Elasticity of Transversely Isotropic Materials

SOLID MECHANICS AND ITS APPLICATIONS
Volume 126

Series Editor: G.M.L. GLADWELL
Department of Civil Engineering
University of Waterloo
Waterloo, Ontario, Canada N2L 3GI

Aims and Scope of the Series

The fundamental questions arising in mechanics are: *Why?, How?,* and *How much?* The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids.

The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies: vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

For a list of related mechanics titles, see final pages.

Elasticity of Transversely Isotropic Materials

by

Haojiang Ding

Zhjiang University,
Hangzhou, China

Weiqiu Chen

Zhjiang University,
Hangzhou, China

and

L. Zhang

The University of Sydney,
NSW, Australia

Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN-10 1-4020-4033-4 (HB)
ISBN-13 978-1-4020-4033-7 (HB)
ISBN-10 1-4020-4034-2 (e-book)
ISBN-13 978-1-4020-4034-4 (e-book)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springer.com

Printed on acid-free paper

Printed in the Netherlands.

Contents

Preface

This book aims to provide a comprehensive introduction to the theory and applications of the mechanics of transversely isotropic elastic materials. There are many reasons why it should be written. First, the theory of transversely isotropic elastic materials is an important branch of applied mathematics and engineering science; but because of the difficulties caused by anisotropy, the mathematical treatments and descriptions of individual problems have been scattered throughout the technical literature. This often hinders further development and applications. Hence, a text that can present the theory and solution methodology uniformly is necessary.

Secondly, with the rapid development of modern technologies, the theory of transversely isotropic elasticity has become increasingly important. In addition to the fields with which the theory has traditionally been associated, such as civil engineering and materials engineering, many emerging technologies have demanded the development of transversely isotropic elasticity. Some immediate examples are thin film technology, piezoelectric technology, functionally gradient materials technology and those involving transversely isotropic and layered microstructures, such as multi-layer systems and tribology mechanics of magnetic recording devices. Thus a unified mathematical treatment and presentation of solution methods for a wide range of mechanics models are of primary importance to both technological and economic progress.

The authors aim to achieve a systematic structure for this complex subject in a single volume and provide the reader with state-of-the-art solution strategies for transversely isotropic elasticity under a unified umbrella. The subject matter has been organized into ten chapters to incorporate fundamental theories, solution skills and applications into an organic whole.

Chapter 1 begins with a concise summary of the basic equations of anisotropic elasticity used in the book, including thermo-elasticity. The materials presented here construct the framework for the theories and solutions of transversely isotropic problems.

The success of solutions relies largely on the strategies and mathematical treatments. Chapter 2 is therefore arranged to explain the basic methodologies for obtaining the general elastic solutions of transversely isotropic materials. In this way, the reader becomes clearer about the specific approaches for individual mechanics models in the later chapters.

Point force solutions are fundamental in solving various problems. Hence, Chapter 3 is devoted to establishing the relevant basics using a unified method that avoids the existing confusions in the literature. Meanwhile, this chapter focuses on infinite body problems and serves as an introduction to solution skills for more complex cases.

With the understanding gained and theory developed in the previous chapters, Chapters 4 to 10 discuss the solution of complicated engineering problems, including half-spaces, layered media, cones, thermal stress, frictional contact and bending, vibration and stability of plates and shells. These provide the reader not only with specific methods for tackling mathematical systems involving transverse isotropy, but also the fundamental solutions that can be extended to more complex situations.

This book is suitable for engineers, designers, researchers and postgraduates who are interested in the solution of transversely isotropic elastic materials. The authors believe that the reader who takes time to study this book will find ample reward.

The first author is indebted to Professor Hu Haichang, who introduced him to the field of elasticity and has offered invaluable help and guidance for many years. The authors appreciate very much the valuable comments and suggestions made by Professor Graham Gladwell. The financial support from the National Natural Science Foundation of China, Natural Science Foundation of Zhejiang Province as well as the 151 Talent Project of Zhejiang Province are very much appreciated. Finally, the authors wish to thank their families for their assistance; without their encouragement the book would never have been completed.

Haojiang DING, Weiqiu CHEN
Hangzhou, P. R. China

Liangchi ZHANG
Sydney, Australia

March 2005

1

BASIC EQUATIONS OF ANISOTROPIC ELASTICITY

This chapter introduces the basic equations of anisotropic elasticity which are essential for solving transversely isotropic elasticity problems. For simplicity, we ignore the mathematical details of deriving these equations; the reader can find them in the relevant references provided in **Appendix A**.

1.1 TRANSFORMATION OF STRAINS AND STRESSES

Consider an anisotropic and ideally elastic continuous solid subjected to small deformation. Assume that the solid is free of stress before deformation. The *stress-strain relationship* in this case is linear, *i.e.*, it follows the *generalized Hooke's law*. If the solid is homogeneous, the coefficients in the stress-strain relationship are constant, but if it is inhomogeneous, they will vary because the elastic properties at different points in the solid are different; they will be functions of the coordinates.

We can use various coordinate systems when studying the stresses and strains in a solid generated by external loading. In this book, we use a *Cartesian coordinate system*, (x, y, z), a *cylindrical coordinate system*, (r, α, z) or a *spherical coordinate system*, (R, θ, α). There are simple relationships between these coordinates, as listed in Tables 1.1 and 1.2, where angle α $(0 \le \alpha \le 2\pi)$ is measured from the positive direction of the x-axis to that of the y-axis, and angle θ $(0 \le \theta \le \pi)$ is measured from the positive direction of the z-axis to the negative direction of the same axis.

Correspondingly, we use (u, v, w), (u_r, u_α, w) or $(u_R, u_\theta, u_\alpha)$, respectively, to denote the displacements at a point in the solid in Cartesian, cylindrical or spherical coordinate systems. In tensor form, the displacements will be written as u_i $(i = 1, 2, 3)$, but in matrix form, we write them as $\{u\} = [u, v, w]^{\mathrm{T}}$, $[u_r, u_\alpha, w]^{\mathrm{T}}$ or $[u_R, u_\theta, u_\alpha]^{\mathrm{T}}$, respectively, here the superscript T stands for transpose.

