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B. Malcolm Brown
Jan Lang
Ian G. Wood
Editors

Spectral Theory, Function Spaces and Inequalities

New Techniques and Recent Trends

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Contents

Preface

This is a a collection of contributed papers by David and Des's friends and colleagues and is issued to mark their respective 80^{th} and 70^{th} birthday. For the past forty years they have made fundamental contributons in the area of differential equations, operator theory and function space theory, and it is fitting that these contributions reflect that.

Our thanks must also go to Dr Thomas Hempfling, Executive Editor, Mathematics Birkhäuser and Ms Sylvia Lotrovsky for the help and assistance given to us during the preparation of this volume. Finally we thank all the authors who contributed papers to this special edition to mark David and Des's birthdays.

B.M. Brown
J. Lang
I.G. Wood

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David Edmunds' Mathematical Work

B.M. Brown, J. Lang and I.G. Wood

David Edmunds has influenced and made major contributions to numerous branches of mathematics. These include spectral theory, functional analysis, approximation theory, the theory of function spaces, operator theory, ordinary and partial differential equations. The breadth of his impact is demonstrated by his publication record, which consists of 5 books and more than 190 research papers, and by his winning the LMS Pólya prize in 1996 and the Bolzano Medal of the Czech Academy of Sciences, in 1998. He was awarded the Ph.D. degree by the University of Wales in 1955, having been supervised by R.M. Morris. After some years working for EMI Electronics on guided missiles, he held positions of Lecturer and then Senior Lecturer at the University of Wales, Cardiff, leaving in 1966 to take up a Readership at the University of Sussex. He was awarded a Personal Chair there in 1970 and is still affiliated to Sussex as well as additionally being appointed Honorary Professor, School of Mathematics, Cardiff University, 2004.

In his early works he focused on problems of fluid dynamics, including moving aerofoils, magneto-hydrodynamics and the nature of solutions of the Navier-Stokes equations (studying questions of stability, backward uniqueness, asymptotic behaviour and removable singularities). Then his interests shifted towards the study of more general non-linear problems, elliptic equations and inequalities, and functional analysis.

His first joint paper with W.D. Evans appeared in 1973. In this, by deriving new weighted embeddings on unbounded domains in L^p spaces, results were obtained for the Dirichlet problem concerning elliptic equations. This work began their long and fruitful collaboration and established their common interest in the properties of function spaces, embedding theorems, integral operators and spectral theory. These matters were also the main topics of their two joint books, the well-known 'Spectral theory and differential operators' (OUP) and the more recent 'Hardy operators, function spaces and embeddings' (Springer).

Edmunds' work on the properties of Besov and Lizorkin spaces has often been motivated by his interest in the nature of eigenvalues and eigenvectors of operators acting on non-Hilbert spaces. Many of his papers on this topic are concerned with the qualitative and quantitative properties of embeddings of function spaces, such as the behaviour of their entropy and s-numbers. These results, including those obtained with Hans Triebel, formed the basis of their joint book, ‘Function spaces, entropy numbers, differential operators’ (CUP).

Interest in functional analysis led him to study interpolation theory. His recent results with Yuri Netrusov settled a long-standing conjecture concerning the behaviour of entropy numbers under real interpolation. In the theory of integral operators he has concentrated on maps of Hardy or Volterra type, acting on function spaces with and without weights; many of his results in this area are presented in his book ‘Bounded and compact integral operators’, with V. Kokilashvili and A. Meskhi. His very recent book ‘Eigenvalues, embeddings and generalised trigonometric functions’, with J. Lang, has as its basis their work on the properties of s-numbers of Hardy-type operators, which involves the study of eigenfunctions of integral and differential operators together with certain generalisations of the trigonometric functions that are of importance in non-linear analysis. Another of his interests is the currently popular theory of $L^{p(x)}$spaces, the so-called variable exponent spaces, fundamental work on which was carried out by J. Rákosník, and together with whom results were obtained that have proved stimulating for many analysts.

His standing attracted a number of mathematicians, such as D. Vassiliev and A. Sobolev, who started their professional careers in the UK at Sussex. He was also active in making contacts with colleagues from other countries. He has supervised 18 Ph.D. students, including J.M. Ball (Oxford), J.R.L. Webb (Glasgow) and V. Mustonen (Oulu); according to the Mathematics Genealogy project he has 73 descendants.

David Edmunds’ contribution to mathematics is not only long, but wide and deep. His superb professional work and his warm personality have deeply influenced a large part of the mathematical community.

