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Multiforms, Dyadics, and Electromagnetic Media

Ismo V. Lindell

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Contents

LIST OF TABLES

LIST OF ILLUSTRATIONS

Chapter 2

Figure 2.1 The double-cross product $\overset{\times}{\times}$ corresponding to the double-wedge product $\overset{\wedge}{\wedge}$ was originally introduced by Gibbs in 1886 [6]. Here the bar corresponds to the dyadic product (later replaced by "no sign" by Gibbs [5]). The expression on the last line stands for the *n*th double-cross power of the dyadic $\mathbf{\Phi}$.

Figure 2.2 The notation (bracketed number in superscript) for the double-wedge power was introduced by Reichardt [7]. There is no sign denoting the bar product.

Preface

This book is a continuation of a previous one by the same author (*Differential Forms in Electromagnetics*, [1]) on the application of multivectors, multiforms, and dyadics to electromagnetic problems. Main attention is focused on applying the formalism to the analysis of electromagnetic media, as inspired by the ongoing engineering interest in constructing novel metamaterials and metaboundaries. In this respect the present exposition can also be seen as an enlargement of a chapter in a recent book on metamaterials [2] by including substance from more recent studies by this author and collaborators. The present four-dimensional (4D) formalism has proved of advantage in simplifying expressions in the analysis of general media in comparison to the classical three-dimensional (3D) Gibbsian formalism. However, the step from electromagnetic media, defined by medium parameters, to actual metamaterials and metaboundaries, defined by physical structures, is beyond the scope of this book.

The first four chapters are devoted to the algebra of multiforms and dyadics in order to introduce the formalism and useful analytic tools. Similar material presented in [1] has been extended. Chapter 5 summarizes basic electromagnetic concepts in the light of the present formalism. Chapter 6 discusses transformations useful for simplifying problems. In the final Chapters 7–10 different classes of electromagnetic media are defined on the basis of their various properties. Because the most general linear electromagnetic medium requires 36 parameters for its definition, it is not easy to understand the effect of all these parameters. This is why it becomes necessary to define medium classes with reduced numbers of parameters. In

Chapter 7 the classes are defined in terms of a natural decomposition of the medium bidyadic in three components, independent of any basis representation. Chapter 8 considers media whose medium bidyadic can be expressed in terms of quadratic functions of dyadics defined by 16 parameters. In Chapter 9 medium classes are defined by the degree of the algebraic equation satisfied by the medium bidyadic. Finally, in Chapter 10 media are defined by certain restrictions imposed on plane waves propagating in the media.

Main emphasis lies on the application of the present formalism in the definition and analysis of media. It turns out that certain concepts cannot be easily defined through the 3D Gibbsian vector and dyadic representation. For example, the perfect electromagnetic conductor (PEMC) medium generalizing both perfect electric conductor (PEC) and perfect magnetic conductor (PMC) media appears as the simplest possible medium in the present formalism while in terms of conventional engineering representation with Gibbsian medium dyadics it requires parameters of infinite magnitude. As another example, decomposable bi-anisotropic media, defined to generalize uniaxially anisotropic media in which fields can be decomposed in transverse electric (TE) and transverse magnetic (TM) components, can be represented in a compact 4D form while the original analysis applying 3D Gibbsian formulation produced extensive expressions. In addition to the economy in expression, the present analysis is able to reveal novel additional solutions. A number of details in the analysis has been skipped in the text and left as problems for the reader. Solutions to the problems can be found at the end of the book, which allows the book to be used for self-study.

Because of the background of the author, the book is mainly directed to electrical engineers, although physicists and applied mathematicians may find the contents of interest as well. It has been attempted to make the transition from 3D Gibbsian vector and dyadic formalism, familiar to most electrical engineers, to the 4D exterior calculus involving multivectors, multiforms, and dyadics, as small as possible by showing connections to the corresponding Gibbsian quantities in an appendix. The main idea for adopting the 4D formalism is not to emphasize time-domain analysis of electromagnetic fields but to obtain compactness in expression and analysis. In fact, harmonic time dependence exp ($j\omega t$) is often tacitly assumed by allowing complex magnitudes for the medium parameters.

Compared to the previous book [1], the present approach shows some changes in the terminology followed by an effort to make the presentation more accessible. For example, to emphasize the most important dyadics defining electromagnetic media, they have been called bidyadics because they represent mappings between two-forms and/or bivectors.

The author thanks students of the postgraduate courses based on the material of this book for their comments and responses. Special thanks are due to professors Ari Sihvola and Friedrich Hehl and for doctors Alberto Favaro and Luzi Bergamin for their long-lasting interest and help in treating questions during the years behind this book.

ISMO V. LINDELL

CHAPTER 1
Multivectors and Multiforms

1.1 VECTORS AND ONE-FORMS

Let us consider two four-dimensional (4D) linear spaces, that of vectors, $\mathbb{E}_1$ and that of one-forms $\mathbb{F}_1$. The elements of $\mathbb{E}_1$ are most generally denoted by boldface lowercase Latin letters,

$$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots \in \mathbb{E}_1, \tag{1.1}$$

while the elements of $\mathbb{F}_1$ are most generally denoted by boldface lowercase Greek letters

$$\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots \in \mathbb{F}_1. \tag{1.2}$$

The space of scalars is denoted by $\mathbb{E}_0$ or $\mathbb{F}_0$ and its elements are in general represented by nonboldface Latin or Greek letters a, b, c, ..., α, β, γ, ...

Exceptions are made for quantities with established conventional notation. For example, the electric and magnetic fields are one-forms which are respectively denoted by the boldface uppercase Latin letters **E** and **H**.

1.1.1 Bar Product |

The product of a vector **a** and a one-form $\boldsymbol{\alpha}$ yielding a scalar is denoted by the “bar” sign | as $\mathbf{a}|\boldsymbol{\alpha} \in \mathbb{E}_0$. The product is assumed symmetric,

$$\mathbf{a}|\boldsymbol{\alpha} = \boldsymbol{\alpha}|\mathbf{a}. \tag{1.3}$$

Because of the sign, it will be called as the bar product. In the past it has also been known as the duality product or the inner product. The bar product should not be confused with the dot product. The dot product can be defined for two vectors as $\mathbf{a} \cdot \mathbf{b}$ or two one-forms as $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}$ and it depends on a particular metric dyadic as will be discussed later.