Table 1.1 Direction cosines between coordinate axes in Cartesian and cylindrical coordinates.

	x	y	z
r	$\cos\alpha$	$\sin\alpha$	0
α	$-\sin\alpha$	$\cos\alpha$	0
z	0	0	1

Table 1.2 Direction cosines between coordinate axes in Cartesian and spherical coordinates.

	x	y	z
R	$\sin\theta\cos\alpha$	$\sin\theta\sin\alpha$	$\cos\theta$
θ	$\cos\theta\cos\alpha$	$\cos\theta\sin\alpha$	$-\sin\theta$
α	$-\sin\alpha$	$\cos\alpha$	0

In the three coordinate systems, the stresses and strains can be written, respectively, as

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{zx} \\ \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix},$$

$$\begin{bmatrix} \sigma_r & \tau_{r\alpha} & \tau_{zr} \\ \tau_{r\alpha} & \sigma_\alpha & \tau_{\alpha z} \\ \tau_{zr} & \tau_{\alpha z} & \sigma_z \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \varepsilon_r & \frac{1}{2}\gamma_{r\alpha} & \frac{1}{2}\gamma_{zr} \\ \frac{1}{2}\gamma_{r\alpha} & \varepsilon_\alpha & \frac{1}{2}\gamma_{\alpha z} \\ \frac{1}{2}\gamma_{zr} & \frac{1}{2}\gamma_{\alpha z} & \varepsilon_z \end{bmatrix},$$

$$\begin{bmatrix} \sigma_R & \tau_{R\theta} & \tau_{\alpha R} \\ \tau_{R\theta} & \sigma_\theta & \tau_{\theta\alpha} \\ \tau_{\alpha R} & \tau_{\theta\alpha} & \sigma_\alpha \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \varepsilon_R & \frac{1}{2}\gamma_{R\theta} & \frac{1}{2}\gamma_{\alpha R} \\ \frac{1}{2}\gamma_{R\theta} & \varepsilon_\theta & \frac{1}{2}\gamma_{\theta\alpha} \\ \frac{1}{2}\gamma_{\alpha R} & \frac{1}{2}\gamma_{\theta\alpha} & \varepsilon_\alpha \end{bmatrix},$$

where γ_{ij} are the *engineering shear strains* and have the relationship, $\gamma_{ij} = 2\varepsilon_{ij}\ (i \neq j)$, with the *strain tensor*, ε_{ij}. Thus, in Cartesian coordinates $\varepsilon_{12} = \varepsilon_{xy} = \gamma_{xy}/2$, in cylindrical coordinates $\varepsilon_{12} = \varepsilon_{r\alpha} = \gamma_{r\alpha}/2$, and in spherical coordinates $\varepsilon_{12} = \varepsilon_{R\theta} = \gamma_{R\theta}/2$.

In tensor form, the *stress tensor*, σ_{ij}, and the strain tensor, ε_{ij}, are expressed respectively as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}.$$

Both the stress and strain are symmetric tensors of rank two, *i.e.*, $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$ and they follow the following *transformation rule*:

$$\sigma_{p'q'} = l_{p'i} l_{q'j} \sigma_{ij}, \tag{1.1.1}$$

$$\varepsilon_{p'q'} = l_{p'i} l_{q'j} \varepsilon_{ij}, \tag{1.1.2}$$

where σ_{ij} stands for the stresses in Cartesian coordinates, (x, y, z), $\sigma_{p'q'}$ represents the stresses in a new Cartesian coordinate system after rotation, (x', y', z'), and $l_{p'i}$ are the direction cosines between two coordinate axes, as listed in Table 1.3. For example, $l_{1'1} = \cos(x', x)$, $l_{1'2} = \cos(x', y)$, $l_{2'3} = \cos(y', z)$.

Table 1.3 Direction cosines between coordinate axes.

	x	y	z
x'	$l_{1'1}$	$l_{1'2}$	$l_{1'3}$
y'	$l_{2'1}$	$l_{2'2}$	$l_{2'3}$
z'	$l_{3'1}$	$l_{3'2}$	$l_{3'3}$

In Eqs. (1.1.1) and (1.1.2), the repetition of a subscript in a term denotes a summation with respect to the index over its range from 1 to 3. In tensor analysis such an index is called a *dummy index*, while one that is not summed out is called a *free index*. This *summation convention* will apply throughout the book unless otherwise stated.

Using Eq. (1.1.1) and Tables 1.1 and 1.3, we can easily obtain the relationships between the stresses in cylindrical and Cartesian systems, *i.e.*,

$$\begin{aligned}
&\sigma_r = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy} \sin\alpha\cos\alpha, \\
&\sigma_\alpha = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha - 2\tau_{xy} \sin\alpha\cos\alpha, \\
&\tau_{r\alpha} = (\sigma_y - \sigma_x)\sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha), \\
&\tau_{zr} = \tau_{zx}\cos\alpha + \tau_{yz}\sin\alpha, \\
&\tau_{\alpha z} = -\tau_{zx}\sin\alpha + \tau_{yz}\cos\alpha, \\
&\sigma_z = \sigma_z.
\end{aligned} \tag{1.1.3}$$

Note that in deriving the above equations, we have used the symmetry of the stress tensor and simplified the formulae because some of the direction cosines are zero. These reduce the nine summation items in Eq. (1.1.1) to six and bring about the constant factor 2 in the first two equations and terms $\cos^2\alpha$ and $-\sin^2\alpha$ in the third.

On any infinitesimal area in a solid with an external normal $\vec{n}$, if the projections of the stress in *x*-, *y*- and *z*-directions in a Cartesian coordinate system are p_x, p_y and p_z, then

$$\begin{aligned}
p_x &= \sigma_x \cos(n,x) + \tau_{xy}\cos(n,y) + \tau_{zx}\cos(n,z), \\
p_y &= \tau_{xy} \cos(n,x) + \sigma_y\cos(n,y) + \tau_{yz}\cos(n,z), \\
p_z &= \tau_{zx} \cos(n,x) + \tau_{yz}\cos(n,y) + \sigma_z\cos(n,z).
\end{aligned} \tag{1.1.4}$$

We can get similar formulae for stresses in cylindrical and spherical coordinate systems.

1.2 BASIC EQUATIONS

The basic equations of elasticity are *geometric equations* (strain-displacement relations), *equations of motion* and *constitutive equations* (stress-strain relations). Using the coordinate transformation discussed above, we can easily get the basic equations in different coordinate systems, as listed below.

