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ADVANCES IN HEAVY TAILED RISK MODELING

A HANDBOOK OF OPERATIONAL RISK

Gareth W. Peters
Pavel V. Shevchenko

WILEY

Advances in Heavy
Tailed Risk Modeling
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GARETH W. PETERS

PAVEL V. SHEVCHENKO

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Gareth W. Peters

This is dedicated to three very inspirational women in my life: Chen Mei—Peters, my mother Laraine Peters and Youxiang Wu; your support, encouragement and patience has made this possible. Mum, you instilled in me the qualities of scientific inquiry, the importance of questioning ideas and scientific rigour. This is especially for my dear Chen who bore witness to all the weekends in the library, the late nights reading papers and the ups and downs of toiling with mathematical proofs across many continents over the past few years.

Pavel V. Shevchenko
To my dear wife Elena

Embarking upon writing this book has proven to be an adventure through the landscape of ideas. Bringing forth feelings of adventure analogous to those that must have stimulated explorers such as Columbus to voyage to new lands.

In the depth of winter, I finally learned that within me there lay an invincible summer.
Albert Camus.

Contents in Brief

1	MOTIVATION FOR HEAVY-TAILED MODELS	1
2	FUNDAMENTALS OF EXTREME VALUE THEORY FOR OPRISK	17
3	HEAVY-TAILED MODEL CLASS CHARACTERIZATIONS FOR LDA	105
4	FLEXIBLE HEAVY-TAILED SEVERITY MODELS: α-STABLE FAMILY	139
5	FLEXIBLE HEAVY-TAILED SEVERITY MODELS: TEMPERED STABLE AND QUANTILE TRANSFORMS	227
6	FAMILIES OF CLOSED-FORM SINGLE RISK LDA MODELS	279
7	SINGLE RISK CLOSED-FORM APPROXIMATIONS OF ASYMPTOTIC TAIL BEHAVIOUR	353
8	SINGLE LOSS CLOSED-FORM APPROXIMATIONS OF RISK MEASURES	433
9	RECURSIONS FOR DISTRIBUTIONS OF LDA MODELS	517
A	MISCELLANEOUS DEFINITIONS AND LIST OF DISTRIBUTIONS	587

Contents

PREFACE	XIX
ACRONYMS	XXI
SYMBOLS	XXIII
LIST OF DISTRIBUTIONS	XXV
1 MOTIVATION FOR HEAVY-TAILED MODELS	1
1.1 Structure of the Book	1
1.2 Dominance of the Heaviest Tail Risks	3
1.3 Empirical Analysis Justifying Heavy-Tailed Loss Models in OpRisk	6
1.4 Motivating Parametric, Spliced and Non-Parametric Severity Models	9
1.5 Creating Flexible Heavy-Tailed Models via Splicing	11
2 FUNDAMENTALS OF EXTREME VALUE THEORY FOR OPRISK	17
2.1 Introduction	17
2.2 Historical Perspective on EVT and Risk	18
2.3 Theoretical Properties of Univariate EVT–Block Maxima and the GEV Family	20
2.4 Generalized Extreme Value Loss Distributional Approach (GEV-LDA)	40
2.4.1 Statistical Considerations for Applicability of the GEV Model	40
2.4.2 Various Statistical Estimation Procedures for the GEV Model Parameters in OpRisk Settings	42
2.4.3 GEV Sub-Family Approaches in OpRisk LDA Modeling	54
2.4.4 Properties of the Frechet–Pareto Family of Severity Models	54
	ix

2.4.5	Single Risk LDA Poisson-Generalized Pareto Family	55
2.4.6	Single Risk LDA Poisson-Burr Family	60
2.4.7	Properties of the Gumbel family of Severity Models	65
2.4.8	Single Risk LDA Poisson-LogNormal Family	65
2.4.9	Single Risk LDA Poisson-Benktander II Models	68
2.5	Theoretical Properties of Univariate EVT—Threshold Exceedances	72
2.5.1	Understanding the Distribution of Threshold Exceedances	74
2.6	Estimation Under the Peaks Over Threshold Approach via the Generalized Pareto Distribution	85
2.6.1	Maximum-Likelihood Estimation Under the GPD Model	87
2.6.2	Comments on Probability-Weighted Method of Moments Estimation Under the GPD Model	93
2.6.3	Robust Estimators of the GPD Model Parameters	95
2.6.4	EVT—Random Number of Losses	101