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Operator Theory:
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Desmond Evans' Mathematical Work

B. M. Brown, J. Lang and I. G. Wood

Desmond Evans has made major influential contributions to numerous branches of mathematical analysis which include the spectral theory of both ordinary and partial differential equations, mathematical physics and functional analysis. The breadth and depth of his achievements are recorded in his 142 published papers and 3 books. Following a B.Sc (Wales) from University College, Swansea in 1961 he went up to Jesus College, Oxford to work for a D.Phil. under the guidance of E. C. Titchmarsh, one of the leading analysts of the day. After the sudden death of Titchmarsh in 1963, Des completed his studies under the direction of another leading analyst, J.B.McLeod. The degree was awarded in 1965. In 1964 he was appointed to a lectureship in Pure Mathematics at the then University College of South Wales and Monmouthshire, which later became Cardiff University, and has remained at Cardiff all his working life, progressing through the grades of senior lecturer and reader before being awarded a Personal Chair by the University of Wales in 1977. During this time at Cardiff he has supervised 13 Ph.D. students.

Following some early work on the Dirac system he worked on the *limit-point, limit-circle* classification problem for ordinary differential equations, inequalities related to differential and difference equations (in particular the HELP inequality and its later variants), and on spectral problems associated with non-selfadjoint differential systems.

His work on partial differential equations has often been motivated by physical questions, a significant portion having been concerned with problems arising in the study of non-relativistic quantum mechanics. This research contains work on the spectrum of relativistic one-electron atoms and on the zero modes of Pauli and Weyl-Dirac operators. His many papers in this and related areas cover an impressive range of topics. These include the spectral analysis of N-body operators for atoms and molecules; quantum graphs; Hardy and Rellich inequalities with magnetic potentials; Schrödinger operators and biharmonic operators with magnetic fields.

He has been active also in functional analysis and operator theory, especially in areas concerning the properties of Hardy-type operators acting between function spaces, estimates and asymptotic results for their approximation numbers and related inequalities. Much of this work was motivated by his study of the properties of embedding maps between Sobolev spaces defined on irregular (including fractal) domains, in which he and Desmond Harris introduced the notion of a generalised ridged domain and developed techniques for reducing problems to analogous ones on associated trees.

His two research monographs with David Edmunds, "Spectral Theory and Differential Equations" and "Hardy Operators, Function Spaces and Embeddings" have become standard texts in his main areas of activity. Recently, with Alex Balinsky, he has published "Spectral analysis of relativistic operators", which includes, in particular, an account of their numerous contributions to problems concerning the stability of matter.

As well as his mathematical contributions, Des has been active in various administrative roles both within and outside Cardiff University. In particular, he has been Head of School at Cardiff on several occasions, and served the London Mathematical Society over many years as Editorial Advisor, Editor of the Proceedings and Council member. Recently he has played an important part in establishing the Wales Institute of Mathematical and Computational Sciences. In recognition of these achievements he was elected a fellow of the Learned Society of Wales in 2011.

(This is an expanded version of the Laudatum in JCAM V208 1 November 2007)

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Generalised Meissner Equations with an Eigenvalue-inducing Interface

B.M. Brown and M.S.P. Eastham

To David and Des

Abstract. An interface situation is considered, where a periodic differential equation is given on one side $x > 0$ of the interface and a general Sturm–Liouville equation is given on a finite interval $(-X, 0)$ on the other side of the interface. A boundary condition is imposed at $-X$. The emphasis is on a periodic discontinuous weight function, which has the effect of widening the spectral gaps (instability intervals). It is shown that the interface can induce eigenvalues in all the gaps beyond some point. The dependence on X of the number of eigenvalues in each gap is noted. The general theory is supported by step-function examples.

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Keywords. Periodic, interface, Meissner.

1. Introduction

A basic property of the periodic differential equation

$$y''(x) + \{\lambda w(x) - q(x)\}y(x) = 0 \tag{1.1}$$

on an unbounded x-interval I is the existence of stability and instability λ-intervals on the real λ-axis. Here the weight function $w(x)$ and the potential function $q(x)$ are real-valued and $L_{loc}(I)$ with $w(x) > 0$, and they have a common period a; I can be either $(-\infty, \infty)$ or a semi-infinite interval, let us say $[0, \infty)$. The stability intervals are specified in terms of the eigenvalues λ_n and μ_n $(n \geq 0)$ of (1.1) arising respectively from the periodic boundary conditions

$$y(0) = y(a), \quad y'(0) = y'(a) \tag{1.2}$$

and the semi-periodic boundary conditions

$$y(0) = -y(a), \quad y'(0) = -y'(a). \tag{1.3}$$

These eigenvalues have the ordering

$$\lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \lambda_3 \cdots . \tag{1.4}$$