1.2.1 Geometric Equations

In Cartesian coordinates, we have

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \\
\varepsilon_y &= \frac{\partial v}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \\
\varepsilon_z &= \frac{\partial w}{\partial z}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.
\end{aligned} \tag{1.2.1}$$

In cylindrical coordinates, they become

$$
\begin{aligned}
&\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \gamma_{\alpha z} = \frac{\partial u_\alpha}{\partial z} + \frac{\partial w}{r\partial \alpha}, \\
&\varepsilon_\alpha = \frac{1}{r}\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_r}{r}, \quad \gamma_{zr} = \frac{\partial w}{\partial r} + \frac{\partial u_r}{\partial z}, \\
&\varepsilon_z = \frac{\partial w}{\partial z}, \quad \gamma_{r\alpha} = \frac{1}{r}\frac{\partial u_r}{\partial \alpha} + \frac{\partial u_\alpha}{\partial r} - \frac{u_\alpha}{r}.
\end{aligned}
\tag{1.2.2}
$$

In spherical coordinates, they can be written as

$$
\begin{aligned}
&\varepsilon_R = \frac{\partial u_R}{\partial R}, \quad \varepsilon_\theta = \frac{1}{R}\frac{\partial u_\theta}{\partial \theta} + \frac{u_R}{R}, \\
&\varepsilon_\alpha = \frac{1}{R\sin\theta}\frac{\partial u_\alpha}{\partial \alpha} + \frac{u_R}{R} + \frac{u_\theta}{R}\cot\theta, \\
&\gamma_{\theta\alpha} = \frac{1}{R}\left(\frac{\partial u_\alpha}{\partial \theta} - u_\alpha \cot\theta\right) + \frac{1}{R\sin\theta}\frac{\partial u_\theta}{\partial \alpha}, \\
&\gamma_{\alpha R} = \frac{1}{R\sin\theta}\frac{\partial u_R}{\partial \alpha} + \frac{\partial u_\alpha}{\partial R} - \frac{u_\alpha}{R}, \\
&\gamma_{R\theta} = \frac{1}{R}\frac{\partial u_R}{\partial \theta} + \frac{\partial u_\theta}{\partial R} - \frac{u_\theta}{R}.
\end{aligned}
\tag{1.2.3}
$$

In tensor form, the geometric equations in Cartesian coordinate system can be written concisely as

$$
\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.2.4}
$$

with $2\varepsilon_{ij} = \gamma_{ij}$ when $i \neq j$. Here $u_{i,j}$ means $\partial u_i / \partial x_j$. Sometimes, it is also useful to express the equations in matrix form, *i.e.*,

$$
\{\varepsilon\} = E^{\mathrm{T}}(\nabla)\{u\}, \tag{1.2.5}
$$

where $\{u\} = [u, v, w]^{\mathrm{T}}$ and $E(\nabla)$ is an operator matrix defined by

$$E(\nabla)=\begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}. \tag{1.2.6}$$

The geometric equations specify strains when displacements are known. They can also be regarded as the first order partial differential equations for solving displacements when strains are known. In this case, however, we will be using six equations to solve for three displacement components. This cannot yield a solution unless the six strain components follow certain conditions, called the *compatibility conditions*, which can be obtained easily as

$$\varepsilon_{ij,kl}+\varepsilon_{kl,ij}-\varepsilon_{lj,ki}-\varepsilon_{ki,lj}=0\,. \tag{1.2.7}$$

Equation (1.2.7) represents six independent equations. In Cartesian coordinate system, they can be written as

$$\begin{aligned}
&\frac{\partial^2\varepsilon_y}{\partial z^2}+\frac{\partial^2\varepsilon_z}{\partial y^2}-\frac{\partial^2\gamma_{yz}}{\partial y\partial z}=0,\\
&\frac{\partial^2\varepsilon_z}{\partial x^2}+\frac{\partial^2\varepsilon_x}{\partial z^2}-\frac{\partial^2\gamma_{zx}}{\partial z\partial x}=0,\\
&\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=0,\\
&\frac{\partial^2\varepsilon_x}{\partial y\partial z}+\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)=0,\\
&\frac{\partial^2\varepsilon_y}{\partial z\partial x}+\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}-\frac{\partial\gamma_{yz}}{\partial x}\right)=0,\\
&\frac{\partial^2\varepsilon_z}{\partial x\partial y}+\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{xy}}{\partial z}-\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}\right)=0.
\end{aligned} \tag{1.2.8}$$

In other coordinate systems, the explicit form of the compatibility conditions is lengthy. For convenience, we list only those in cylindrical coordinates when deformation is axisymmetric, *i.e.*,

$$\varepsilon_r - \varepsilon_\alpha - r\frac{\partial \varepsilon_\alpha}{\partial r} = 0, \quad \frac{\partial \varepsilon_z}{\partial r} + r\frac{\partial^2 \varepsilon_\alpha}{\partial z^2} - \frac{\partial \gamma_{zr}}{\partial z} = 0 \,. \tag{1.2.9}$$

1.2.2 Equations of Motion

In Cartesian coordinates, the equations of motion can be expressed as

$$\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = \rho\frac{\partial^2 u}{\partial t^2}, \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + F_y = \rho\frac{\partial^2 v}{\partial t^2}, \\
&\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = \rho\frac{\partial^2 w}{\partial t^2},
\end{aligned} \tag{1.2.10}$$

where ρ is the density of the material and F_i is the component in i-direction of the body force per unit volume.

In cylindrical coordinates, these equations become

$$\begin{aligned}
&\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\alpha}}{\partial \alpha} + \frac{\partial \tau_{zr}}{\partial z} + \frac{\sigma_r - \sigma_\alpha}{r} + F_r = \rho\frac{\partial^2 u_r}{\partial t^2}, \\
&\frac{\partial \tau_{r\alpha}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\alpha}{\partial \alpha} + \frac{\partial \tau_{\alpha z}}{\partial z} + \frac{2\tau_{r\alpha}}{r} + F_\alpha = \rho\frac{\partial^2 u_\alpha}{\partial t^2}, \\
&\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\alpha z}}{\partial \alpha} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{zr}}{r} + F_z = \rho\frac{\partial^2 w}{\partial t^2}.
\end{aligned} \tag{1.2.11}$$

In spherical coordinates, these equations can be written as

$$\begin{aligned}
&\frac{\partial \sigma_R}{\partial R} + \frac{1}{R}\frac{\partial \tau_{R\theta}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \tau_{\alpha R}}{\partial \alpha} + \frac{1}{R}(2\sigma_R - \sigma_\theta - \sigma_\alpha + \tau_{R\theta}\cot\theta) + F_R = \rho\frac{\partial^2 u_R}{\partial t^2}, \\
&\frac{\partial \tau_{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \tau_{\theta\alpha}}{\partial \alpha} + \frac{1}{R}[(\sigma_\theta - \sigma_\alpha)\cot\theta + 3\tau_{R\theta})] + F_\theta = \rho\frac{\partial^2 u_\theta}{\partial t^2}, \\
&\frac{\partial \tau_{\alpha R}}{\partial R} + \frac{1}{R}\frac{\partial \tau_{\theta\alpha}}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial \sigma_\alpha}{\partial \alpha} + \frac{1}{R}[2\tau_{\theta\alpha}\cot\theta + 3\tau_{\alpha R})] + F_\alpha = \rho\frac{\partial^2 u_\alpha}{\partial t^2}.
\end{aligned} \tag{1.2.12}$$