3 HEAVY-TAILED MODEL CLASS CHARACTERIZATIONS FOR LDA **105**

3.1	Landau Notations for OpRisk Asymptotics: Big and Little ‘Oh’	106
3.2	Introduction to the Sub-Exponential Family of Heavy-Tailed Models	113
3.3	Introduction to the Regular and Slow Variation Families of Heavy-Tailed Models	121
3.4	Alternative Classifications of Heavy-Tailed Models and Tail Variation	129
3.5	Extended Regular Variation and Matuszewska Indices for Heavy-Tailed Models	135

4 FLEXIBLE HEAVY-TAILED SEVERITY MODELS: α -STABLE FAMILY **139**

4.1	Infinitely Divisible and Self-Decomposable Loss Random Variables	140
4.1.1	Basic Properties of Characteristic Functions	140
4.1.2	Divisibility and Self-Decomposability of Loss Random Variables	143
4.2	Characterizing Heavy-Tailed α -Stable Severity Models	148
4.2.1	Characterisations of α -Stable Severity Models via the Domain of Attraction	152
4.3	Deriving the Properties and Characterizations of the α -Stable Severity Models	156
4.3.1	Unimodality of α -Stable Severity Models	158
4.3.2	Relationship between \mathbb{L} Class and α -Stable Distributions	160
4.3.3	Fundamentals of Obtaining the α -Stable Characteristic Function	163
4.3.4	From Lévy–Khinchin’s Canonical Representation to the α -Stable Characteristic Function Parameterizations	167

4.4	Popular Parameterizations of the α -Stable Severity Model	
	Characteristic Functions	171
4.4.1	Univariate α -Stable Parameterizations of Zolotarev A, M, B, W, C and E Types	172
4.4.2	Univariate α -Stable Parameterizations of Nolan S0 and S1	178
4.5	Density Representations of α -Stable Severity Models	181
4.5.1	Basics of Moving from a Characteristic Function to a Distribution or Density	182
4.5.2	Density Approximation Approach 1: Quadrature Integration via Transformation and Clenshaw–Curtis Discrete Cosine Transform Quadrature	187
4.5.3	Density Approximation Approach 2: Adaptive Quadrature Integration via Fast Fourier Transform (Midpoint Rule) and Bergstrom Series Tail Expansion	189
4.5.4	Density Approximation Approach 3: Truncated Polynomial Series Expansions	196
4.5.5	Density Approximation Approach 4: Reparameterization	198
4.5.6	Density Approximation Approach 5: Infinite Series Expansion Density and Distribution Representations	200
4.6	Distribution Representations of α -Stable Severity Models	207
4.6.1	Quadrature Approximations for Distribution Representations of α -Stable Severity Models	208
4.6.2	Convergent Series Representations of the Distribution for α -Stable Severity Models	209
4.7	Quantile Function Representations and Loss Simulation for α -Stable Severity Models	210
4.7.1	Approximating the Quantile Function of Stable Loss Random Variables	210
4.7.2	Sampling Realizations of Stable Loss Random Variables	214
4.8	Parameter Estimation in an α -Stable Severity Model	215
4.8.1	McCulloch’s Quantile-Based α -Stable Severity Model Estimators	216
4.8.2	Zolotarev’s Transformation to W-Class-Based α -stable Severity Model Estimators	217
4.8.3	Press’s Method-of-Moments-Based α -stable Severity Model Estimators	218
4.9	Location of the Most Probable Loss Amount for Stable Severity Models	219
4.10	Asymptotic Tail Properties of α -Stable Severity Models and Rates of Convergence to Paretian Laws	220