Then the stability intervals are the open intervals

$$(\lambda_{2m}, \mu_{2m}), \quad (\mu_{2m+1}, \lambda_{2m+1}) \quad (m \ge 0) \tag{1.5}$$

which, together possibly with their end-points, comprise the values of λ for which all solutions of (1.1) are bounded in I. In terms of the spectral theory of (1.1) in the Hilbert space $L^2(I)$, the intervals (1.5) exhibit the band structure of the associated spectrum in the sense that the essential spectrum is formed by the closures of (1.5). We refer to the standard books [4, chapters 8–9], [6], [17, chapter 13] and [22, chapter 21] for all this basic theory of (1.1). We also refer to [8] for (1.4) in a wider context of more general boundary conditions.

Moving on to the particular topic considered in this paper, we note first that, when I is $(-\infty, \infty)$, (1.1) has no eigenvalues in the spectral gaps between the intervals (1.5) but, when I is $[0, \infty)$ and the usual type of boundary condition is imposed at $x = 0$, any given λ in a gap can be an eigenvalue arising from an associated boundary condition [4, p. 257], [9, Theorem 2.5.3]. If however the periodicity of (1.1) is compromised by some perturbation, there is the possibility that eigenvalues can appear in the gaps as a result of the perturbation, that is, new spectral points may arise.

Historically, the first type of perturbation to be considered in this context is the addition to $q(x)$ of a non-periodic function $p(x)$ which is small in some sense when $|x|$ is large. On the one hand, we have the result that if $p(x) = O(|x|)^{-2})$ $(|x| \to \infty)$, then only a finite number of eigenvalues can appear in sufficiently distant spectral gaps [19, Corollary 3] (see also [18]). On the other hand, if a fixed spectral gap is considered and $p(x)$ contains a coupling parameter c, the number of eigenvalues in the gap can become large for large c, and the asymptotic distribution is investigated in [3], [20], [21] (see also [1]).

Another type of perturbation is based on the idea of introducing an interface where (1.1) holds on one side of the interface (say for $x > 0$) and a different periodic equation holds on the other side $x < 0$. In [13] a dislocation situation is considered in which $w(x) = 1$ and the potential for $x < 0$ is $q(x+t)$ where $t \in \mathbf{R}$ is the dislocation parameter. It is shown that a spectral point (an eigenvalue or a resonance) $\lambda(t)$ is produced in each spectral gap, and its behaviour in terms of t is discussed [13, pp. 474, 480]. A biperiodic situation is introduced in [14] where q has a different period for $x < 0$, and here it is shown that up to two spectral points appear in each spectral gap. Recently, similar interface problems have been considered in [5] again with $w(x) = 1$ and using the method of C^1 gluing across the interface. In [5, section 3] explicit conditions on q are derived which guarantee the appearance of up to two eigenvalues in the first two spectral gaps $(-\infty, \lambda_0)$ and (μ_0, μ_1).

In this paper we consider a different interface situation where a new feature is that arbitrarily many eigenvalues can occur in the spectral gaps (μ_{2m}, μ_{2m+1}) and $(\lambda_{2m+1}, \lambda_{2m+2})$ $(m \ge 0)$. Our main focus is on the weight function $w(x)$ and, in (1.1), we take $w(x)$ to be non-constant for $x > 0$ and, in particular, $w(x)$ has discontinuities. This has the effect of widening the gaps [6, section 4.5], [7], [16]. On the other side ($x < 0$) of the interface the differential equation is

$$y''(x) + \{\lambda - q_1(x)\}y(x) = 0 \tag{1.6}$$

on a finite interval $[-X, 0)$ with an arbitrary $q_1(x)$ and a non-trivial boundary condition

