of real numbers, the set $\mathbb{C}$ of complex numbers is geometrically realized as the plane $\mathbb{R}^2$. The addition and modulus of complex numbers parallel the addition and magnitude of planar vectors. The triangle inequality gives an important bound on sums.

In 1797, a Norwegian surveyor named Caspar Wessel was the first of many to consider the geometric interpretation of the complex numbers, but his work was largely unknown as was that of Jean-Robert Argand in 1806. (The complex plane is often referred to as the *Argand plane*.) The work of Carl Friedrich Gauss in the first half of the 19$^{\text{th}}$ century brought the concept to the masses.

Exercises for Section 1.3.

1. Geometrically illustrate the parallelogram rule for the complex numbers $2 + i$ and $1 + 3i$.
2. Geometrically illustrate the relationship between the complex number $-1 + 2i$ and its conjugate.
3. Describe the following sets geometrically. Sketch each.

(a) $\{z \in \mathbb{C} : |z - 1 + i| = 2\}$
(b) $\{z \in \mathbb{C} : |z - 1|^2 + |z + 1|^2 \leq 6\}$
(c) $\{z \in \mathbb{C} : \operatorname{Im} z > \operatorname{Re} z\}$
(d) $\{z \in \mathbb{C} : \operatorname{Re}(iz + 1) < 0\}$

4. Provide a geometric description of complex multiplication for nonzero $z, w \in \mathbb{C}$. It is helpful to write $z = r(\cos\theta + i\sin\theta)$ and $w = \rho(\cos\varphi + i\sin\varphi)$ and use trigonometric identities.
5. For which pairs of complex numbers is equality attained in the triangle inequality? Prove your answer.
6. $\triangleright$ Prove that for all $z, w \in \mathbb{C}$, $||z| - |w|| \leq |z - w|$.
7. Show that for all $z \in \mathbb{C}$, $|\operatorname{Re} z| + |\operatorname{Im} z| \leq \sqrt{2}\,|z|$.
8. This exercise concerns lines in $\mathbb{C}$.

(a) Let $a, b \in \mathbb{C}$ with $b \neq 0$. Prove that the set

$$L = \left\{z \in \mathbb{C} : \operatorname{Im}\left(\frac{z-a}{b}\right) = 0\right\}$$

is a line in $\mathbb{C}$. Explain the role of a and b in the geometry of L. (*Hint*: Recall the vector form of a line from multivariable calculus.)

(b) Let C be a circle in $\mathbb{C}$ with center c and radius $r > 0$. If a lies on C, write the line tangent to C at a in the form given in part (a).

1.4 The Topology of the Complex Plane

The *topology* of a certain space (in our case $\mathbb{C}$) gives a useful alternative to traditional geometry to describe relationships between points and sets. The key concepts of limits and continuity from calculus are tied to the topology of the real line, as we will see is also true in the plane. That connection just scratches the surface of how powerful a tool we will find planar topology to be for analyzing functions.

We begin by observing that with respect to a given subset of $\mathbb{C}$, each point of $\mathbb{C}$ is of one of three types.

1.4.1 Definition. Let $A \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. Then a is an *interior point* of A if A contains a disk centered at a, a is an *exterior point* of A if it is an interior point of the complement $\mathbb{C} \setminus A$, and a is a *boundary point* of A if it is neither an interior point of A nor an exterior point of A. (See Figure 1.2.)

These points form the following sets.

1.4.2 Definition. Let $A \subseteq \mathbb{C}$. The set of interior points of A is called the *interior* of A and is denoted A°. The set of boundary points of A is called the *boundary* of A and is denoted ∂A.

Note that the set of exterior points of A is the interior of $\mathbb{C} \setminus A$, and so we need not define a new symbol for this set. We have that $\mathbb{C}$ can be decomposed into the disjoint union

$$\mathbb{C} = A^\circ \cup \partial A \cup (\mathbb{C} \setminus A)^\circ.$$

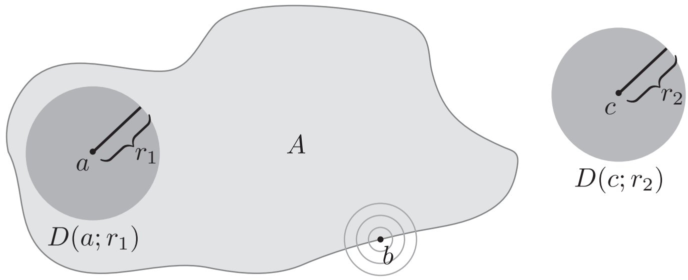

Figure 1.2 a is an interior point of A, b is a boundary point of A, and c is an exterior point of A

The set A contains all of its interior points, none of its exterior points, and none, some, or all of its boundary points. The extremal cases are special.