5 FLEXIBLE HEAVY-TAILED SEVERITY MODELS: TEMPERED STABLE AND QUANTILE TRANSFORMS

5.1	Tempered and Generalized Tempered Stable Severity Models	227
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5.1.1	Understanding the Concept of Tempering Stable Severity Models	228
5.1.2	Families and Representations of Tempering in Stable Severity Models	231
5.1.3	Density of the Tempered Stable Severity Model	241
5.1.4	Properties of Tempered Stable Severity Models	243
5.1.5	Parameter Estimation of Loss Random Variables from a Tempered Stable Severity Model	246
5.1.6	Simulation of Loss Random Variables from a Tempered Stable Severity Model	248
5.1.7	Tail Behaviour of the Tempered Stable Severity Model	252
5.2	Quantile Function Heavy-Tailed Severity Models	253
5.2.1	g -and- h Severity Model Family in OpRisk	257
5.2.2	Tail Properties of the g -and- h , g , h and h - h Severity in OpRisk	268
5.2.3	Parameter Estimation for the g -and- h Severity in OpRisk	270
5.2.4	Bayesian Models for the g -and- h Severity in OpRisk	273

6 FAMILIES OF CLOSED-FORM SINGLE RISK LDA MODELS **279**

6.1	Motivating the Consideration of Closed-Form Models in LDA Frameworks	279
6.2	Formal Characterization of Closed-Form LDA Models: Convolutional Semi-Groups and Doubly Infinitely Divisible Processes	281
6.2.1	Basic Properties of Convolution Operators and Semi-Groups for Distribution and Density Functions	282
6.2.2	Domain of Attraction of Lévy Processes: Stable and Tweedie Convergence	303
6.3	Practical Closed-Form Characterization of Families of LDA Models for Light-Tailed Severities	309
6.3.1	General Properties of Exponential Dispersion and Poisson-Tweedie Models for LDA Structures	309
6.4	Sub-Exponential Families of LDA Models	321
6.4.1	Properties of Discrete Exponential Dispersion Models	322
6.4.2	Closed-Form LDA Models for Large Loss Number Processes	326
6.4.3	Closed-Form LDA Models for the α -Stable Severity Family	333
6.4.4	Closed-Form LDA Models for the Tempered α -Stable Severity Family	349

7	SINGLE RISK CLOSED-FORM APPROXIMATIONS OF ASYMPTOTIC TAIL BEHAVIOUR	353
7.1	Tail Asymptotics for Partial Sums and Heavy-Tailed Severity Models	356
7.1.1	Partial Sum Tail Asymptotics with Heavy-Tailed Severity Models: Finite Number of Annual Losses $N = n$	357
7.1.2	Partial Sum Tail Asymptotics with Heavy-Tailed Severity Models: Large Numbers of Loss Events	362
7.2	Asymptotics for LDA Models: Compound Processes	367
7.2.1	Asymptotics for LDA Models Light Frequency and Light Severity Tails: SaddlePoint Tail Approximations	368
7.3	Asymptotics for LDA Models Dominated by Frequency Distribution Tails	372
7.3.1	Heavy-Tailed Frequency Distribution and LDA Tail Asymptotics (Frechet Domain of Attraction)	374
7.3.2	Heavy-Tailed Frequency Distribution and LDA Tail Asymptotics (Gumbel Domain of Attraction)	375
7.4	First-Order Single Risk Loss Process Asymptotics for Heavy-Tailed LDA Models: Independent Losses	376
7.4.1	First-Order Single Risk Loss Process Asymptotics for Heavy-Tailed LDA Models: General Sub-exponential Severity Model Results	377
7.4.2	First-Order Single Risk Loss Process Asymptotics for Heavy-Tailed LDA Models: Regular and O-Regularly Varying Severity Model Results	380
7.4.3	Remainder Analysis: First-Order Single Risk Loss Process Asymptotics for Heavy-Tailed LDA Models	385
7.4.4	Summary: First-Order Single Risk Loss Process Asymptotics for Heavy-Tailed LDA Models	388
7.5	Refinements and Second-Order Single Risk Loss Process Asymptotics for Heavy-Tailed LDA Models: Independent Losses	389
7.6	Single Risk Loss Process Asymptotics for Heavy-Tailed LDA Models: Dependent Losses	393
7.6.1	Severity Dependence Structures that Do Not Affect LDA Model Tail Asymptotics: Stochastic Bounds	402
7.6.2	Severity Dependence Structures that Do Not Affect LDA Model Tail Asymptotics: Sub-exponential, Partial Sums and Compound Processes	405