1.1.2 Basis Expansions

A set of four vectors, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_4$, is called a basis if any vector $\mathbf{a}$ can be expressed as

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_4\mathbf{e}_4, \tag{1.4}$$

in terms of some scalars a_i. Similarly, any one-form can be expanded in a basis of one-forms, $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_4$ as

$$\boldsymbol{\alpha} = \alpha_1\boldsymbol{\varepsilon}_1 + \alpha_2\boldsymbol{\varepsilon}_2 + \alpha_3\boldsymbol{\varepsilon}_3 + \alpha_4\boldsymbol{\varepsilon}_4. \tag{1.5}$$

The expansion of the bar product yields

$$\mathbf{a}|\boldsymbol{\alpha} = \sum_{i=1}^{4}\sum_{j=1}^{4} a_i\alpha_j\mathbf{e}_i|\boldsymbol{\varepsilon}_j. \tag{1.6}$$

The vector and one-form bases are called reciprocal to one another if they satisfy

$$\mathbf{e}_i|\boldsymbol{\varepsilon}_j = \delta_{i,j}, \tag{1.7}$$

with

$$\delta_{i,i} = 1, \qquad \delta_{i,j} = 0 \quad i \neq j. \tag{1.8}$$

In this case the scalar coefficients in (1.4) and (1.5) satisfy

$$a_i = \boldsymbol{\varepsilon}_i|\mathbf{a}, \qquad \alpha_i = \mathbf{e}_i|\boldsymbol{\alpha}, \tag{1.9}$$

and the bar product can be expanded as

$$\mathbf{a}|\boldsymbol{\alpha} = \sum_{i=1}^{4} a_i\alpha_i = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4. \tag{1.10}$$

From here onwards we always assume that when the two bases are denoted by $\mathbf{e}_i$ and $\boldsymbol{\varepsilon}_j$, they are reciprocal.

Vectors can be visualized as yardsticks in the 4D spacetime, and they can be used for measuring one-forms. For example, measuring the electric field one-form $\mathbf{E} \in \mathbb{F}_1$ by a vector **a** yields the voltage U between the endpoints of the vector

$$\mathbf{a}|\mathbf{E} = U, \tag{1.11}$$

provided **E** is constant in space or **a** is small in terms of wavelength.

The bar product $\mathbf{a}|\boldsymbol{\alpha}$ is a bilinear function of **a** and $\boldsymbol{\alpha}$. Thus, $\mathbf{a}|\boldsymbol{\alpha}$ can be conceived as a linear scalar-valued function of $\boldsymbol{\alpha}$ for a given vector **a**. Conversely, any linear scalar-valued function $f(\boldsymbol{\alpha})$ can be expressed as a bar product $\mathbf{a}|\boldsymbol{\alpha}$ in terms of some vector **a**. To prove this, we express $\boldsymbol{\alpha}$ in a basis $\{\boldsymbol{\varepsilon}_i\}$ and apply linearity, whence we have

$$\mathbf{a}|\boldsymbol{\alpha} = f(\boldsymbol{\alpha}) = f\left(\sum \alpha_i\boldsymbol{\varepsilon}_i\right) = \sum \alpha_i f(\boldsymbol{\varepsilon}_i) = \sum f(\boldsymbol{\varepsilon}_i)\mathbf{e}_i|\boldsymbol{\alpha}, \tag{1.12}$$

in terms of the reciprocal vector basis $\{\mathbf{e}_i\}$. Thus, the vector **a** can be defined as

$$\mathbf{a} = \sum f(\boldsymbol{\varepsilon}_i)\mathbf{e}_i. \tag{1.13}$$

1.2 BIVECTORS AND TWO-FORMS

1.2.1 Wedge Product ∧

The antisymmetric wedge product ∧ between two vectors **a** and **b** yields a bivector, an element of the space $\mathbb{E}_2$ of bivectors,

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}. \tag{1.14}$$

This implies

$$\mathbf{a} \wedge \mathbf{a} = 0, \tag{1.15}$$

for any vector **a**. In general, bivectors are denoted by boldface uppercase Latin letters,

$$\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots \in \mathbb{E}_2, \tag{1.16}$$

and they can be represented by a sum of wedge products of vectors,

$$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{d} + \cdots . \tag{1.17}$$

Similarly, the wedge product of two one-forms $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ produces a two-form

$$\boldsymbol{\alpha} \wedge \boldsymbol{\beta} = -\boldsymbol{\beta} \wedge \boldsymbol{\alpha}. \tag{1.18}$$

Two-forms are denoted by boldface uppercase Greek letters whenever it appears possible,

$$\boldsymbol{\Gamma}, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \ldots \in \mathbb{F}_2, \tag{1.19}$$

and they are linear combinations of wedge products of one-forms,

$$\boldsymbol{\Gamma} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta} + \boldsymbol{\gamma} \wedge \boldsymbol{\delta} + \cdots . \tag{1.20}$$

A bivector which can be expressed as a wedge product of two vectors, in the form

$$\mathbf{A} = \mathbf{a} \wedge \mathbf{b}, \tag{1.21}$$

is called a simple bivector. Similarly, two-forms of the special form

$$\boldsymbol{\Gamma} = \boldsymbol{\alpha} \wedge \boldsymbol{\beta}, \tag{1.22}$$

are called simple two-forms.

For the 4D vector space as considered here, the bivectors form a space of six dimensions as will be seen below. It is not possible to express the general bivector in the form of a simple bivector.