$$c_1 y(-X) + c_2 y'(-X) = 0. \tag{1.7}$$

Our spectral setting is therefore on the x-interval $[-X, \infty)$ with (1.1) for $x > 0$, (1.6) for $x < 0$ and the boundary condition (1.7). A simple relative compactness argument shows that the essential spectrum of our interface problem retains the band structure noted above. In section 2, we recall the basic Floquet theory from (for example) [6] which we require, and formulate the eigenvalue equation arising from (1.7). Then in section 3 we formulate and prove a general theorem (Theorem 3.1) on the existence of interface-induced eigenvalues in sufficiently distant spectral gaps, subject to a condition concerning the length of the gaps. The dependence of the number of these eigenvalues on the value of X is noted. In sections 4 and 5, we consider the case where $w(x)$ has two discontinuities in its period, a two-valued step-function being an example, and we lead up to situations where the length condition in Theorem 3.1 is satisfied. We mention here that the eigenvalues λ_n and μ_n for step-function examples have been discussed in [6, section 2.2], [10, section 50], [12], and one contribution of our paper in sections 4 and 5 is to develop properties of these eigenvalues which are not confined to the step-function case. In section 6, we discuss briefly the case where $w(x)$ has just one discontinuity in its period, that is, $w(0) \neq w(a)$. Finally in section 7 we discuss in more detail some step-function examples where induced eigenvalues appear in all the spectral gaps (except $(-\infty, \lambda_0)$), not just the distant ones.

We conclude this introduction by mentioning that the adjective Meissner is applied to any periodic equation (1.1) in which w and q are step functions (cf. [2], [10], [12]). This follows the original equation of this kind formulated by Meissner [15] (concerning locomotive coupling rods). In our paper we are not confined to step-functions, but the discontinuities in $w(x)$ are essential.

2. Formulation of the eigenvalue problem

We begin with the solutions $\phi_1(x)$ and $\phi_2(x)$ of (1.1) which have the initial values

$$1, 0 \quad \text{and} \quad 0, 1 \tag{2.1}$$

respectively at $x = 0$, with the dependence on λ not indicated until necessary. Since we are dealing with λ in a spectral gap, the basic theory of [6, chapters 1 and 2] shows that there are also solutions $\psi_k(x)$ ($k = 1, 2$) of (1.1) such that

$$\psi_k(x + a) = \rho_k \psi_k(x) \quad (x \geq 0) \tag{2.2}$$

where the ρ_k are the two distinct and real solutions of the quadratic

$$\rho^2 - D\rho + 1 = 0$$

with

$$D = \phi_1(a) + \phi_2'(a). \tag{2.3}$$

Further [6, section 1.1], $\psi_k(x)$ can be written as either

$$\psi_k(x) = \phi_2(a)\phi_1(x) - \{\phi_1(a) - \rho_k\}\phi_2(x). \tag{2.4}$$

or, in case (2.4) is a trivial linear combination of $\phi_1(x)$ and $\phi_2(x)$,

$$\psi_k(x) = \{\phi_2'(a) - \rho_k\}\phi_1(x) - \phi_1'(a)\phi_2(x). \tag{2.5}$$

We shall generally keep to (2.4) and comment on the change to (2.5) as necessary. Since $\rho_1\rho_2 = 1$, we take it that

$$|\rho_1| < 1 \quad \text{and} \quad |\rho_2| > 1. \tag{2.6}$$

We are looking for a solution of (1.1) which is $L^2(0,\infty)$, and it follows from (2.2) and (2.6) that this solution must be

$$y(x) = \psi_1(x) \qquad (x \geq 0) \tag{2.7}$$

to within a constant multiple. We have now to continue this solution into $x < 0$ and substitute the result into the boundary condition (1.7) to complete the formulation of the eigenvalue problem. We continue to denote by $\phi_1(x)$ and $\phi_2(x)$ the solutions now of (1.6) in $[-X, 0)$ but still satisfying (2.1). We also note that, by (2.1) and (2.4), $\psi_1(x)$ has the initial values

$$\phi_2(a), \quad -\{\phi_1(a) - \rho_1\}$$

at $x = 0$. Hence, as a linear combination of $\phi_1(x)$ and $\phi_2(x)$, (2.7) is continued into $x < 0$ as

$$y(x) = \phi_2(a)\phi_1(x) - \{\phi_1(a) - \rho_1\}\phi_2(x) \quad (x < 0). \tag{2.8}$$

Then (1.7) gives the equation to determine the eigenvalues in the spectral gaps as

$$\begin{aligned} &\phi_2(a)\{c_1\phi_1(-X) + c_2\phi_1'(-X)\} \\ &\qquad - \{\phi_1(a) - \rho_1\}\{c_1\phi_2(-X) + c_2\phi_2'(-X)\} = 0. \end{aligned} \tag{2.9}$$

We shall examine (2.9) firstly for sufficiently distant gaps and then, in an example, for all gaps. To prepare for the former in the next section, we require a slightly more precise version of a familiar result for the Sturm-Liouville equation

$$y''(x) + \{\lambda - Q(x)\}y(x) = 0 \quad (0 \leq x \leq A) \tag{2.10}$$

with any Q in $L_{loc}[0, A]$. We consider $\lambda > 0$ and write $\nu = \sqrt{\lambda}$.