1.4.3 Definition. Let $A \subseteq \mathbb{C}$. If $\partial A \cap A = \varnothing$, then A is *open*. If $\partial A \subseteq A$, then A is *closed*.

The following properties can be deduced from the above definitions. Their proofs are left as an exercise.

1.4.4 Theorem. *The following hold for $A \subseteq \mathbb{C}$.*

(a) *The set A is closed if and only if its complement $\mathbb{C} \setminus A$ is open.*

(b) *The set A is open if and only if for every $a \in A$, there exists $r > 0$ such that $D(a; r) \subseteq A$.*

(c) *A point $a \in \mathbb{C}$ is in ∂A if and only if $D(a; r) \cap A \neq \varnothing$ and $D(a; r) \setminus A \neq \varnothing$ for all $r > 0$.*

1.4.5 Example. We study the disk $D(a; r)$ for some $a \in \mathbb{C}$ and $r > 0$. If $z_0 \in D(a; r)$, then let $\rho = r - |z_0 - a|$. Then $0 < \rho \leq r$. For all $z \in D(z_0; \rho)$,

$$|z - a| = |z - z_0 + z_0 - a| \leq |z - z_0| + |z_0 - a| < \rho + (r - \rho) = r$$

by the triangle inequality, showing $z \in D(a; r)$. Therefore $D(z_0; \rho) \subseteq D(a; r)$. This implies that $D(a; r)$ is an open set.

It is left as an exercise (using an argument quite similar to the one just presented) to show that the exterior points of $D(a; r)$ form the set $\{z \in \mathbb{C} : |z - a| > r\}$. Therefore the boundary of the disk is

$$\partial D(a; r) = \{z \in \mathbb{C} : |z - a| = r\}, \tag{1.4.1}$$

which is exactly the circle of radius r centered at a.

1.4.6 Definition. Given any set $A \subseteq \mathbb{C}$, the set

$$\overline{A} = A \cup \partial A \tag{1.4.2}$$

is called the *closure* of A.

Many of the properties of the closure are addressed in the exercises.

1.4.7 Example. If $D(a;r)$ is the disk in Example 1.4.5, then its closure is the *closed disk*

$$\overline{D}(a;r) = \overline{D(a;r)} = \{z \in \mathbb{C} : |z-a| \leq r\}. \tag{1.4.3}$$

1.4.8 Definition. Let $A \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. We say that a is a *limit point* of A provided that $D(a;r) \cap A \setminus \{a\} \neq \varnothing$ for every $r > 0$.

In other words, a is a limit point of A if every disk centered at a intersects A at a point *other than* a. This leads to another useful characterization of closed sets.

1.4.9 Theorem. *A set $E \subseteq \mathbb{C}$ is closed if and only if E contains all of its limit points.*

Proof. Suppose that E is closed and that a is a limit point of E. Were a an exterior point of E, we would have $D(a;r) \subseteq \mathbb{C} \setminus E$ for some $r > 0$. Since this contradicts that a is a limit point of E, it must be that a is an interior point or boundary point of E. Either way, $a \in E$.

Conversely, assume that E contains all of its limit points. Suppose that $a \in \partial E \setminus E$. For any $r > 0$, $D(a;r) \cap E \neq \varnothing$ by Theorem 1.4.4. Since $a \notin E$, $D(a;r) \cap E \setminus \{a\} \neq \varnothing$ for all $r > 0$, and thus a is a limit point of E. This shows that $a \in E$, a contradiction. Thus $\partial E \subseteq E$, and hence E is closed. □

We continue with two more definitions.

1.4.10 Definition. A set $A \subseteq \mathbb{C}$ is *bounded* if $A \subseteq D(0;R)$ for some $R > 0$.

1.4.11 Definition. A set $K \subseteq \mathbb{C}$ is *compact* if K is closed and bounded.

One must be careful not to be misled by the simplicity of the above definition and underestimate the importance of compact sets to the study of analysis. In fact, many properties of complex functions depend on compactness.

A reader with some previous exposure to topological concepts may have seen compactness defined in terms of “open covers.” This definition is of great importance to the study of topology, but does not serve our purpose in this text. That our definition is equivalent is the content of the Heine–Borel theorem. An outline of the proof of this theorem is included in the exercises.

We now consider our final topological concept.

1.4.12 Definition. Nonempty sets $A, B \subseteq \mathbb{C}$ are *separated* if $\overline{A} \cap B = A \cap \overline{B} = \varnothing$. A nonempty set $E \subseteq \mathbb{C}$ is *connected* if E is not equal to the union of separated sets. Otherwise, E is *disconnected*.

While this definition may seem complicated, it should bring the reader comfort that the intuitive notion of connectedness matches the rigorous definition. The following observation is key to a method of detecting connectedness that will be sufficient for most circumstances we will encounter.