1.2.2 Basis Expansions

Expanding vectors in a vector basis $\{\mathbf{e}_i\}$ induces a basis expansion of bivectors where the basis bivectors can be denoted by $\mathbf{e}_{ij} = \mathbf{e}_i \wedge \mathbf{e}_j$. Because $\mathbf{e}_{ii} = 0$ and six of the remaining twelve bivectors are linearly dependent of the other six,

$$\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2 = -\mathbf{e}_{21}, \qquad \mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3 = -\mathbf{e}_{32}, \quad \text{etc.}, \tag{1.23}$$

the space of bivectors is six dimensional. Actually, the bivector basis need not be based on any vector basis. Any set of six linearly independent bivectors could do.

A bivector can be expanded in the bivector basis as

$$\begin{aligned} \mathbf{A} &= \sum_J A_J \mathbf{e}_J \\ &= A_{12}\mathbf{e}_{12} + A_{23}\mathbf{e}_{23} + A_{31}\mathbf{e}_{31} + A_{14}\mathbf{e}_{14} + A_{24}\mathbf{e}_{24} + A_{34}\mathbf{e}_{34}. \end{aligned} \tag{1.24}$$

Here, $J = ij$ is a bi-index containing two indices i, j taken in a suitable order. In the following we will apply the order

$$J = 12,\ 23,\ 31,\ 14,\ 24,\ 34. \tag{1.25}$$

Similarly, a basis of two-forms can be built upon the basis of one-forms as $\boldsymbol{\varepsilon}_J = \boldsymbol{\varepsilon}_{ij} = \boldsymbol{\varepsilon}_i \wedge \boldsymbol{\varepsilon}_j$.

It helps in memorizing if we assume that the index 4 corresponds to the temporal basis element and 1, 2, 3 to the three spatial elements. In this case the spatial indices appear in cyclical order $1 \to 2 \to 3 \to 1$ in J while the index 4 occupies the last position.

It is useful to define an operation $K(J)$ yielding the complementary bi-index of a given bi-index J as

$$K(12) = 34, \qquad K(23) = 14, \qquad K(31) = 24, \tag{1.26}$$

$$K(14) = 23, \qquad K(24) = 31, \qquad K(34) = 12. \tag{1.27}$$

Obviously, the complementary index operation satisfies

$$K(K(J)) = J. \tag{1.28}$$

The basis expansion (1.24) can be used to show that any bivector can be expressed as a sum of two simple bivectors, in the form

$$\mathbf{A} = \mathbf{a} \wedge \mathbf{b} + \mathbf{c} \wedge \mathbf{d}. \tag{1.29}$$

Such a representation is not unique. As an example, assuming $A_{23} \neq 0$ in (1.24), we can write

$$\mathbf{A} = \frac{1}{A_{23}}(A_{31}\mathbf{e}_1 - A_{23}\mathbf{e}_2) \wedge (A_{12}\mathbf{e}_1 - A_{23}\mathbf{e}_3) + \left(\sum_{i=1}^{3} A_{i4}\mathbf{e}_i\right) \wedge \mathbf{e}_4. \tag{1.30}$$

Thus, any bivector can be expressed in the form

$$\mathbf{A} = \mathbf{a}_1 \wedge \mathbf{a}_2 + \mathbf{a}_3 \wedge \mathbf{e}_4, \tag{1.31}$$

where the vectors $\mathbf{a}_i$ are spatial, that is, they satisfy $\mathbf{a}_i | \boldsymbol{\varepsilon}_4 = 0$. $\mathbf{a}_1 \wedge \mathbf{a}_2$ is called the spatial part of $\mathbf{A}$ and $\mathbf{a}_3 \wedge \mathbf{e}_4$ its temporal part. Similar rules are valid for two-forms. In particular, any two-form can be expanded in terms of spatial and temporal one-forms as

$$\mathbf{\Gamma} = \boldsymbol{\alpha}_1 \wedge \boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3 \wedge \boldsymbol{\varepsilon}_4, \quad \mathbf{e}_4|\boldsymbol{\alpha}_i = 0. \tag{1.32}$$

1.2.3 Bar Product

We can extend the definition of the bar product of a vector and a one-form to that of a bivector and a two-form, $\mathbf{A}|\mathbf{\Phi} = \mathbf{\Phi}|\mathbf{A}$. Starting from a simple bivector **a∧b** and a simple one-form $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ the bar product is a quadrilinear scalar function of the two vectors and two one-forms and it can be expressed in terms of the four possible bar products of vectors and one-forms as

$$(\mathbf{a} \wedge \mathbf{b})|(\boldsymbol{\alpha} \wedge \boldsymbol{\beta}) = (\mathbf{a}|\boldsymbol{\alpha})(\mathbf{b}|\boldsymbol{\beta}) - (\mathbf{a}|\boldsymbol{\beta})(\mathbf{b}|\boldsymbol{\alpha}) = \det\begin{pmatrix} \mathbf{a}|\boldsymbol{\alpha} & \mathbf{a}|\boldsymbol{\beta} \\ \mathbf{b}|\boldsymbol{\alpha} & \mathbf{b}|\boldsymbol{\beta} \end{pmatrix}. \tag{1.33}$$

Such an expansion follows directly from the antisymmetry of the wedge product and assuming orthogonality of the basis bivectors and two-forms as

$$\mathbf{e}_{ij}|\boldsymbol{\varepsilon}_{k\ell} = \delta_{i,k}\delta_{j,\ell}. \tag{1.34}$$

by assuming ordered indices. Equation (1.33) can be memorized from the corresponding rule for three-dimensional (3D) Gibbsian vectors denoted by $\mathbf{a}_g, \mathbf{b}_g, \mathbf{c}_g, \mathbf{d}_g \in \mathbb{E}_1$,

$$(\mathbf{a}_g \times \mathbf{b}_g) \cdot (\mathbf{c}_g \times \mathbf{d}_g) = (\mathbf{a}_g \cdot \mathbf{c}_g)(\mathbf{b}_g \cdot \mathbf{d}_g) - (\mathbf{a}_g \cdot \mathbf{d}_g)(\mathbf{b}_g \cdot \mathbf{c}_g). \tag{1.35}$$

Relations of multivectors and multiforms to Gibbsian vectors are summarized in Appendix B.