The equations of motion in Cartesian coordinates can be also written concisely in tensor form as

$$\sigma_{ij,j} + F_i = \rho \ddot{u}_i\,, \quad (i = 1, 2, 3)\,, \tag{1.2.13}$$

where a dot indicates partial differentiation with respect to time t. If the motion of the solid does not involve acceleration, Eq. (1.2.13) reduces to the *equations of equilibrium*, *i.e.*,

$$\sigma_{ij,j} + F_i = 0\,, \quad (i = 1, 2, 3)\,. \tag{1.2.14}$$

1.2.3 Constitutive Equations

The constitutive equations in linear elasticity are represented by the generalized Hooke's law. If the state of vanishing strain corresponds to zero stress, then in Cartesian coordinates the generalized Hooke's law can be written as

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}\,, \tag{1.2.15}$$

where c_{ijkl} are components of a fourth-rank tensor, representing the properties of a material, which generally varies from one point to another in the material. If c_{ijkl} do not change across a material, it is called a homogeneous material. This book will only consider homogeneous and elastic materials whose c_{ijkl} are independent of coordinates.

Since σ_{ij} is symmetric, the exchange of indices i and j in Eq. (1.2.15) does not alter the result, which gives rise to

$$c_{ijkl} = c_{jikl}\,,$$

Without losing the generality, c_{ijkl} can be regarded symmetric with respect to the last two indices as detailed below.

First, define

$$c'_{ijkl} = (c_{ijkl} + c_{ijlk})/2\,, \quad c''_{ijkl} = (c_{ijkl} - c_{ijlk})/2\,,$$

which shows $c'_{ijkl} = c'_{ijlk}$ and $c''_{ijkl} = -c''_{ijlk}$, *i.e.* c'_{ijkl} are symmetric and c''_{ijkl} are antisymmetric with respect to the last two indices. Then, c_{ijkl} can be expressed as

$$c_{ijkl} = c'_{ijkl} + c''_{ijkl} .$$

Thus, Eq. (1.2.15) can be written as

$$\sigma_{ij} = c'_{ijkl}\varepsilon_{kl} + c''_{ijkl}\varepsilon_{kl} .$$

Noting that the second term of the right-hand side vanishes because $\varepsilon_{kl} = \varepsilon_{lk}$ and $c''_{ijkl} = -c''_{ijlk}$, we have

$$\sigma_{ij} = c'_{ijkl}\varepsilon_{kl} ,$$

where c'_{ijkl} are symmetric with respect to either the first two indices or the last two indices.

It is therefore reasonable to assume that c_{ijkl} in Eq. (1.2.15) has the following symmetry:

$$c_{ijkl} = c_{jikl} = c_{ijlk} .$$

Thus, among the total 81 components of c_{ijkl}, the maximum number of independent ones is 36.

To avoid the double summation over k and l in Eq. (1.2.15), introduce the following notations

$$\sigma_{11} = \sigma_1 , \quad \sigma_{22} = \sigma_2 , \quad \sigma_{33} = \sigma_3 , \quad \sigma_{23} = \sigma_4 , \quad \sigma_{31} = \sigma_5 , \quad \sigma_{12} = \sigma_6 ,$$
$$\varepsilon_{11} = \varepsilon_1 , \quad \varepsilon_{22} = \varepsilon_2 , \quad \varepsilon_{33} = \varepsilon_3 , \quad 2\varepsilon_{23} = \varepsilon_4 , \quad 2\varepsilon_{31} = \varepsilon_5 , \quad 2\varepsilon_{12} = \varepsilon_6 ,$$

where ε_4, ε_5 and ε_6 are the engineering shear strains. Equation (1.2.15) can then be rewritten as

$$\sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + c_{13}\varepsilon_3 + c_{14}\varepsilon_4 + c_{15}\varepsilon_5 + c_{16}\varepsilon_6 ,$$
$$\sigma_2 = c_{21}\varepsilon_1 + c_{22}\varepsilon_2 + c_{23}\varepsilon_3 + c_{24}\varepsilon_4 + c_{25}\varepsilon_5 + c_{26}\varepsilon_6 ,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\sigma_6 = c_{61}\varepsilon_1 + c_{62}\varepsilon_2 + c_{63}\varepsilon_3 + c_{64}\varepsilon_4 + c_{65}\varepsilon_5 + c_{66}\varepsilon_6 ,$$

or in a more concise form,

$$\sigma_i = c_{ij}\varepsilon_j, \quad (i,\ j = 1,\ 2,\ \cdots,\ 6)\ .$$

The corresponding matrix form is

$$\{\sigma\} = [c]\{\varepsilon\}, \tag{1.2.16}$$

where $\{\sigma\}$ and $\{\varepsilon\}$ are vectors of stress and engineering strain, respectively. In Cartesian coordinates, they become

$$\{\sigma\} = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{zx}, \tau_{xy}]^{\mathrm{T}},$$
$$\{\varepsilon\} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{zx}, \gamma_{xy}]^{\mathrm{T}}.$$

In Eq. (1.2.16), $[c] = [c_{ij}]$ should be a nonsingular and reversible matrix, *i.e.* $\det[c] \neq 0$. Hence, Eq. (1.2.16) can also be written as

$$\{\varepsilon\} = [s]\{\sigma\}, \tag{1.2.17}$$

where $[s] = [s_{ij}]$ is the inverse of $[c]$, *i.e.* $[s] = [c]^{-1}$. In the above, c_{ij} are called the *elastic stiffnesses* (or moduli) of a material, having the dimension of stresses (F/L^2) because strains are dimensionless, and s_{ij} are called the *elastic compliances* of the material with the dimension of L^2/F.