1.4.13 Lemma. *Let $A, B \subseteq \mathbb{C}$ be separated sets, $a \in A$, $b \in B$, and L be the line segment with endpoints a and b. Then $L \not\subseteq A \cup B$.*

Proof. Suppose $L \subseteq A \cup B$. If $u = (b-a)/|b-a|$, then $L = \{a + tu : 0 \le t \le |b-a|\}$. Set

$$t_0 = \sup\{t \in [0, |b-a|] : a + tu \in A\}, \qquad c = a + t_0 u \in L.$$

(See Appendix B for properties of the supremum.)

If $c \in A$, then $c \notin \overline{B}$, and hence $D(c; r) \cap B = \varnothing$ for some $r > 0$. Furthermore, $t_0 < |b-a|$, and if $t_0 < t < \min\{t_0 + r, |b-a|\}$, then $a + tu \in L \setminus A \subseteq B$. But $|(a+tu) - c| = t - t_0 < r$, a contradiction.

If $c \in B$, then $c \notin \overline{A}$, and so $D(c; r) \cap A = \varnothing$ for some $r > 0$. But $t_0 > 0$, and there must exist $\max\{t_0 - r, 0\} < t < t_0$ such that $a + tu \in A$. (See Theorem B.4.) But $|(a+tu) - c| = t_0 - t < r$, a contradiction. Hence $L \not\subseteq A \cup B$. □

We now see that if a nonempty set $E \subseteq \mathbb{C}$ is such that the line segment connecting two arbitrary points in E lies in E, then E is connected. *For instance, all open and closed disks are connected, as are all lines, rays, and line segments.* This can be taken a step further. The proof of the following theorem and related results are considered in the exercises. See Figure 1.3.

1.4.14 Theorem. *Let $E \subseteq \mathbb{C}$ be nonempty. If for all $a, b \in E$, there are $a_0, \ldots, a_n \in E$ such that $a_0 = a$, $a_n = b$, and for each $k = 1, \ldots, n$, the line segment with endpoints a_{k-1} and a_k lies in E, then E is connected.*

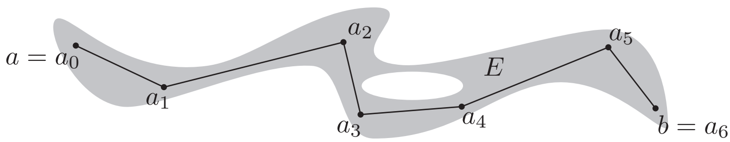

Figure 1.3 An illustration of Theorem 1.4.14 with $n = 6$

In our upcoming work, the most important connected sets are also open and warrant their own name.

1.4.15 Definition. A connected open subset of $\mathbb{C}$ is called a *domain*.

1.4.16 Theorem. *Let $\Omega \subseteq \mathbb{C}$ be a nonempty open set. Then Ω is a domain if and only if it is not equal to the union of disjoint nonempty open sets.*

Proof. It is equivalent to show that Ω is disconnected if and only if $\Omega = A \cup B$, where both A and B are open, nonempty, and $A \cap B = \varnothing$.

If Ω is disconnected, then let $\Omega = A \cup B$, where A and B are separated. Let $a \in A$. Since a is an exterior point of B, there is $r_1 > 0$ such that $D(a; r_1) \cap B = \varnothing$. Since $a \in \Omega$, there is $r_2 > 0$ such that $D(a; r_2) \subseteq \Omega$. If $r = \min\{r_1, r_2\} > 0$, then $D(a; r) \subseteq \Omega \setminus B = A$, showing A is open. A symmetric argument shows B is open.

Conversely, suppose $\Omega = A \cup B$ for disjoint nonempty open sets A, B. Every point of A is an exterior point of B and hence $A \cap \overline{B} = \varnothing$. Similarly, $\overline{A} \cap B = \varnothing$, showing A and B are separated. Thus Ω is disconnected. □

We conclude this section with one more definition.

1.4.17 Definition. A maximal connected subset of a set $E \subseteq \mathbb{C}$ is called a *component* of E.

This means that $A \subseteq E$ is a component of E if A is connected and for any connected set $B \subseteq E$ such that $A \subseteq B$, it must be that $A = B$. It is an exercise to show that every point of E lies in a component of E (and hence components exist).

1.4.18 Example. We know that intervals in $\mathbb{R}$ are connected. If $E = (-1, 0) \cup (0, 1)$, then E is disconnected using the separated sets $A = (-1, 0)$ and $B = (0, 1)$. In fact, A and B are components of E.

One may easily decompose any set with at least two elements into the union of two nonempty disjoint subsets, showing the importance of $\overline{A}$ and $\overline{B}$ in Definition 1.4.12. Observing that $\overline{A} \cap \overline{B} = \{0\} \neq \varnothing$ in this example shows why only one closure is considered at a time.