As examples of spatial two-forms we may consider the electric and magnetic flux densities, for which we use the established symbols **D** and **B**. Bivectors can be visualized as surface regions with orientation (sense of rotation). They can be used to measure the flux of a two-form through the surface region. For example, the magnetic flux Φ (a scalar) of the magnetic spatial two-form **B** through the bivector $\mathbf{a}\wedge\mathbf{b}$ is obtained as

$$\Phi = (\mathbf{a}\wedge\mathbf{b})|\mathbf{B}. \tag{1.36}$$

For more details on geometric interpretation of multiforms see, for example, [3, 4].

1.2.4 Contraction Products ⌋ and ⌊

Considering a bivector $\mathbf{a}\wedge\mathbf{b}$ and a two-form $\boldsymbol{\Phi}$, the bar product $(\mathbf{a}\wedge\mathbf{b})|\boldsymbol{\Phi}$ can be conceived as a linear scalar-valued function of the vector **a**. Thus, there must exist a one-form $\boldsymbol{\alpha}$ in terms of which we can express

$$\mathbf{a}|\boldsymbol{\alpha} = (\mathbf{a}\wedge\mathbf{b})|\boldsymbol{\Phi} = \boldsymbol{\Phi}|(\mathbf{a}\wedge\mathbf{b}) = -\boldsymbol{\Phi}|(\mathbf{b}\wedge\mathbf{a}) = \boldsymbol{\alpha}|\mathbf{a}. \tag{1.37}$$

Obviously, the one-form $\boldsymbol{\alpha}$ is a linear function of both **b** and $\boldsymbol{\Phi}$ so that we can express it as a product of the vector **b** and the two-form $\boldsymbol{\Phi}$ and denote it either

$$\boldsymbol{\alpha} = \mathbf{b}\rfloor\boldsymbol{\Phi}, \tag{1.38}$$

or

$$\boldsymbol{\alpha} = -\boldsymbol{\Phi}\lfloor\mathbf{b}. \tag{1.39}$$

The operation denoted by the multiplication sign ⌋ or ⌊ will be called contraction, because the two-form $\boldsymbol{\Phi}$ is contracted ("shortened") by the vector **b** from the left or from the right to yield a one-form. Thus, the contraction product obeys the simple rules

$$\mathbf{a}|(\mathbf{b}\rfloor\boldsymbol{\Phi}) = (\mathbf{a}\wedge\mathbf{b})|\boldsymbol{\Phi}, \tag{1.40}$$

$$(\mathbf{\Phi}\lfloor\mathbf{b})|\mathbf{a} = \mathbf{\Phi}|(\mathbf{b}\wedge\mathbf{a}). \tag{1.41}$$

Contraction of a bivector $\mathbf{A}$ by a one-form $\boldsymbol{\alpha}$ can be defined similarly. Applied to (1.33), with slightly changed symbols, yields

$$\begin{aligned}(\mathbf{d}\wedge\mathbf{a})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) - ((\mathbf{d}|\boldsymbol{\beta})(\mathbf{a}|\boldsymbol{\gamma}) - (\mathbf{d}|\boldsymbol{\gamma})(\mathbf{a}|\boldsymbol{\beta})) =\\ \mathbf{d}|[\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})) - ((\boldsymbol{\beta}(\mathbf{a}|\boldsymbol{\gamma}) - \boldsymbol{\gamma}(\mathbf{a}|\boldsymbol{\beta})))] = 0,\end{aligned} \tag{1.42}$$

which is valid for any vector $\mathbf{d}$. Choosing $\mathbf{d} = \mathbf{e}_i$ for $i = 1$, …, 4, all components of the one-form expression in square brackets vanish. Thus, we immediately obtain the “bac-cab rule” valid for any vector $\mathbf{a}$ and one-forms $\boldsymbol{\beta}, \boldsymbol{\gamma}$,

$$\mathbf{a}\rfloor(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = \boldsymbol{\beta}(\mathbf{a}|\boldsymbol{\gamma}) - \boldsymbol{\gamma}(\mathbf{a}|\boldsymbol{\beta}) = (\boldsymbol{\gamma}\wedge\boldsymbol{\beta})\lfloor\mathbf{a}. \tag{1.43}$$

Equation (1.43) corresponds to the well-known bac-cab rule of 3D Gibbsian vectors, Appendix B,

$$\mathbf{a}_g\times(\mathbf{b}_g\times\mathbf{c}_g) = \mathbf{b}_g(\mathbf{a}_g\cdot\mathbf{c}_g) - \mathbf{c}_g(\mathbf{a}_g\cdot\mathbf{b}_g) = (\mathbf{c}_g\times\mathbf{b}_g)\times\mathbf{a}_g, \tag{1.44}$$

which helps in memorizing the 4D rule (1.43).

Useful contraction rules for basis vectors and one-forms can be obtained as special cases of (1.43) as

$$\mathbf{e}_j\rfloor(\boldsymbol{\varepsilon}_i\wedge\boldsymbol{\varepsilon}_j) = \mathbf{e}_j\rfloor\boldsymbol{\varepsilon}_{ij} = \boldsymbol{\varepsilon}_i, \tag{1.45}$$

$$(\mathbf{e}_i\wedge\mathbf{e}_j)\lfloor\boldsymbol{\varepsilon}_i = \mathbf{e}_{ij}\lfloor\boldsymbol{\varepsilon}_i = \mathbf{e}_j. \tag{1.46}$$

They can be easily memorized as a way of canceling basis vectors and one-forms with the same index from the contraction operation.