7.6.3	Severity Dependence Structures that Do Not Affect LDA Model Tail Asymptotics: Consistent Variation	410
7.6.4	Dependent Severity Models: Partial Sums and Compound Process Second-Order Tail Asymptotics	412
7.7	Third-order and Higher Order Single Risk Loss Process Asymptotics for Heavy-Tailed LDA Models: Independent Losses	414
7.7.1	Background Understanding on Higher Order Tail Decomposition Approaches	414
7.7.2	Decomposition Approach 1: Higher Order Tail Approximation Variants	415
7.7.3	Decomposition Approach 2: Higher Order Tail Approximations	426
7.7.4	Explicit Expressions for Higher Order Recursive Tail Decompositions Under Different Assumptions on Severity Distribution Behaviour	430

8 SINGLE LOSS CLOSED-FORM APPROXIMATIONS OF RISK MEASURES 433

8.1	Summary of Chapter Key Results on Single-Loss Risk Measure Approximation (SLA)	433
8.2	Development of Capital Accords and the Motivation for SLAs	436
8.3	Examples of Closed-Form Quantile and Conditional Tail Expectation Functions for OpRisk Severity Models	440
8.3.1	Exponential Dispersion Family Loss Models	441
8.3.2	g -and- h Distribution Family Loss Models	445
8.3.3	Extended GPD: the Asymmetric Power Family Loss Models	446
8.4	Non-Parametric Estimators for Quantile and Conditional Tail Expectation Functions	448
8.5	First- and Second-Order SLA of the VaR for OpRisk LDA Models	451
8.5.1	Second-Order Refinements of the SLA VaR for Heavy-Tailed LDA Models	457
8.6	EVT-Based Penultimate SLA	468
8.7	Motivation for Expected Shortfall and Spectral Risk Measures	475
8.8	First- and Second-Order Approximation of Expected Shortfall and Spectral Risk Measure	478
8.8.1	Understanding the First-Order SLA for ES for Regularly Varying Loss Models	481
8.8.2	Second-Order SLA for Expected Shortfall for Regularly Varying Loss Models	485

8.8.3	Empirical Process and EVT Approximations of Expected Shortfall	488
8.8.4	SLA for Spectral Risk Measures	492
8.9	Assessing the Accuracy and Sensitivity of the Univariate SLA	496
8.9.1	Understanding the Impact of Parameter Estimation Error on a SLA	498
8.9.2	Understanding the SLA Error	502
8.10	Infinite Mean-Tempered Tail Conditional Expectation Risk Measure Approximations	503

9 RECURSIONS FOR DISTRIBUTIONS OF LDA MODELS **517**

9.1	Introduction	517
9.2	Discretization Methods for Severity Distribution	519
9.2.1	Discretization Method 1: Rounding	520
9.2.2	Discretization Method 2: Localized Moment Matching	522
9.2.3	Discretization Method 3: Lloyd's Algorithm	524
9.2.4	Discretization Method 4: Minimizing Kolmogorov Statistic	524
9.3	Classes of Discrete Distributions: Discrete Infinite Divisibility and Discrete Heavy Tails	525
9.4	Discretization Errors and Extrapolation Methods	533
9.5	Recursions for Convolutions (Partial Sums) with Discretized Severity Distributions (Fixed n)	535
9.5.1	De Pril Transforms for n -Fold Convolutions (Partial Sums) with Discretized Severity Distributions	537
9.5.2	De Pril's First Method	538
9.5.3	De Pril's Second Method	539
9.5.4	De Pril Transforms and Convolutions of Infinitely Divisible Distributions	540
9.5.5	Recursions for n -Fold Convolutions (Partial Sum) Distribution Tails with Discretized Severity	542
9.6	Estimating Higher Order Tail Approximations for Convolutions with Continuous Severity Distributions (Fixed n)	544
9.6.1	Approximation Stages to be Studied	547
9.7	Sequential Monte Carlo Sampler Methodology and Components	550
9.7.1	Choice of Mutation Kernel and Backward Kernel	553
9.7.2	Incorporating Partial Rejection Control into SMC Samplers	556