Lemma 2.1. *Let $y(x)$ satisfy (2.10) and let*

$$\nu \geq \int_0^A |Q(t)|dt. \tag{2.11}$$

Then

$$y(x) = \{y(0) + \frac{1}{\nu}E(x)\}\cos\nu x + \frac{1}{\nu}\{y'(0) + F(x)\}\sin\nu x, \tag{2.12}$$

$$y'(x) = -\nu\{y(0) + \frac{1}{\nu}E(x)\}\sin\nu x + \{y'(0) + F(x)\}\cos\nu x, \tag{2.13}$$

where

$$|E(x)|,\ |F(x)| \le (|y(0)| + \frac{1}{\nu}|y'(0)|)(e-1)\int_0^A |Q(t)|dt. \tag{2.14}$$

Proof. The integral formulation of (2.10) is

$$y(x) = y(0)\cos\nu x + \frac{1}{\nu}y'(0)\sin\nu x + \frac{1}{\nu}\int_0^x \sin\{\nu(x-t)\}Q(t)y(t)dt. \tag{2.15}$$

Hence

$$|y(x)| \le |y(0)| + \frac{1}{\nu}|y'(0)| + \frac{1}{\nu}\int_0^x |Q(t)||y(t)|dt,$$

and this Gronwall inequality gives

$$|y(x)| \le (|y(0)| + \frac{1}{\nu}|y'(0)|)\exp\left(\frac{1}{\nu}\int_0^x |Q(t)|dt\right). \tag{2.16}$$

We now write (2.15) in the form (2.12) with

$$E(x) = -\int_0^x (\sin\nu t)Q(t)y(t)dt$$

and similarly for $F(x)$ with $\cos\nu t$ in place of $-\sin\nu t$. Then, by (2.16),

$$\begin{aligned}|E(x)|,\ |F(x)| &\le (|y(0)| + \frac{1}{\nu}|y'(0)|)\int_0^x |Q(t)|\exp\big(\frac{1}{\nu}\int_0^t |Q(u)|du\big)dt \\ &= (|y(0)| + \frac{1}{\nu}|y'(0)|)\nu\{\exp\big(\frac{1}{\nu}\int_0^x |Q(t)|dt\big) - 1\}\end{aligned}$$

yielding (2.14) when (2.11) holds. This proves (2.12), and (2.13) follows similarly from differentiation of (2.15). □

We note that the lemma also holds for $A < 0$ if the integration range in (2.11) and (2.14) is replaced by $(A, 0)$.

3. Dirichlet and Neumann boundary conditions

The Dirichlet condition is the case $c_2 = 0$ of (1.7), and then (2.9) becomes

$$\phi_2(a,\lambda)\phi_1(-X,\lambda) - \{\phi_1(a,\lambda) - \rho_1(\lambda)\}\phi_2(-X,\lambda) = 0, \tag{3.1}$$

where we are now indicating the dependence on λ. The Neumann condition is the case $c_1 = 0$ of (1.7) and, as usual, it is typical of the situation when $c_2 \ne 0$. When $c_1 = 0$, (2.9) becomes

$$\phi_2(a,\lambda)\phi_1'(-X,\lambda) - \{\phi_1(a,\lambda) - \rho_1(\lambda)\}\phi_2'(-X,\lambda) = 0. \tag{3.2}$$

Before proceeding further with (3.1) and (3.2), it is convenient at this point to refer to the familiar Dirichlet and Neumann problems for (1.1) over the basic periodicity interval $(0, a)$ [6, section 3.1]. In any instability interval, each of these problems has a unique eigenvalue Λ_D and Λ_N for which

$$\phi_2(a, \Lambda_D) = 0, \quad \phi_1'(a, \Lambda_N) = 0$$

[6, Theorem 3.1]. As we indicated in section 2, there is therefore the possibility that (2.4) is a trivial linear combination when $\lambda = \Lambda_D$. But both cannot occur together should $\Lambda_D = \Lambda_N$. In case (2.4) is trivial, (2.9), (3.1) and (3.2) can be expressed instead interms of (2.5). Then (3.2) for example becomes

$$\{\phi_2'(a,\lambda) - \rho_1(\lambda)\}\phi_1'(-X,\lambda) - \phi_1'(a,\lambda)\phi_2'(-X,\lambda) = 0. \tag{3.3}$$

In what follows, we avoid this slight complication by simply excluding the value Λ_D from our considerations. Thus we work with (3.2) $(\lambda \neq \Lambda_D)$ rather than (3.3).