If there exists a *strain energy density function*,

$$W = \frac{1}{2}c_{ij}\varepsilon_i\varepsilon_j, \quad (i,\ j = 1,\ 2,\ \cdots,\ 6) \tag{1.2.18}$$

then

$$\frac{\partial W}{\partial \varepsilon_i} = \sigma_i = c_{ij}\varepsilon_j, \tag{1.2.19}$$

and

$$\frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = c_{ij}\ .$$

Similarly,

$$\frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i} = c_{ji}\ .$$

Therefore $c_{ij} = c_{ji}$, since the order of differentiation is immaterial. This indicates that the number of independent elastic stiffnesses c_{ij} is further reduced from 36 to 21. Similarly, we have $s_{ij} = s_{ji}$. Thus, for a general anisotropic elastic material, there are only 21 independent elastic stiffness constants or elastic compliance coefficients. Because the strain energy density W is always non-negative and becomes zero only when $\varepsilon_i = 0$, $(i = 1, 2, \cdots, 6)$, it is clear that the stiffness matrix $[c]$ and its inverse, the compliance matrix $[s]$, are both positive definite.

In a different coordinate system (x', y', z'), the constitutive equations will have the same form as Eq. (1.2.16) or (1.2.17), *i.e.*,

$$\{\sigma'\} = [c']\{\varepsilon'\}, \tag{1.2.20}$$

or

$$\{\varepsilon'\} = [s']\{\sigma'\}. \tag{1.2.21}$$

Using Eqs. (1.1.1) and (1.1.2), we can transfer Eq. (1.2.20) into the linear relationship between $\{\sigma\}$ and $\{\varepsilon\}$. Then by comparing it with Eq. (1.2.16), we can easily obtain the transformation formula between $[c']$ and $[c]$.

As just mentioned, for a general anisotropic material, $[c]$ or $[s]$ has twenty-one independent elements and hence the application of the constitutive equation (1.2.16) or (1.2.17) will bring about tremendous difficulties in solving a problem. Fortunately, the equation can be much simplified when the elastic properties of a material possess certain symmetries. We will now introduce the simplified constitutive equations for various materials with special properties.

(1) Plane of elastic symmetry

At any point in a solid, if there exists a plane about which the elastic properties are symmetrical, the number of independent elements in $[c]$ will reduce to thirteen. The direction perpendicular to this *plane of elastic symmetry* is often called the *principal elastic direction* or the *principal direction of the material*. Consider a substance elastically symmetric with respect to the xOy coordinate plane. The symmetry is expressed by the statement that $[c]$ is invariant under the transformation $x' = x$, $y' = y$, and $z' = -z$. Thus, according to Eq. (1.2.20), we have

$$\{\sigma'\} = [c]\{\varepsilon'\}. \tag{1.2.22}$$

For this transformation, we have

$$l_{1'1} = l_{2'2} = 1,\ l_{3'3} = -1,\ l_{1'2} = l_{2'1} = l_{2'3} = l_{3'2} = l_{3'1} = l_{1'3} = 0.$$

Substituting these direction cosines into Eqs. (1.1.1) and (1.1.2) yields

$$\begin{aligned}
&\sigma_{x'} = \sigma_x,\ \sigma_{y'} = \sigma_y,\ \sigma_{z'} = \sigma_z,\ \tau_{y'z'} = -\tau_{yz},\ \tau_{z'x'} = -\tau_{zx},\ \tau_{x'y'} = \tau_{xy},\\
&\varepsilon_{x'} = \varepsilon_x,\ \varepsilon_{y'} = \varepsilon_y,\ \varepsilon_{z'} = \varepsilon_z,\ \gamma_{y'z'} = -\gamma_{yz},\ \gamma_{z'x'} = -\gamma_{zx},\ \gamma_{x'y'} = \gamma_{xy}.
\end{aligned}$$

Using these relations, we get from Eqs. (1.2.22) and (1.2.16)

$$c_{14} = c_{15} = c_{24} = c_{25} = c_{34} = c_{35} = c_{46} = c_{56} = 0. \tag{1.2.23}$$

Thus the generalized Hooke's law of Eq. (1.2.16) becomes

$$\begin{aligned}
\sigma_x &= c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z + c_{16}\gamma_{xy},\\
\sigma_y &= c_{12}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z + c_{26}\gamma_{xy},\\
\sigma_z &= c_{13}\varepsilon_x + c_{23}\varepsilon_y + c_{33}\varepsilon_z + c_{36}\gamma_{xy},\\
\tau_{yz} &= c_{44}\gamma_{yz} + c_{45}\gamma_{zx},\\
\tau_{zx} &= c_{45}\gamma_{yz} + c_{55}\gamma_{zx},\\
\tau_{xy} &= c_{16}\varepsilon_x + c_{26}\varepsilon_y + c_{36}\varepsilon_z + c_{66}\gamma_{xy}.
\end{aligned} \tag{1.2.24}$$

(2) Orthotropic material

If there exist three orthogonal planes of elastic symmetry at any point in a solid, then there are nine independent elements in $[c]$, and the material is said to be orthotropic. Let the three coordinate planes of a Cartesian system, *xOy*, *xOz* and *yOz*, coincide with these planes of symmetry. Performing the similar transformation with respect to each coordinate plane, we will get, in addition to Eq. (1.2.23),

$$\begin{aligned} &c_{14} = c_{16} = c_{24} = c_{26} = c_{34} = c_{36} = c_{45} = c_{56} = 0\,, \\ &c_{15} = c_{16} = c_{25} = c_{26} = c_{35} = c_{36} = c_{45} = c_{46} = 0\,. \end{aligned} \tag{1.2.25}$$

Hence Eq. (1.2.24) further reduces to

$$\begin{aligned} &\sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z\,, \\ &\sigma_y = c_{12}\varepsilon_x + c_{22}\varepsilon_y + c_{23}\varepsilon_z\,, \\ &\sigma_z = c_{13}\varepsilon_x + c_{23}\varepsilon_y + c_{33}\varepsilon_z\,, \\ &\tau_{yz} = c_{44}\gamma_{yz}\,, \quad \tau_{zx} = c_{55}\gamma_{zx}\,, \quad \tau_{xy} = c_{66}\gamma_{xy}\,. \end{aligned} \tag{1.2.26}$$

In this case, all the coordinate axes are in the principal directions of the material. Equation (1.2.26) shows that in an orthotropic material, normal stresses depend only on normal strains; a shear stress on a plane depends only on the shear strain on the same plane. This makes the stress and deformation analysis of an orthotropic solid much easier than that of a general anisotropic material.