1.2.5 Decomposition of Vectors and One-Forms

Two vectors $\mathbf{a}, \mathbf{b} \neq 0$ are called parallel if they satisfy the relation

$$\mathbf{a} \wedge \mathbf{b} = 0. \tag{1.47}$$

Applying the bac-cab rule (1.43) for parallel vectors **a**, **b**,

$$\boldsymbol{\alpha}\rfloor(\mathbf{a} \wedge \mathbf{b}) = \mathbf{a}(\boldsymbol{\alpha}|\mathbf{b}) - \mathbf{b}(\boldsymbol{\alpha}|\mathbf{a}) = 0, \tag{1.48}$$

implies that parallel vectors are linearly dependent, that is, one is a multiple of the other one. Assuming $\mathbf{a} \wedge \mathbf{b} \neq 0$ and $\boldsymbol{\alpha}|\mathbf{a} \neq 0$, we can write the following decomposition for a given vector **b**:

$$\mathbf{b} = \mathbf{b}_{\|} + \mathbf{b}_{\perp}, \tag{1.49}$$

$$\mathbf{b}_{\|} = \frac{\boldsymbol{\alpha}|\mathbf{b}}{\boldsymbol{\alpha}|\mathbf{a}}\mathbf{a}, \qquad \mathbf{b}_{\perp} = \frac{\boldsymbol{\alpha}\rfloor(\mathbf{b} \wedge \mathbf{a})}{\boldsymbol{\alpha}|\mathbf{a}}. \tag{1.50}$$

Here, $\mathbf{b}_{\|}$ can be interpreted as the component parallel to a given vector **a**, while $\mathbf{b}_{\perp}$ can be called as the component perpendicular to a given one-form $\boldsymbol{\alpha}$, because it satisfies

$$\boldsymbol{\alpha}|\mathbf{b}_{\perp} = \boldsymbol{\alpha}|(\boldsymbol{\alpha}\rfloor(\mathbf{a} \wedge \mathbf{b})) = (\boldsymbol{\alpha} \wedge \boldsymbol{\alpha})|(\mathbf{a} \wedge \mathbf{b}) = 0. \tag{1.51}$$

Similarly, we can decompose a one-form $\boldsymbol{\beta}$ as

$$\boldsymbol{\beta} = \boldsymbol{\beta}_{\|} + \boldsymbol{\beta}_{\perp}, \tag{1.52}$$

$$\boldsymbol{\beta}_{\|} = \frac{\mathbf{a}|\boldsymbol{\beta}}{\mathbf{a}|\boldsymbol{\alpha}}\boldsymbol{\alpha}, \qquad \boldsymbol{\beta}_{\perp} = \frac{\mathbf{a}\rfloor(\boldsymbol{\alpha} \wedge \boldsymbol{\beta})}{\mathbf{a}|\boldsymbol{\alpha}}, \tag{1.53}$$

in terms of a given one-form $\boldsymbol{\alpha}$ and a given vector **a** satisfying $\boldsymbol{\alpha}|\mathbf{a} \neq 0$.

1.3 MULTIVECTORS AND MULTIFORMS

Higher-order multivectors and multiforms are produced through wedge multiplication. The wedge product is associative so that we have

$$\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c})=(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c}=\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}, \tag{1.54}$$

and the brackets can be omitted. Thus, trivectors and three-forms are obtained as

$$\mathbf{k}=\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3+\mathbf{b}_1\wedge\mathbf{b}_2\wedge\mathbf{b}_3+\cdots\in\mathbb{E}_3, \tag{1.55}$$

$$\boldsymbol{\pi}=\boldsymbol{\alpha}_1\wedge\boldsymbol{\alpha}_2\wedge\boldsymbol{\alpha}_3+\boldsymbol{\beta}_1\wedge\boldsymbol{\beta}_2\wedge\boldsymbol{\beta}_3+\cdots\in\mathbb{F}_3. \tag{1.56}$$

They will be denoted by lowercase Latin and Greek characters taken, if possible, from the end of the alphabets. Quadrivectors and four-forms can be constructed as

$$\mathbf{q}_N=\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3\wedge\mathbf{a}_4+\mathbf{b}_1\wedge\mathbf{b}_2\wedge\mathbf{b}_3\wedge\mathbf{b}_4+\cdots\in\mathbb{E}_4, \tag{1.57}$$

$$\boldsymbol{\kappa}_N=\boldsymbol{\alpha}_1\wedge\boldsymbol{\alpha}_2\wedge\boldsymbol{\alpha}_3\wedge\boldsymbol{\alpha}_4+\boldsymbol{\beta}_1\wedge\boldsymbol{\beta}_2\wedge\boldsymbol{\beta}_3\wedge\boldsymbol{\beta}_4+\cdots\in\mathbb{F}_4. \tag{1.58}$$

The subscript N denoting the quadri-index $N = 1234$ is used to mark a quadrivector or a four-form.

Because the space of vectors is 4D, there are no multivectors of higher order than four. In fact, because any vector $\mathbf{a}_5$ can be expressed as a linear combination of a basis $\mathbf{a}_1 \ldots \mathbf{a}_4$ satisfying $\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3\wedge\mathbf{a}_4\neq 0$, as will be shown below, we have $\mathbf{a}_1\wedge\mathbf{a}_2\wedge\mathbf{a}_3\wedge\mathbf{a}_4\wedge\mathbf{a}_5 = 0$. The spaces of trivectors and three-forms are 4D and, those of quadrivectors and four-forms, one dimensional.

1.3.1 Basis of Multivectors

The vector basis $\{\mathbf{e}_i\}$ induces the trivector basis

$$\mathbf{e}_{123}=\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_3,\quad \mathbf{e}_{234}=\mathbf{e}_{23}\wedge\mathbf{e}_4=\mathbf{e}_2\wedge\mathbf{e}_3\wedge\mathbf{e}_4,$$

$$\mathbf{e}_{314}=\mathbf{e}_{31}\wedge\mathbf{e}_4=\mathbf{e}_3\wedge\mathbf{e}_1\wedge\mathbf{e}_4,\quad \mathbf{e}_{124}=\mathbf{e}_{12}\wedge\mathbf{e}_4=\mathbf{e}_1\wedge\mathbf{e}_2\wedge\mathbf{e}_4, \tag{1.59}$$

whence the space of trivectors is 4D. There is only a single basis quadrivector denoted by

$$\mathbf{e}_N = \mathbf{e}_{1234} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4, \tag{1.60}$$

based upon the vector basis. Similar definitions apply for the basis three-forms $\boldsymbol{\varepsilon}_{ijk} = \boldsymbol{\varepsilon}_i \wedge \boldsymbol{\varepsilon}_j \wedge \boldsymbol{\varepsilon}_k$ and the basis four-form $\boldsymbol{\varepsilon}_N = \boldsymbol{\varepsilon}_{1234} = \boldsymbol{\varepsilon}_1 \wedge \boldsymbol{\varepsilon}_2 \wedge \boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\varepsilon}_4$.