9.8	Multi-Level Sequential Monte Carlo Samplers for Higher Order Tail Expansions and Continuous Severity Distributions (Fixed n)	560
9.8.1	Key Components of Multi-Level SMC Samplers	562
9.9	Recursions for Compound Process Distributions and Tails with Discretized Severity Distribution (Random N)	565
9.9.1	Panjer Recursions for Compound Distributions with Discretized Severity Distributions	566
9.9.2	Alternatives to Panjer Recursions: Recursions for Compound Distributions with Discretized Severity Distributions	573
9.9.3	Higher Order Recursions for Discretized Severity Distributions in Compound LDA Models	575
9.9.4	Recursions for Discretized Severity Distributions in Compound Mixed Poisson LDA Models	577
9.10	Continuous Versions of the Panjer Recursion	581
9.10.1	The Panjer Recursion via Volterra Integral Equations of the Second Kind	581
9.10.2	Importance Sampling Solutions to the Continuous Panjer Recursion	583

A MISCELLANEOUS DEFINITIONS AND LIST OF DISTRIBUTIONS 587

A.1	Indicator Function, 587
A.2	Gamma Function, 587
A.3	Discrete Distributions, 587
A.3.1	Poisson Distribution, 587
A.3.2	Binomial Distribution, 588
A.3.3	Negative Binomial Distribution, 588
A.3.4	Doubly Stochastic Poisson Process (Cox Process), 589
A.4	Continuous Distributions, 589
A.4.1	Uniform Distribution, 589
A.4.2	Normal (Gaussian) Distribution, 590
A.4.3	Inverse Gaussian Distribution, 590
A.4.4	LogNormal Distribution, 591
A.4.5	Student's t -Distribution, 591
A.4.6	Gamma Distribution, 591
A.4.7	Weibull Distribution, 592
A.4.8	Inverse Chi-Squared Distribution, 592
A.4.9	Pareto Distribution (One Parameter), 592
A.4.10	Pareto Distribution (Two Parameter), 593
A.4.11	Generalized Pareto Distribution, 593

- A.4.12 Beta Distribution, 594
- A.4.13 Generalized Inverse Gaussian Distribution, 594
- A.4.14 d -Variate Normal Distribution, 595
- A.4.15 d -Variate t -Distribution, 595

REFERENCES	597
INDEX	623

Preface

This book covers key mathematical and statistical aspects of the quantitative modeling of heavy tailed loss processes in operational risk (OpRisk) and insurance settings. OpRisk has been through significant changes in the past few years with increased regulatory pressure for more comprehensive frameworks. Nowadays, every mid-sized and larger financial institution across the planet would have an OpRisk department. Despite the growing awareness and understanding of the importance of OpRisk modeling throughout the banking and insurance industry there is yet to be a convergence to a standardization of the modeling frameworks for this new area of risk management. In fact to date the majority of general texts on this topic of OpRisk have tended to cover basic topics of modeling that are typically standard in the majority of risk management disciplines. We believe that this is where the combination of the two books *Fundamental Aspects of Operational Risk and Insurance Analytics: A Handbook of Operational Risk* (Cruz, Peters and Shevchenko, 2015) and the companion book *Advances in Heavy Tailed Risk Modeling: A Handbook of Operational Risk* will play an important role in better understanding specific details of risk modeling directly aimed to specifically capture fundamental and core features specific to OpRisk loss processes.

These two texts form a sequence of books which provide a detailed and comprehensive guide to the state of the art OpRisk modeling approaches. In particular, this second book on heavy tailed modeling provides one of the few detailed texts which is aimed to be accessible to both practitioners and graduate students with quantitative background to understand the significance of heavy tailed modeling in risk and insurance, particularly in the setting of OpRisk. It covers a range of modeling frameworks from general concepts of heavy tailed loss processes, to extreme value theory, how dependence plays a role in joint heavy tailed models, risk measures and capital estimation behaviors in the presence of heavy tailed loss processes and finishes with simulation and estimation methods that can be implemented in practice. This second book on heavy tailed modeling is targeted at a PhD or advanced graduate level quantitative course in OpRisk and insurance and is suitable for quantitative analysts working in OpRisk and insurance wishing to understand more fundamental properties of heavy tailed modeling that is directly relevant to practice. This is where the *Advances in Heavy-Tailed Risk Modeling: A Handbook of Operational Risk* can add value to the industry. In particular, by providing a clear and detailed coverage of modeling for heavy tailed OpRisk losses from both a rigorous mathematical as well as a statistical perspective.