We can also express the strains in Eq. (1.2.26) in terms of the stresses using the compliance matrix $[s]=[c]^{-1}$,

$$\begin{aligned} &\varepsilon_x = s_{11}\sigma_x + s_{12}\sigma_y + s_{13}\sigma_z\,, \\ &\varepsilon_y = s_{12}\sigma_x + s_{22}\sigma_y + s_{23}\sigma_z\,, \\ &\varepsilon_z = s_{13}\sigma_x + s_{23}\sigma_y + s_{33}\sigma_z\,, \\ &\gamma_{yz} = s_{44}\tau_{yz}\,, \quad \gamma_{zx} = s_{55}\tau_{zx}\,, \quad \gamma_{xy} = s_{66}\tau_{xy}\,. \end{aligned} \tag{1.2.27a}$$

In the literature, one often finds engineering constants, E_i, G_{ij} and ν_{ij}, in the stress-strain relationship of Eq. (1.2.27*a*), *i.e.*,

$$\varepsilon_x = \frac{1}{E_1}\sigma_x - \frac{\nu_{21}}{E_2}\sigma_y - \frac{\nu_{31}}{E_3}\sigma_z\,,$$

$$\varepsilon_y = -\frac{\nu_{12}}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y - \frac{\nu_{32}}{E_3}\sigma_z ,$$
$$\varepsilon_z = -\frac{\nu_{13}}{E_1}\sigma_x - \frac{\nu_{23}}{E_2}\sigma_y + \frac{1}{E_3}\sigma_z , \tag{1.2.27b}$$
$$\gamma_{yz} = \frac{1}{G_{23}}\tau_{yz} , \quad \gamma_{zx} = \frac{1}{G_{31}}\tau_{zx} , \quad \gamma_{xy} = \frac{1}{G_{12}}\tau_{xy} ,$$

where

$$\nu_{21}/E_2 = \nu_{12}/E_1, \quad \nu_{31}/E_3 = \nu_{13}/E_1, \quad \nu_{32}/E_3 = \nu_{23}/E_2 .$$

The most important and frequently encountered type of curvilinear anisotropy is *cylindrical anisotropy*. Natural wood is a representative example of such materials. In this case, if cylindrical coordinates (r, α, z) are adopted with the z-axis coincident with the axis of anisotropy, then the generalized Hooke's law still holds its form of Eq. (1.2.16) or (1.2.17), while the stress and strain vectors should now be written as $\{\sigma\} = [\sigma_r, \sigma_\alpha, \sigma_z, \tau_{\alpha z}, \tau_{zr}, \tau_{r\alpha}]^{\mathrm{T}}$ and $\{\varepsilon\} = [\varepsilon_r, \varepsilon_\alpha, \varepsilon_z, \gamma_{\alpha z}, \gamma_{zr}, \gamma_{r\alpha}]^{\mathrm{T}}$. Accordingly, the number of independent elastic stiffness constants c_{ij} or compliance coefficients s_{ij} is still 21. If any plane perpendicular to the z-axis is a plane of elastic symmetry, then Eq. (1.2.23) also holds, and the number of independent elastic stiffness constants c_{ij} (or compliance coefficients s_{ij}) reduces to 13. Moreover, if, at an arbitrary point in the material, there exist three planes of elastic symmetry that are perpendicular to r-, α- and z-directions, respectively, then Eqs. (1.2.23) and (1.2.25) keep unchanged, and the number of independent elastic stiffness constants c_{ij} (or compliance coefficients s_{ij}) further reduces to 9. Materials having this type of anisotropy are cylindrically orthotropic. Similar to Eqs. (1.2.26) and (1.2.27), for a *cylindrically orthotropic material*, we have

$$\begin{aligned}
\sigma_r &= c_{11}\varepsilon_r + c_{12}\varepsilon_\alpha + c_{13}\varepsilon_z ,\\
\sigma_\alpha &= c_{12}\varepsilon_r + c_{22}\varepsilon_\alpha + c_{23}\varepsilon_z ,\\
\sigma_z &= c_{13}\varepsilon_r + c_{23}\varepsilon_\alpha + c_{33}\varepsilon_z ,\\
\tau_{\alpha z} &= c_{44}\gamma_{\alpha z} , \quad \tau_{zr} = c_{55}\gamma_{zr} , \quad \tau_{r\alpha} = c_{66}\gamma_{r\alpha} ,
\end{aligned} \tag{1.2.28}$$

or in an inverse form

$$
\begin{aligned}
&\varepsilon_r = s_{11}\sigma_r + s_{12}\sigma_\alpha + s_{13}\sigma_z, \\
&\varepsilon_\alpha = s_{12}\sigma_r + s_{22}\sigma_\alpha + s_{23}\sigma_z, \\
&\varepsilon_z = s_{13}\sigma_r + s_{23}\sigma_\alpha + s_{33}\sigma_z, \\
&\gamma_{\alpha z} = s_{44}\tau_{\alpha z}\ , \quad \gamma_{zr} = s_{55}\tau_{zr}\ , \quad \gamma_{r\alpha} = s_{66}\tau_{r\alpha}.
\end{aligned}
\tag{1.2.29a}
$$

Using engineering symbols, we can rewrite Eq. (1.2.29*a*) as

$$
\begin{aligned}
&\varepsilon_r = \frac{1}{E_r}\sigma_r - \frac{\nu_{\alpha r}}{E_\alpha}\sigma_\alpha - \frac{\nu_{zr}}{E_z}\sigma_z, \quad \gamma_{\alpha z} = \frac{1}{G_{\alpha z}}\tau_{\alpha z}, \\
&\varepsilon_\alpha = -\frac{\nu_{r\alpha}}{E_r}\sigma_r + \frac{1}{E_\alpha}\sigma_\alpha - \frac{\nu_{z\alpha}}{E_z}\sigma_z, \quad \gamma_{zr} = \frac{1}{G_{zr}}\tau_{zr}, \\
&\varepsilon_z = -\frac{\nu_{rz}}{E_r}\sigma_r - \frac{\nu_{\alpha z}}{E_\alpha}\sigma_\alpha + \frac{1}{E_z}\sigma_z, \quad \gamma_{r\alpha} = \frac{1}{G_{r\alpha}}\tau_{r\alpha},
\end{aligned}
\tag{1.2.29b}
$$

where

$$
E_i\nu_{ji} = E_j\nu_{ij}, \qquad (i,\ j = r,\ \alpha,\ z). \tag{1.2.30}
$$

Another frequently encountered type of curvilinear anisotropy is *spherical anisotropy,* of which a typical example is the model of Earth considering the effect of curvature in its constitutive description[1]. In this case, if spherical coordinates (R, θ, α) are used, the generalized Hooke's law is still in the form of Eq. (1.2.16) or (1.2.17), but the stress and strain vectors need to be replaced by $\{\sigma\} = [\sigma_R, \sigma_\theta, \sigma_\alpha, \tau_{\theta\alpha}, \tau_{\alpha R}, \tau_{R\theta}]^{\mathrm{T}}$ and $\{\varepsilon\} = [\varepsilon_R, \varepsilon_\theta, \varepsilon_\alpha, \gamma_{\theta\alpha}, \gamma_{\alpha R}, \gamma_{R\theta}]^{\mathrm{T}}$. For a general case, there are only 21 independent elastic stiffness constants c_{ij} (or compliance coefficients s_{ij}). If, at any point in the material, there are three planes of elastic symmetry that are perpendicular to R-, θ- and α-directions (Fig. 1.1), respectively, then Eqs. (1.2.23) and (1.2.25) still hold and the number of independent elastic stiffness constants c_{ij} (or compliance coefficients s_{ij}) becomes 9. Materials with this type of anisotropy are spherically orthotropic.