Recalling the definition of the complementary bi-index (1.26), (1.27), and applying the antisymmetry of the wedge product we obtain

$$\mathbf{e}_{23} \wedge \mathbf{e}_{K(23)} = \mathbf{e}_{23} \wedge \mathbf{e}_{14} = \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_4 = \mathbf{e}_{1234} = \mathbf{e}_N. \tag{1.61}$$

More generally, we can write

$$\mathbf{e}_J \wedge \mathbf{e}_{K(J)} = \mathbf{e}_{K(J)} \wedge \mathbf{e}_J = \mathbf{e}_N. \tag{1.62}$$

Defining the bi-index Kronecker delta by

$$\delta_{I,J} = 0, \;\; I \neq J, \qquad \delta_{I,J} = 1, \;\; I = J, \tag{1.63}$$

we can write even more generally,

$$\mathbf{e}_I \wedge \mathbf{e}_J = \delta_{I,K(J)} \mathbf{e}_N. \tag{1.64}$$

This means that, unless I equals $K(J)$, that is, J equals $K(I)$, the wedge product yields zero.

1.3.2 Bar Product of Multivectors and Multiforms

Extending the bar product to multivectors and multiforms, we can define the orthogonality relations for the reciprocal basis multivector and multiforms as

$$\mathbf{e}_{ijk} | \boldsymbol{\varepsilon}_{rst} = \delta_{i,r}\delta_{j,s}\delta_{k,t}, \tag{1.65}$$

$$\mathbf{e}_{ijk\ell} | \boldsymbol{\varepsilon}_{rstu} = \delta_{i,r}\delta_{j,s}\delta_{k,t}\delta_{\ell,u}, \tag{1.66}$$

when the indices are ordered. From the antisymmetry of the wedge product we obtain the expansion rules

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})|(\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma}) = \det \begin{pmatrix} \mathbf{a}|\boldsymbol{\alpha} & \mathbf{a}|\boldsymbol{\beta} & \mathbf{a}|\boldsymbol{\gamma} \\ \mathbf{b}|\boldsymbol{\alpha} & \mathbf{b}|\boldsymbol{\beta} & \mathbf{b}|\boldsymbol{\gamma} \\ \mathbf{c}|\boldsymbol{\alpha} & \mathbf{c}|\boldsymbol{\beta} & \mathbf{c}|\boldsymbol{\gamma} \end{pmatrix}, \tag{1.67}$$

$$(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d})|(\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma} \wedge \boldsymbol{\delta}) = \det \begin{pmatrix} \mathbf{a}|\boldsymbol{\alpha} & \mathbf{a}|\boldsymbol{\beta} & \mathbf{a}|\boldsymbol{\gamma} & \mathbf{a}|\boldsymbol{\delta} \\ \mathbf{b}|\boldsymbol{\alpha} & \mathbf{b}|\boldsymbol{\beta} & \mathbf{b}|\boldsymbol{\gamma} & \mathbf{b}|\boldsymbol{\delta} \\ \mathbf{c}|\boldsymbol{\alpha} & \mathbf{c}|\boldsymbol{\beta} & \mathbf{c}|\boldsymbol{\gamma} & \mathbf{c}|\boldsymbol{\delta} \\ \mathbf{d}|\boldsymbol{\alpha} & \mathbf{d}|\boldsymbol{\beta} & \mathbf{d}|\boldsymbol{\gamma} & \mathbf{d}|\boldsymbol{\delta} \end{pmatrix}. \tag{1.68}$$

All quadrivectors are multiples of a given basis quadrivector $\mathbf{e}_N$ and all four-forms are multiples of a given basis four-form $\boldsymbol{\varepsilon}_N$:

$$\mathbf{q}_N = q\mathbf{e}_N, \qquad \boldsymbol{\kappa}_N = \kappa\boldsymbol{\varepsilon}_N, \tag{1.69}$$

with

$$q = \mathbf{q}_N|\boldsymbol{\varepsilon}_N, \qquad \kappa = \mathbf{e}_N|\boldsymbol{\kappa}_N. \tag{1.70}$$

Applying the expansion rule for the determinant in (1.67), we can expand the bar product

$$\begin{aligned}(\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c})|(\boldsymbol{\alpha} \wedge \boldsymbol{\beta} \wedge \boldsymbol{\gamma}) &= \mathbf{a}|\boldsymbol{\alpha} \det \begin{pmatrix} \mathbf{b}|\boldsymbol{\beta} & \mathbf{b}|\boldsymbol{\gamma} \\ \mathbf{c}|\boldsymbol{\beta} & \mathbf{c}|\boldsymbol{\gamma} \end{pmatrix} - \mathbf{a}|\boldsymbol{\beta} \det \begin{pmatrix} \mathbf{b}|\boldsymbol{\alpha} & \mathbf{b}|\boldsymbol{\gamma} \\ \mathbf{c}|\boldsymbol{\alpha} & \mathbf{c}|\boldsymbol{\gamma} \end{pmatrix} \\ &\quad + \mathbf{a}|\boldsymbol{\gamma} \det \begin{pmatrix} \mathbf{b}|\boldsymbol{\alpha} & \mathbf{b}|\boldsymbol{\beta} \\ \mathbf{c}|\boldsymbol{\alpha} & \mathbf{c}|\boldsymbol{\beta} \end{pmatrix},\end{aligned} \tag{1.71}$$

whence from (1.33) we obtain the rule

$$(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = (\mathbf{a}|\boldsymbol{\alpha})(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) + (\mathbf{a}|\boldsymbol{\beta})(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}) + (\mathbf{a}|\boldsymbol{\gamma})(\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})]. \tag{1.72}$$