1.4.19 Example. We conclude by analyzing a set with regard to all concepts introduced in this section. Filling in the details of the statements made is left as an exercise. Let

$$E = \{z \in \mathbb{C} : |\operatorname{Im} z| < |\operatorname{Re} z|\}.$$

(See Figure 1.4.) Each point $a \in E$ is an interior point of E, and hence E is open. Indeed, one may show that $D(a; r) \subseteq E$, where $r = (|\operatorname{Re} a| - |\operatorname{Im} a|)/2$, using the triangle inequality. Likewise, if $a \in \mathbb{C}$ is such that $|\operatorname{Re} a| < |\operatorname{Im} a|$, then $D(a; r) \subseteq \mathbb{C} \setminus E$ if $r = (|\operatorname{Im} a| - |\operatorname{Re} a|)/2$, showing a is an exterior point of E. We also have that ∂E consists of those $a \in \mathbb{C}$ for which $|\operatorname{Re} a| = |\operatorname{Im} a|$ since for such a and $r > 0$, at least one of $a \pm r/2$ lies in E and a lies in $\mathbb{C} \setminus E$. We conclude from this reasoning that the limit points of E are precisely the points in $\overline{E}$. Since $\partial E \nsubseteq E$, E is not closed. Moreover, E is not bounded, as $(0, \infty) \subseteq E$. Hence E fails both conditions required of compactness. Lastly, we note that $A = \{z \in E : \operatorname{Re} z < 0\}$ and $B = \{z \in E : \operatorname{Re} z > 0\}$ are connected because any pair of points in one set is connected by a line segment contained within that set. The above logic shows A and B are open, and hence E is disconnected by Theorem 1.4.16. It follows that A and B are connected components of E and are each domains.

Summary and Notes for Section 1.4.

We have defined basic topological concepts such as open, closed, compact, bounded, and connected sets in the complex plane $\mathbb{C}$.

The field of topology is vast and deep and comes in many flavors. Point set topology is the study of abstract topological spaces where open sets are defined by a set of axioms. Despite its importance, it is a relatively young area, only coming into its own in the early 20^{th} century. This particular flavor of topology was motivated

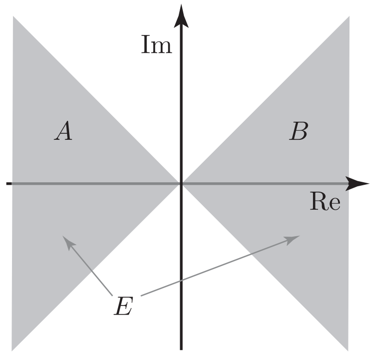

Figure 1.4 The set E and its components A and B from Example 1.4.19

by problems in abstract analysis. Indeed, John L. Kelley, in the preface to his classic book on general topology [14], wrote, "I have, with difficulty, been prevented by my friends from labeling [this book]: What Every Young Analyst Should Know."

Exercises for Section 1.4.

1. For each of the following sets E, determine whether E is open, closed, bounded, compact, connected, a domain. In addition, identify E°, ∂E, $\overline{E}$, the collection of limit points of E, and the components of E. Do not include proofs.

   (a) $E = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$

   (b) $E = \{z \in \mathbb{C} : 0 < |z| < 1\}$

   (c) $E = \{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1,\ |z| > 2\}$

   (d) $E = \bigcup_{n \in \mathbb{Z}} D(n; 1/2)$

   (e) $E = \{z \in \mathbb{C} : \operatorname{Re} z \neq |z|\}$

   (f) $E = \{z \in \mathbb{C} : \operatorname{Re} z, \operatorname{Im} z \in \mathbb{Q}\}$

   (g) $E = \{1/n + i/m : n, m \in \mathbb{N}\}$

2. Justify the statements made in Example 1.4.19.
3. Let $a \in \mathbb{C}$ and $r > 0$. Show that the set of exterior points of the disk $D(a; r)$ is $\{z \in \mathbb{C} : |z - a| > r\}$.
4. Prove Theorem 1.4.4.
5. Prove the following for a set $E \subseteq \mathbb{C}$.

   (a) The set $\overline{E}$ is a closed set.

   (b) The set E is closed if and only if $E = \overline{E}$.

6. Let $A \subseteq \mathbb{C}$ and $a \in \mathbb{C}$. Prove that a is a limit point of A if and only if $D(a; r) \cap A$ is infinite for all $r > 0$.
7. Let $A \subseteq \mathbb{C}$. Show that if $B \subseteq A$, then $\overline{B} \subseteq \overline{A}$.
8. $\triangleright$ Prove that a set $A \subseteq \mathbb{C}$ is open if and only if for every $a \in A$, there is a closed disk $\overline{D}(a; r) \subseteq A$ for some $r > 0$.