The generalized Hooke's law for a *spherically orthotropic material* takes a similar form to Eq. (1.2.26), *i.e.*

[1] See the ACY400 Earth model in Montagner and Anderson (1989). The analytical determination of stress fields in the interior of the Earth using this model was presented and discussed by Ding, Zou and Ding (1996).

$$\begin{aligned}
\sigma_R &= c_{11}\varepsilon_R + c_{12}\varepsilon_\theta + c_{13}\varepsilon_\alpha, \\
\sigma_\theta &= c_{12}\varepsilon_R + c_{22}\varepsilon_\theta + c_{23}\varepsilon_\alpha, \\
\sigma_\alpha &= c_{13}\varepsilon_R + c_{23}\varepsilon_\theta + c_{33}\varepsilon_\alpha, \\
\tau_{\theta\alpha} &= c_{44}\gamma_{\theta\alpha}, \quad \tau_{\alpha R} = c_{55}\gamma_{\alpha R}, \quad \tau_{R\theta} = c_{66}\gamma_{R\theta}.
\end{aligned} \tag{1.2.31}$$

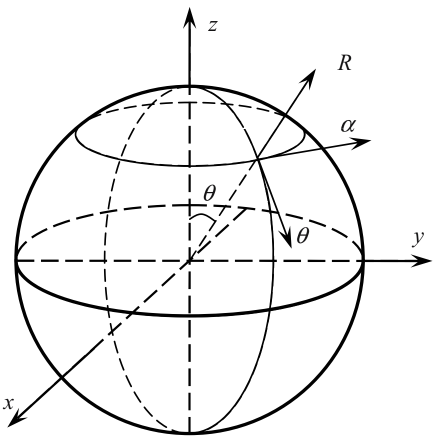

Fig. 1.1 The directions of coordinates at a point on the spherical surface.

(3) Transversely isotropic material

If at any point there is an axis of symmetry such that the elastic properties in any direction within a plane perpendicular to the axis are all the same, the total number of independent elements in [c] will reduce to five. The plane is called an *isotropic plane* and the material is called a *transversely isotropic material*. The hexagonal crystals, like Cadmium and Zinc, are transversely isotropic.

If we take the coordinate plane xOy to coincide with the isotropic plane, then the z-axis is the axis of symmetry. Taking a new Cartesian system such that $x' = y$, $y' = -x$, and $z' = z$, then we have

$$l_{1'2} = l_{3'3} = 1, \; l_{2'1} = -1, \; l_{1'1} = l_{1'3} = l_{2'2} = l_{2'3} = l_{3'1} = l_{3'2} = 0.$$

Substituting into Eqs. (1.1.1) and (1.1.2), yields

$$\begin{aligned}
&\sigma_{x'} = \sigma_y, \; \sigma_{y'} = \sigma_x, \; \sigma_{z'} = \sigma_z, \; \tau_{y'z'} = -\tau_{zx}, \; \tau_{z'x'} = -\tau_{yz}, \; \tau_{x'y'} = -\tau_{xy}, \\
&\varepsilon_{x'} = \varepsilon_y, \; \varepsilon_{y'} = \varepsilon_x, \; \varepsilon_{z'} = \varepsilon_z, \; \gamma_{y'z'} = -\gamma_{zx}, \; \gamma_{z'x'} = -\gamma_{yz}, \; \gamma_{x'y'} = -\gamma_{xy}.
\end{aligned}$$

Using these relations as well as Eqs. (1.2.23) and (1.2.25), we get from Eqs. (1.2.22) and (1.2.26)

$$c_{11} = c_{22}, \quad c_{13} = c_{23}, \quad c_{44} = c_{55}. \tag{1.2.32a}$$

In view of this equation, there are only six independent constants in the constitutive relations in Eq. (1.2.26). Now taking another transformation by rotating the original coordinate system 45 degree about the z-axis, we have the relations

$$x' = (x+y)/\sqrt{2}, \quad y' = (y-x)/\sqrt{2}, \quad z' = z,$$

and

$$l_{1'1} = l_{1'2} = l_{2'2} = 1/\sqrt{2}, \quad l_{2'1} = -1/\sqrt{2}, \quad l_{3'3} = 1, \quad l_{1'3} = l_{2'3} = l_{3'1} = l_{3'2} = 0.$$

Substituting these direction cosines into Eqs. (1.1.1) and (1.1.2), yields

$$\begin{aligned}
&\sigma_{x'} = (\sigma_x + \sigma_y + 2\tau_{xy})/2, \quad \sigma_{y'} = (\sigma_x + \sigma_y - 2\tau_{xy})/2, \quad \sigma_{z'} = \sigma_z, \\
&\tau_{y'z'} = (\tau_{yz} - \tau_{zx})/\sqrt{2}, \quad \tau_{z'x'} = (\tau_{yz} + \tau_{zx})/\sqrt{2}, \quad \tau_{x'y'} = (\sigma_y - \sigma_x)/2, \\
&\varepsilon_{x'} = (\varepsilon_x + \varepsilon_y + \gamma_{xy})/2, \quad \varepsilon_{y'} = (\varepsilon_x + \varepsilon_y - \gamma_{xy})/2, \quad \varepsilon_{z'} = \varepsilon_z, \\
&\gamma_{y'z'} = (\gamma_{yz} - \gamma_{zx})/\sqrt{2}, \quad \gamma_{z'x'} = (\gamma_{yz} + \gamma_{zx})/\sqrt{2}, \quad \gamma_{x'y'} = \varepsilon_y - \varepsilon_x.
\end{aligned}$$

Using these relations as well as Eqs. (1.2.23), (1.2.25) and (1.2.32a), we get from Eqs. (1.2.22) and (1.2.26)

$$2c_{66} = c_{11} - c_{12}. \tag{1.2.32b}$$

Thus, Eq. (1.2.26) becomes

$$\begin{aligned}
&\sigma_x = c_{11}\varepsilon_x + c_{12}\varepsilon_y + c_{13}\varepsilon_z, \\
&\sigma_y = c_{12}\varepsilon_x + c_{11}\varepsilon_y + c_{13}\varepsilon_z, \\
&\sigma_z = c_{13}\varepsilon_x + c_{13}\varepsilon_y + c_{33}\varepsilon_z, \\
&\tau_{yz} = c_{44}\gamma_{yz}, \ \tau_{zx} = c_{44}\gamma_{zx}, \ \tau_{xy} = c_{66}\gamma_{xy},
\end{aligned} \tag{1.2.33}$$

where $c_{66} = (c_{11} - c_{12})/2$.