More specifically, this book covers advanced topics on risk modeling in high consequence low frequency loss processes. This includes splice loss models and motivation for heavy tailed risk models. The key aspects of extreme value theory and their development in loss distributional approach modeling are considered. Classification and understanding of different classes of heavy tailed risk process models is discussed; this leads to topics on heavy tailed closed-form

loss distribution approach models and flexible heavy tailed risk models such as α -stable, tempered stable, g-and-h, GB2 and Tukey quantile transform based models. The remainder of the chapters covers advanced topics on risk measures and asymptotics for heavy tailed compound process models. Then the final chapter covers advanced topics including forming links between actuarial compound process recursions and Monte Carlo numerical solutions for capital risk measure estimations.

The book is primarily developed for advanced risk management practitioners and quantitative analysts. In addition, it is suitable as a core reference for an advanced mathematical or statistical risk management masters course or a PhD research course on risk management and asymptotics.

As mentioned, this book is a companion book of *Fundamental Aspects of Operational Risk and Insurance Analytics: A Handbook of Operational Risk* (Cruz, Peters and Shevchenko, 2015). The latter covers fundamentals of the building blocks of OpRisk management and measurement related to Basel II/III regulation, modeling dependence, estimation of risk models and the four-data elements (internal data, external data, scenario analysis and business environment and internal control factors) that need to be used in the OpRisk framework.

Overall, these two books provide a consistent and comprehensive coverage of all aspects of OpRisk management and related insurance analytics as they relate to loss distribution approach modeling and OpRisk – organizational structure, methodologies, policies and infrastructure – for both financial and non-financial institutions. The risk measurement and modeling techniques discussed in the book are based on the latest research. They are presented, however, with considerations based on practical experience of the authors with the daily application of risk measurement tools. We have incorporated the latest evolution of the regulatory framework. The books offer a unique presentation of the latest OpRisk management techniques and provide a unique source of knowledge in risk management ranging from current regulatory issues, data collection and management, technological infrastructure, hedging techniques and organizational structure.

We would like to thank our families for their patience with our absence whilst we were writing this book.

Gareth W. Peters and Pavel V. Shevchenko
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Acronyms

ABC	approximate Bayesian computation
ALP	accumulated loss policy
a.s.	almost surely
AMA	advanced measurement approach
APT	arbitrage pricing theory
BCRLB	Bayesian Cramer–Rao lower bound
BCBS	Basel Committee on Banking Supervision
BIS	Bank for International Settlements
CV	co-variation
CD	co-difference
CRLB	Cramer–Rao lower bound
CLP	combined loss policy
CVaR	conditional value at risk
DFT	discrete Fourier transform
EVT	extreme value theory
EVI	extreme value index
ES	expected shortfall
FFT	fast Fourier transform
GLM	generalized linear models
GAM	generalized additive models
GLMM	generalized linear mixed models
GAMM	generalized additive mixed models
GAMLSS	generalized additive models for location scale and shape
HMCR	higher moment coherent risk measure
HILP	haircut individual loss policy
ILPU	individual loss policy uncapped
ILPC	individual loss policy capped
i.i.d.	independent and identically distributed
LDA	loss distribution approach
MCMC	Markov chain Monte Carlo
MC	Monte Carlo
MLE	maximum likelihood estimator
MPT	modern portfolio theory
OpRisk	operational risk
PMCMC	particle Markov chain Monte Carlo
r.v.	random variable
SMC	sequential Monte Carlo

SRM	spectral risk measure
SLA	Single Loss Approximation
s.t.	such that
TCE	tail conditional expectation
TTCE	tempered tail conditional expectation
VaR	value at risk
Vco	variational coefficient
w.r.t.	with respect to