1.3.3 Contraction of Trivectors and Three-Forms

Defining the contraction of a three-form by a bivector as arising from

$$\begin{aligned}(\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) &= \mathbf{a}|((\mathbf{b}\wedge\mathbf{c})\rfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}))\\ &= ((\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\lfloor(\mathbf{b}\wedge\mathbf{c}))|\mathbf{a},\end{aligned} \tag{1.73}$$

from (1.72) we obtain the expansion rule

$$\begin{aligned}&(\mathbf{b}\wedge\mathbf{c})\rfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) = (\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\lfloor(\mathbf{b}\wedge\mathbf{c})\\ &\quad = ((\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}))\boldsymbol{\alpha} + ((\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}))\boldsymbol{\beta} + ((\mathbf{b}\wedge\mathbf{c})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}))\boldsymbol{\gamma}.\end{aligned} \tag{1.74}$$

From this it follows that if the three one-forms satisfy $\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}=0$, they must be linearly dependent.

Rewriting (1.72) in the form

$$\begin{aligned}(\mathbf{b}\wedge\mathbf{c})|(\mathbf{a}\rfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma})) &= ((\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\lfloor\mathbf{a})|(\mathbf{b}\wedge\mathbf{c})\\ &= (\mathbf{b}\wedge\mathbf{c})|[(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\mathbf{a}|\boldsymbol{\alpha} + (\boldsymbol{\gamma}\wedge\boldsymbol{\alpha})\mathbf{a}|\boldsymbol{\beta} + (\boldsymbol{\alpha}\wedge\boldsymbol{\beta})\mathbf{a}|\boldsymbol{\gamma}],\end{aligned} \tag{1.75}$$

which remains valid when $\mathbf{b}\wedge\mathbf{c}$ is replaced by any bivector $\mathbf{A}$ because of linearity, we obtain another contraction rule for contracting a three-form by a vector,

$$\begin{aligned}\mathbf{a}\rfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) &= (\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\lfloor\mathbf{a}\\ &= (\mathbf{a}|\boldsymbol{\alpha})(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) + (\mathbf{a}|\boldsymbol{\beta})(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}) + (\mathbf{a}|\boldsymbol{\gamma})(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}).\end{aligned} \tag{1.76}$$

If $\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}=0$, the three two-forms $\boldsymbol{\beta}\wedge\boldsymbol{\gamma}$, $\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}$, and $\boldsymbol{\alpha}\wedge\boldsymbol{\beta}$ must be linearly dependent, which also follows from the linear dependence of the three one-forms.

The contraction rules (1.74) and (1.76) are similar to the bac-cab rule (1.43) and they can be easily memorized because of the cyclic symmetry. Other similar forms are obtained by replacing vectors by one-forms and one-forms by vectors in (1.74) and (1.76). Commutation rules for the contraction product can be summarized as

$$\boldsymbol{\alpha}\rfloor\mathbf{k} = \mathbf{k}\lfloor\boldsymbol{\alpha}, \qquad \boldsymbol{\Phi}\rfloor\mathbf{k} = \mathbf{k}\lfloor\boldsymbol{\Phi}, \tag{1.77}$$

$$\mathbf{a}\rfloor\boldsymbol{\pi} = \boldsymbol{\pi}\lfloor\mathbf{a}, \qquad \mathbf{A}\rfloor\boldsymbol{\pi} = \boldsymbol{\pi}\lfloor\mathbf{A}. \tag{1.78}$$

Here, $\mathbf{a}$ is a vector, $\mathbf{A}$ is a bivector and $\mathbf{k}$ is a trivector while $\boldsymbol{\alpha}$ is a one-form, $\boldsymbol{\Phi}$ is a two-form and $\boldsymbol{\pi}$ is a three-form. Useful rules for the contraction operations involving basis trivectors and three-forms can be formed as

$$\mathbf{e}_{jk}\rfloor\boldsymbol{\varepsilon}_{ijk} = \mathbf{e}_j\rfloor\boldsymbol{\varepsilon}_{ij} = \boldsymbol{\varepsilon}_i, \tag{1.79}$$

$$\mathbf{e}_k\rfloor\boldsymbol{\varepsilon}_{ijk} = \boldsymbol{\varepsilon}_{ij}, \tag{1.80}$$

showing how similar indices are canceled in contraction operations.

1.3.4 Contraction of Quadrivectors and Four-Forms

Following the same path of reasoning, starting from (1.68) we can expand the contraction of a four-form by a trivector

$$\begin{aligned}(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})\rfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta}) &= -(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta})\lfloor(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}) \\ &= -(\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma})\boldsymbol{\delta} + (\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta})\boldsymbol{\alpha} \\ &\quad + (\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}\wedge\boldsymbol{\delta})\boldsymbol{\beta} + (\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\delta})\boldsymbol{\gamma},\end{aligned} \tag{1.81}$$

the contraction of a four-form by a bivector,

$$(\mathbf{c}\wedge\mathbf{d})\rfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta}) = (\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta})\lfloor(\mathbf{c}\wedge\mathbf{d})$$
$$= (\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\beta}\wedge\boldsymbol{\gamma})(\boldsymbol{\alpha}\wedge\boldsymbol{\delta}) + (\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\beta}\wedge\boldsymbol{\alpha})(\boldsymbol{\gamma}\wedge\boldsymbol{\delta})$$
$$+ (\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha})(\boldsymbol{\beta}\wedge\boldsymbol{\delta}) + (\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\gamma}\wedge\boldsymbol{\delta})(\boldsymbol{\alpha}\wedge\boldsymbol{\beta})$$
$$+ (\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\alpha}\wedge\boldsymbol{\delta})(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) + (\mathbf{c}\wedge\mathbf{d})|(\boldsymbol{\beta}\wedge\boldsymbol{\delta})(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}), \quad (1.82)$$

and the contraction of a four-form by a vector,

$$\mathbf{d}\rfloor(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta}) = -(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta})\lfloor\mathbf{d}$$
$$= (\mathbf{d}|\boldsymbol{\alpha})(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta}) + (\mathbf{d}|\boldsymbol{\beta})(\boldsymbol{\gamma}\wedge\boldsymbol{\alpha}\wedge\boldsymbol{\delta})$$
$$+ (\mathbf{d}|\boldsymbol{\gamma})(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\delta}) - (\mathbf{d}|\boldsymbol{\delta})(\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}). \quad (1.83)$$

The above expressions appear invariant to cyclic permutation of the one-forms $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, which may help in memorizing and checking the formulas.