Similarly, the expressions for strains in terms of stresses, given by Eq. (1.2.27*a*), become

$$\begin{aligned} \varepsilon_x &= s_{11}\sigma_x + s_{12}\sigma_y + s_{13}\sigma_z, \\ \varepsilon_y &= s_{12}\sigma_x + s_{11}\sigma_y + s_{13}\sigma_z, \\ \varepsilon_z &= s_{13}\sigma_x + s_{13}\sigma_y + s_{33}\sigma_z, \\ \gamma_{yz} &= s_{44}\tau_{yz}, \quad \gamma_{zx} = s_{44}\tau_{zx}, \quad \gamma_{xy} = s_{66}\tau_{xy}, \end{aligned} \tag{1.2.34}$$

where $s_{66} = 2(s_{11} - s_{12})$. We can also express this equation in terms of engineering constants, *i.e.*,

$$\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) - \frac{\nu'}{E'}\sigma_z, \quad \gamma_{yz} = \frac{1}{G'}\tau_{yz}, \\ \varepsilon_y &= \frac{1}{E}(-\nu\sigma_x + \sigma_y) - \frac{\nu'}{E'}\sigma_z, \quad \gamma_{zx} = \frac{1}{G'}\tau_{zx}, \\ \varepsilon_z &= -\frac{\nu'}{E'}(\sigma_x + \sigma_y) + \frac{1}{E'}\sigma_z, \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}, \end{aligned} \tag{1.2.35}$$

where

$$2G = E/(1+\nu). \tag{1.2.36}$$

In cylindrical coordinates, Eq. (1.2.33) becomes

$$\begin{aligned} \sigma_r &= c_{11}\varepsilon_r + c_{12}\varepsilon_\alpha + c_{13}\varepsilon_z, \\ \sigma_\alpha &= c_{12}\varepsilon_r + c_{11}\varepsilon_\alpha + c_{13}\varepsilon_z, \\ \sigma_z &= c_{13}\varepsilon_r + c_{13}\varepsilon_\alpha + c_{33}\varepsilon_z, \\ \tau_{\alpha z} &= c_{44}\gamma_{\alpha z}, \quad \tau_{zr} = c_{44}\gamma_{zr}, \quad \tau_{r\alpha} = c_{66}\gamma_{r\alpha}, \end{aligned} \tag{1.2.37}$$

where $c_{66} = (c_{11} - c_{12})/2$, and Eq. (1.2.34) becomes

$$\begin{aligned} \varepsilon_r &= s_{11}\sigma_r + s_{12}\sigma_\alpha + s_{13}\sigma_z, \\ \varepsilon_\alpha &= s_{12}\sigma_r + s_{11}\sigma_\alpha + s_{13}\sigma_z, \\ \varepsilon_z &= s_{13}\sigma_r + s_{13}\sigma_\alpha + s_{33}\sigma_z, \\ \gamma_{\alpha z} &= s_{44}\tau_{\alpha z}, \quad \gamma_{zr} = s_{44}\tau_{zr}, \quad \gamma_{r\alpha} = s_{66}\tau_{r\alpha}, \end{aligned} \tag{1.2.38}$$

where $s_{66} = 2(s_{11} - s_{12})$.

The constitutive relations for a spherically orthotropic material are given by Eq. (1.2.31). Further, if the elasticity property in any direction is the same at the point of intersection in a plane perpendicularly intersecting the radial R, the material is said to be spherically isotropic[2]. By a derivation similar to that of Eq. (1.2.33), we get the generalized Hooke's law of a *spherically isotropic material* in spherical coordinates

$$\begin{aligned}
\sigma_R &= c_{11}\varepsilon_R + c_{12}\varepsilon_\theta + c_{12}\varepsilon_\alpha, \\
\sigma_\theta &= c_{12}\varepsilon_R + c_{22}\varepsilon_\theta + c_{23}\varepsilon_\alpha, \\
\sigma_\alpha &= c_{12}\varepsilon_R + c_{23}\varepsilon_\theta + c_{22}\varepsilon_\alpha, \\
\tau_{\theta\alpha} &= c_{44}\gamma_{\theta\alpha}, \tau_{\alpha R} = c_{55}\gamma_{\alpha R}, \tau_{R\theta} = c_{55}\gamma_{R\theta},
\end{aligned} \tag{1.2.39a}$$

where

$$2c_{44} = c_{22} - c_{23}. \tag{1.2.39b}$$

If Eq. (1.2.39) is used to describe a small portion of material where R is very large, the material can be treated approximately as transversely isotropic since the effect of curvature may be ignored. For this approximation, it will be convenient to rewrite Eq. (1.2.39) by rearrangement of the subscripts of elastic constants as

$$\begin{aligned}
\sigma_\theta &= c_{11}\varepsilon_\theta + c_{12}\varepsilon_\alpha + c_{13}\varepsilon_R, \\
\sigma_\alpha &= c_{12}\varepsilon_\theta + c_{11}\varepsilon_\alpha + c_{13}\varepsilon_R, \\
\sigma_R &= c_{13}\varepsilon_\theta + c_{13}\varepsilon_\alpha + c_{33}\varepsilon_R, \\
\tau_{\alpha R} &= c_{44}\gamma_{\alpha R}, \tau_{R\theta} = c_{44}\gamma_{R\theta}, \tau_{\theta\alpha} = c_{66}\gamma_{\theta\alpha},
\end{aligned} \tag{1.2.40a}$$

where

$$2c_{66} = c_{11} - c_{12}. \tag{1.2.40b}$$

The materials described by Eq. (1.2.40) are also called transversely isotropic materials in some references, see Shenderov (1985), Bufler (1998), Khoma (1998), among others.

Using engineering symbols, we have

---

[2] A classical solution of spherically isotropic material was obtained by Saint-Venant for a spherical shell subjected to uniform internal and external pressures, see Love (1927) and Lekhnitskii (1981).