We can now state and prove a theorem which, in general terms, is the main result of the paper. It contains a general condition (3.4) which will be analysed in subsequent sections. The theorem concerns λ-solutions of (3.2) lying in a spectral gap (λ', λ'') of (1.1), and we recall that these solutions are the eigenvalues induced in the gap by the interface represented by (1.6) and (1.7) with a variable X being allowed for. A similar result holds for (3.1). We write $\nu' = \sqrt{\lambda'}$ and $\nu'' = \sqrt{\lambda''}$.

Theorem 3.1. *Suppose that there exist a fixed number K $(K > 0)$ such that*

$$\nu'' - \nu' \geq K \tag{3.4}$$

for a sequence of spectral gaps receding to infinity. Then there is a number $\nu_0(X)$ such that (3.2) *has λ-solutions in (λ', λ'') when $\nu' \geq \nu_0(X)$, and the number of such solutions exceeds*

$$3KX/4\pi - 4. \tag{3.5}$$

Further, if q_1 is defined in $(-\infty, 0)$ and satisfies

$$q_1 \in L(-\infty, 0), \tag{3.6}$$

then $\nu_0(X)$ can be taken to be independent of X.

Proof. We begin by applying Lemma 2.1 to (1.6) on $[-X, 0)$. By (2.11) we are considering values of ν such that

$$\nu \geq \int_{-X}^{0} |q_1(t)|dt. \tag{3.7}$$

Then, recalling the initial values (2.1), we obtain from (2.13) and (2.14)

$$\phi_1'(-X,\lambda) = \nu(1 + \frac{1}{\nu}E_1)\sin\nu X + F_1\cos\nu X$$

$$\phi_2'(-X,\lambda) = E_2\sin\nu X + (1 + F_2)\cos\nu X,$$

where $E_1 = E_1(-X)$ etc, and

$$|E_1|, \quad |F_1| \leq (e-1)\int_{-X}^{0} |q_1(t)|dt := I(X), \tag{3.8}$$

say, and

$$|E_2|, \quad |F_2| \le \frac{1}{\nu} I(X). \tag{3.9}$$

On substituting into (3.2), we obtain

$$\frac{(1+\frac{1}{\nu}E_1)\sin\nu X + \frac{1}{\nu}F_1\cos\nu X}{E_2\sin\nu X + (1+F_2)\cos\nu X} = \frac{1}{\nu}H(\nu), \tag{3.10}$$

where

$$H(\nu) = \{\phi_1(a,\lambda) - \rho_1(\lambda)\}/\phi_2(a,\lambda) \quad (\lambda \ne \Lambda_D). \tag{3.11}$$

□

Let us now denote the left-hand side of (3.10) by $T(\nu)$ $(\nu' < \nu < \nu'')$. We wish to show that $T(\nu)$ behaves sufficiently like $\tan\nu X$, the relevant property of the latter being that it increases from $-\infty$ to $+\infty$ in any ν-interval

$$((k-\frac{1}{2})\pi/X, (k+\frac{1}{2})\pi/X) \tag{3.12}$$

with k an integer. Then, in any such interval which lies within (ν', ν''), the graph of $\tan\nu X$ crosses that of $\frac{1}{\nu}H(\nu)$, producing a solution of the equation $\tan\nu X = \frac{1}{\nu}H(\nu)$ (excepting possibly only an interval (3.12) which contains the point Λ_D). Since, however, we have $T(\nu)$ rather than $\tan\nu X$, we begin by considering instead of (3.12) intervals $\mathcal{I}(k)$ of the form

$$(\{(k-\frac{1}{2})\pi + \eta\}/X, \{(k+\frac{1}{2})\pi - \eta\}/X), \tag{3.13}$$

where

$$\sin\eta = I(X)/\{\nu' - I(X)\}. \tag{3.14}$$

Also, further to (3.7), we assume that

$$\nu' > 3I(X), \tag{3.15}$$

where $I(X)$ is as in (3.8). Then (3.14) and (3.15) imply that

$$0 < \eta < \pi/6. \tag{3.16}$$

Denoting the denominator in $T(\nu)$ by $C(\nu)$, we show first that $C(\nu) > 0$ or $C(\nu) < 0$ in $\mathcal{I}(k)$ according to k being even or odd. Taking k even (odd is similar), it follows from (3.9) that, in $\mathcal{I}(k)$,

$$C(\nu) := E_2\sin\nu X + (1+F_2)\cos\nu X > -\frac{1}{\nu'}I(X) + \{1 - \frac{1}{\nu'}I(X)\}\sin\eta = 0$$

by (3.14), as required.