If $\boldsymbol{\alpha}\wedge\boldsymbol{\beta}\wedge\boldsymbol{\gamma}\wedge\boldsymbol{\delta} = 0$, from (1.81) it follows that the four one-forms are linearly dependent, from (1.82) it further follows that also the six two-forms are linearly dependent and from (1.83) it follows that the four three-forms are linearly dependent. The contraction of a four-form $\boldsymbol{\kappa}_N$ or quadrivector $\mathbf{q}_N$ obeys the commutation rules

$$\mathbf{a}\rfloor\boldsymbol{\kappa}_N = -\boldsymbol{\kappa}_N\lfloor\mathbf{a}, \qquad \boldsymbol{\alpha}\rfloor\mathbf{q}_N = -\mathbf{q}_N\lfloor\boldsymbol{\alpha}, \quad (1.84)$$

$$\mathbf{A}\rfloor\boldsymbol{\kappa}_N = \boldsymbol{\kappa}_N\lfloor\mathbf{A}, \qquad \boldsymbol{\Phi}\rfloor\mathbf{q}_N = \mathbf{q}_N\lfloor\boldsymbol{\Phi}, \quad (1.85)$$

$$\mathbf{k}\rfloor\boldsymbol{\kappa}_N = -\boldsymbol{\kappa}_N\lfloor\mathbf{k}, \qquad \boldsymbol{\pi}\rfloor\mathbf{q}_N = -\mathbf{q}_N\lfloor\boldsymbol{\pi}, \quad (1.86)$$

Equations (1.81)–(1.83) imply the following contraction rules for the basis multivectors and multiforms:

$$\mathbf{e}_\ell\rfloor\boldsymbol{\varepsilon}_{ijk\ell} = \boldsymbol{\varepsilon}_{ijk}, \quad \mathbf{e}_{ijk\ell}\lfloor\boldsymbol{\varepsilon}_i = \mathbf{e}_{jk\ell} \quad (1.87)$$

$$\mathbf{e}_{k\ell}\rfloor\boldsymbol{\varepsilon}_{ijk\ell} = \boldsymbol{\varepsilon}_{ij}, \quad \mathbf{e}_{ijk\ell}\lfloor\boldsymbol{\varepsilon}_{ij} = \mathbf{e}_{k\ell} \quad (1.88)$$

$$\mathbf{e}_{jk\ell}\rfloor\boldsymbol{\varepsilon}_{ijk\ell} = \boldsymbol{\varepsilon}_i, \quad \mathbf{e}_{ijk\ell}\lfloor\boldsymbol{\varepsilon}_{ijk} = \mathbf{e}_\ell, \tag{1.89}$$

which can be applied for canceling indices in expressions involving contraction of basis multivectors and multiforms. From (1.81) to (1.83) we can see that contraction of a four-form can be applied to transform vectors to three-forms, bivectors to two-forms and trivectors to one-forms and conversely. The converse cases can be obtained by applying the rules

$$\mathbf{e}_N\lfloor(\boldsymbol{\varepsilon}_N\lfloor\mathbf{a}) = (\mathbf{a}\rfloor\boldsymbol{\varepsilon}_N)\rfloor\mathbf{e}_N = -\mathbf{a}, \tag{1.90}$$

$$\mathbf{e}_N\lfloor(\boldsymbol{\varepsilon}_N\lfloor\mathbf{A}) = (\mathbf{A}\rfloor\boldsymbol{\varepsilon}_N)\rfloor\mathbf{e}_N = \mathbf{A}, \tag{1.91}$$

$$\mathbf{e}_N\lfloor(\boldsymbol{\varepsilon}_N\lfloor\mathbf{k}) = (\mathbf{k}\rfloor\boldsymbol{\varepsilon}_N)\rfloor\mathbf{e}_N = -\mathbf{k}. \tag{1.92}$$

1.3.5 Construction of Reciprocal Basis

Given a set of basis vectors $\mathbf{a}_i$, $i = 1, \ldots, 4$, and a four-form $\boldsymbol{\kappa}_N$, we can form the reciprocal one-form basis as

$$\boldsymbol{\alpha}_i = \frac{\mathbf{a}_{K(i)}\rfloor\boldsymbol{\kappa}_N}{\mathbf{a}_N|\boldsymbol{\kappa}_N}, \tag{1.93}$$

where the $\mathbf{a}_{K(i)}$ are four three-forms defined by

$$\mathbf{a}_{K(1)} = \mathbf{a}_{234}, \quad \mathbf{a}_{K(2)} = \mathbf{a}_{314}, \quad \mathbf{a}_{K(3)} = \mathbf{a}_{124}, \quad \mathbf{a}_{K(4)} = -\mathbf{a}_{123}. \tag{1.94}$$

satisfying

$$\mathbf{a}_j \wedge \mathbf{a}_{K(i)} = -\mathbf{a}_{K(i)} \wedge \mathbf{a}_j = \mathbf{a}_N\delta_{i,j}. \tag{1.95}$$

The rule (1.93) is easily checked:

$$\mathbf{a}_j|\boldsymbol{\alpha}_i = \frac{(\mathbf{a}_j \wedge \mathbf{a}_{K(i)})|\boldsymbol{\kappa}_N}{\mathbf{a}_N|\boldsymbol{\kappa}_N} = \delta_{i,j}. \tag{1.96}$$