Symbols

\forall	for all
\exists	there exists
\cup	union of two sets
\cap	intersection of two sets
$*$	convolution operator
$F(\cdot)$	probability distribution function
$C(u_1, u_2, \dots, u_d)$	d -dimensional Copula probability distribution function.
\bar{F}	tail function $\bar{F}(x) = 1 - F(x)$
$f(\cdot)$	probability density function
$h(\cdot)$	hazard rate given by $h(\cdot) = \frac{f(\cdot)}{F(\cdot)}$
$\Phi_X[\theta]$	characteristic function for random variable X
$M_X(t)$	moment-generating function of random variable X
$F^{(n)*}(\cdot)$	n -fold convolution of a distribution function with itself
$F^{-1}(\cdot)$	inverse distribution function (quantile function)
$Q(\alpha)$	quantile function
$U(y) = Q(1 - 1/y)$	tail quantile function
$F^{\leftarrow}(\cdot)$	generalized inverse
$g(\cdot) \sim f(\cdot)$	function g is asymptotic equivalent to f at infinity (unless specified otherwise)
$X \sim F(\cdot)$	random variable X is distributed according to F
$X_{(k,n)}$	k th largest sample from n samples, that is, the k th order statistic
$\text{VaR}_\alpha[\cdot]$	value at risk for level α
$\text{ES}_\alpha[\cdot]$	expected shortfall for level α
$\text{SRM}_\alpha[\cdot]$	spectral risk measure for level α
\mathbb{J}	space of integers
\mathbb{R}	real line
\mathbb{C}	complex plane
$\Re\{\cdot\}$	real component of a complex number or function
$\Im\{\cdot\}$	imaginary component of a complex number
$\mathbb{E}[\cdot]$	expectation operator
$\mathbb{P}\text{r}[\cdot]$	probability
$f(\cdot) \in \text{RV}(\rho)$	function f is in the class of Regularly Varying functions with index of regular variation ρ
$L(\cdot)$	slowly varying function, that is, function $L(\cdot) \in \text{RV}(0)$
$\Phi^{(d)}$	d th derivative of function Φ
$\mathbb{I}[\cdot]$	indicator function of an event

$\delta(\cdot)$	Dirac delta function
$o(\cdot)$	little ‘Oh’ Landau asymptotic notation
$O(\cdot)$	big ‘Oh’ Landau asymptotic notation
\ll	Vinogradov’s asymptotic notation for big ‘Oh’ notation
$f \circ g$	f is a composite function of g , that is, $f(g(\cdot))$
$\mathscr{W}[x]$	Lambert W function
$\mathcal{E}[n, x]$	Misra function

List of Distributions

Distribution Name	Distribution Symbol
α -Stable	$S_\alpha(\cdot)$
Asymmetric Laplace	<i>AssymmetricLaplace</i> (\cdot)
Beta	<i>Beta</i> (\cdot)
Chi Squared	<i>ChiSquared</i> (\cdot)
Dagum or Burr Type XII	<i>BurrXII</i> (\cdot)
Double h - h distributions	$T_{h,h}(\cdot)$
Exponential	<i>Exp</i> (\cdot)
Gamma	<i>Gamma</i> (\cdot)
g distributions	$T_g(\cdot)$
g -and- h distributions	$T_{g,h}(\cdot)$
g -and- k distributions	$T_{g,k}(\cdot)$
Generalized gamma	<i>GG</i> (\cdot)
Generalized beta	<i>GB2</i> (\cdot)
Generalized Hyperbolic	<i>GH</i> (\cdot)
Generalized Inverse Guassian	<i>GIG</i> (\cdot)
Generalized Log-Logistic	<i>Gen</i> (\cdot)
Geometric stable	<i>GeoStable</i> (\cdot)
h distributions	$T_h(\cdot)$
Halphen type A	<i>Halphen</i> (\cdot)
Inverse Gamma	<i>InverseGamma</i> (\cdot)
Inverse Gaussian	<i>InverseGaussian</i> (\cdot)
Laplace	<i>Laplace</i> (\cdot)
Log-Cauchy	<i>LogCauchy</i> (\cdot)
LogNormal	<i>LogNormal</i> (\cdot)
Log- t	<i>Log-t</i> (\cdot)
Lomax	<i>Lomax</i> (\cdot)
Normal (Gaussian)	<i>Normal</i> (\cdot)
Normal Inverse Gaussian	<i>NIG</i> (\cdot)
Singh-Maddala or Burr Type III	<i>BurrIII</i> (\cdot)
Standard Normal	$\Phi(\cdot)$
Tempered stable	<i>TS</i> (\cdot)