Next, considering also $\mathcal{I}(k+1)$ in (3.13), it follows from what we have just proved that, in the interval

$$\mathcal{L}(k) = [\{(k+\frac{1}{2})\pi - \eta\}/X, \{(k+\frac{1}{2})\pi + \eta\}/X], \tag{3.17}$$

$C(\nu)$ has a least zero $\nu_l(k)$ and a greatest zero $\nu_g(k)$. Also, in $\mathcal{L}(k)$, the numerator $S(\nu)$ in (3.10) is positive (k even) or negative (k odd). This is easily seen because (if k is even for example)

$$\begin{aligned} S(\nu) &:= (1 + \frac{1}{\nu}E_1)\sin \nu X + \frac{1}{\nu}F_1 \cos \nu X \\ &\geq \{1 - \frac{1}{\nu'}I(X)\}\cos\eta - \frac{1}{\nu'}I(X)\sin\eta \\ &> \frac{1}{3}(2\cos\eta - \sin\eta) > 0. \end{aligned}$$

Altogether, then, we have shown that

$$T(\nu) \to +\infty \quad \text{as} \quad \nu \to \nu_l(k) - 0$$

and

$$T(\nu) \to -\infty \quad \text{as} \quad \nu \to \nu_g(k-1) + 0$$

with $T(\nu)$ continuous in $(\nu_g(k-1), \nu_l(k))$. Further, both $\nu_g(k-1)$ and $\nu_l(k)$ lie in the interval

$$\mathcal{L}(k-1) \cup I(k) \cup \mathcal{L}(k), \tag{3.18}$$

which is an interval of total length $(\pi + 2\eta)/X$. It follows that (3.10) has at least one solution ν in each interval (3.18), with the possible exception of at most two intervals which contain Λ_D. The number of complete intervals (3.18) which lie in $[\nu', \nu'']$ is at least $KX/(\pi + 2\eta) - 2$, by (3.4). Then, discounting at most two intervals which contain Λ_D and using (3.16), we arrive finally at (3.5) with

$$\nu_0(X) = 3(e-1)\int_{-X}^{0} |q_1(t)|dt \tag{3.19}$$

by (3.8) and (3.15). Also, the concluding statement of the theorem, where (3.6) holds, is now clear

There is one further observation to be made concerning the condition (3.6). If we consider $X \to \infty$ in (1.7), our eigenvalue problem approximates to the problem with two singular end-points where the differential equation is (1.6) in $(-\infty, 0)$ and (1.1) in $(0, \infty)$.When (3.6) holds, this latter problem has $[\lambda_0, \infty)$ as an interval of continuous spectrum [23, sections 3.1, 3.8, 5.6 and 5.14]. Our Theorem 3.1 is therefore in accord with this property in that, at least for $\nu' > \nu_0(\infty)$ (see(3.19)), (3.13), (3.17) and (3.18) show that the interface eigenvalues become everywhere dense in (ν', ν'') as $X \to \infty$, thus filling up the spectral gaps of (1.1).

Finally, we note that Theorem 3.1 continues to apply when (3.4) holds, not for a fixed K, but for a sequence $K_n \to 0$ as $n \to \infty$. Then for (3.5) to guarantee at least one eigenvalue, we require $K_n \geq 20\pi/3X$. Thus only a finite number of the K_n is allowed for a fixed X.

4. Two discontinuities

In this section and the next we suppose that the weight function $w(x)$ in (1.1) is given by

$$w(x) = \begin{cases} w_1(x) & (0 \le x < a_1) \\ w_2(x) & (a_1 \le x < a), \end{cases} \tag{4.1}$$

where w_1 and w_2 have continuous second derivatives in $[0, a_1]$ and $[a_1, a]$ respectively, but $w_1(a_1 - 0) \ne w_2(a_1)$ and $w_2(a-0) \ne w_1(0)$. (To be brief, we omit "-0" in the sequel.) We write

$$\sigma_1 = (w_2/w_1)^{1/4}(a_1), \quad \sigma_2 = (w_1/w_2)^{1/4}(a). \tag{4.2}$$

At this point we note that, in the step-function case where w_1 and w_2 are different constants, σ_1 and σ_2 are connected by the relation $\sigma_1\sigma_2 = 1$. We shall refer later to this relation, but our analysis is not dependent on it.