Motivation for Heavy-Tailed Models

1.1 Structure of the Book

This book is split into a few core components covering fundamental concepts:

- Chapter 1 motivates the need to consider heavy-tailed loss process models in operational risk (OpRisk) and insurance modeling frameworks. It provides a basic introduction to the concept of separating the modeling of the central loss process and the tails of the loss process through splice models. It also sets out the key statistical questions one must consider studying when performing analysis and modeling of high consequence rare-event loss processes.
- Chapter 2 covers all the fundamental properties one may require in univariate loss process modeling under an extreme value theory (EVT) approach. Of importance is the detailed discussion on the associated statistical assumptions that must be made regarding the properties of any data utilized in model estimation when working with EVT models. This chapter provides a relatively advanced coverage of generalized extreme value (GEV) family of models, block maximum and peaks over threshold frameworks. It provides detailed discussion on statistical estimation that should be utilized in practice for such models and how one may adapt such methods to small sample settings that may arise in OpRisk settings. In the process, the chapter details clearly how to construct several loss distributional approach models based on EVT analysis. It then concludes with results of EVT in the context of compound processes.
- Chapter 3 provides a set of formal mathematical definitions for different notions regarding a heavy-tailed or fat-tailed loss distribution and its properties. It is important that when modeling such loss processes, especially the asymptotic properties of compound process

models built with heavy-tailed loss models, a clear understanding of the tail properties of such loss models is understood. In this regard, we discuss the family of sub-exponential loss models, the family of regularly varying and slowly varying models. There are within these large classes of models sub-categorizations that are often of use to understand when thinking about risk measures resulting from such loss models, these are also detailed, for example, long-tailed models, subversively varying models and extended regular variation. In addition, the chapter opens with a basic introduction to key notations and properties of asymptotic notations that are utilized throughout the book.

- Chapter 4 begins with a basic introduction to properties of mathematical representations and characterizations of heavy-tailed loss models through the characteristic function and its representation. It then details the notions of divisibility, self-decomposability and the resulting consequences such distributional properties have on loss distributional approach compound process models. The remainder of the chapter provides a detailed coverage of the family of univariate α -stable models, detailing their characterization, the parameterizations, density and distribution representations and parameter estimation. Such a family of models is becoming increasingly interesting for OpRisk modeling and insurance. It is recognized that such a family of models possesses many relevant and useful features that will capture aspects of OpRisk and insurance loss processes accurately and with advantageous features when used in a compound process model under a loss distributional approach structure.
- Chapter 5 provides the representations of flexible severity models based on tempering or exponential tilting of the α -stable family of loss models. Under this concept, there are many families of tempered stable models available; this chapter characterizes each and discusses the mathematical properties of each sub-class of models and how they may be used in compound process models for heavy-tailed loss models in OpRisk and insurance. In addition, it discusses the aspects of model estimation and simulation for such models. The chapter then finishes with a detailed discussion on quantile-transformed-based heavy-tailed loss models for OpRisk and insurance, such as the Tukey transforms and the sub-family of the g -and- h distributions that have been popular in OpRisk.
- Chapter 6 discusses compound processes and convolutional semi-group structures. This then leads to developing representations of closed-form compound process loss distributions and densities that admit heavy-tailed loss processes. The chapter characterizes several classes of such models that can be used in practice, which avoid the need for computationally costly Monte Carlo simulation when working with such models.
- Chapter 7 discusses many properties of different classes of heavy-tailed loss processes with regard to asymptotic representations and properties of the tail of both partial sums and compound random sums. It does so under first-, second- and third-order asymptotic expansions for the tail process of such heavy-tailed loss processes. This is achieved under many different assumptions relating to the frequency and severity distribution and the possible dependence structures in such loss processes.
- Chapter 8 extends the results of the asymptotics for the tail of heavy-tailed loss processes partial sums and compound random sums to the asymptotics of risk measures developed from such loss processes. In particular, it discusses closed-form single-loss approximations and first-order, second-order and higher order expansion representations. It covers value-at-risk, expected shortfall and spectral risk measure asymptotics. This chapter also covers some alternative risk measure asymptotic results based on EVT known as *penultimate approximations*.