In order to examine the spectral gaps in the case (4.1), and in particular to verify (3.4), we need to determine the eigenvalues λ_n and μ_n associated with (1.2) and (1.3). These eigenvalues are the solutions of the equations $D(\lambda) = \pm 2$ with D as in (2.3) [6, chapter 2]. In the following lemma, we obtain the form of $D(\lambda)$, at least for $\nu (= \sqrt{\lambda})$ large enough. In the lemma and its proof we use the general notation $M(\nu)$ to denote any expression satisfying an inequality

$$|M(\nu)| \le C \tag{4.3}$$

where C is independent of ν and is expressible explicitly in terms of w_1, w_2 and q.

Lemma 4.1. *With the notation* (4.1)–(4.3),

$$\begin{aligned} \phi_1(a,\lambda) + \phi_2'(a,\lambda) = \ & (\sigma_1\sigma_2 + \frac{1}{\sigma_1\sigma_2}) \cos\nu A_1 \cos\nu A_2 \\ & -(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}) \sin\nu A_1 \sin\nu A_2 + \frac{1}{\nu} M(\nu), \end{aligned} \tag{4.4}$$

where

$$A_1 = \int_0^{a_1} w_1^{1/2}(x)dx, \quad A_2 = \int_{a_1}^{a} w_2^{1/2}(x)dx \tag{4.5}$$

and

$$\nu \ge \int_0^a |(qw^{-1/2} - w^{-1/4}(w^{-1/4})'')(x)|dx. \tag{4.6}$$

Proof. We make the Liouville transformation of (1.1) in each of the two intervals indicated in (4.1):

$$\left.\begin{array}{ll} y = w_1^{-1/4} z_1, & t_1 = \int_0^x w_1^{1/2}(u)du \ \ (0 \le x < a_1) \\ y = w_2^{-1/4} z_2, & t_2 = \int_{a_1}^x w_2^{1/2}(u)du \ \ (a_1 \le x < a) \end{array}\right\} \tag{4.7}$$

This gives

$$d^2 z_j/dt_j^2 + \{\lambda - Q_j(t_j)\} z_j = 0 \tag{4.8}$$

where

$$Q_j(t_j) = (q/w_j - w_j^{-3/4}(w_j^{-1/4})'')(x) \tag{4.9}$$

and

$$0 \le t_j < A_j \quad (j = 1, 2)$$

[6, section 4.1]. The solutions ϕ_1 and ϕ_2 of (1.1) are transformed into solutions $\Phi_{1,j}$ and $\Phi_{2,j}$ of (4.8), and we shall obtain (4.4) by applying Lemma 2.1 to (4.8) □

We first note the connection between the values of z_1 and z_2 and their derivatives which follows from the continuity of y and y' at $x = a_1$. Thus, from (4.7), we have

$$z_2(0) = [w_2^{1/4} y](a_1) = \sigma_1 z_1(A_1) \tag{4.10}$$

$$\frac{dz_2}{dt_2}(0) = W(a_1) z_1(A_1) + \frac{1}{\sigma_1}\frac{dz_1}{dt_1}(A_1) \tag{4.11}$$

where σ_1 is as in (4.2) and

$$W = w_2^{-1/2}(w_2^{1/4} w_1^{-1/4})'. \tag{4.12}$$

Let us now consider ϕ_1 and its transforms $\Phi_{1,1}$ and $\Phi_{1,2}$. By (2.1) and (4.7), $\Phi_{1,1}(t_1, \lambda)$ has the initial values

$$w_1^{1/4}(0), \quad [w_1^{-1/2}(w_1^{1/4})'](0)$$

at $t_1 = 0$. Then Lemma 2.1 gives

$$\Phi_{1,1}(A_1, \lambda) = w_1^{1/4}(0) \cos \nu A_1 + \frac{1}{\nu}[w_1^{-1/2}(w_1^{1/4})'](0) \sin \nu A_1 + \frac{1}{\nu} E_1(\nu) \tag{4.13}$$

$$\frac{d\Phi_{1,1}}{dt_1}(A_1, \lambda) = -\nu w_1^{1/4}(0) \sin \nu A_1 + [w_1^{-1/2}(w_1^{1/4})'](0) \cos \nu A_1 + F_1(\nu) \tag{4.14}$$

where

$$|E_1(\nu)|,\ |F_1(\nu)| \le \{w_1^{1/4}(0) + \frac{1}{\nu}|[w_1^{-1/2}(w_1^{1/4})'](0)|\}(e-1)\int_0^{A_1} |Q_1(t_1)| dt_1 \tag{4.15}$$

and

$$\nu \ge \int_0^{A_1} |Q_1(t_1)| dt_1 \tag{4.16}$$

by (2.11). Using $|\sin \nu A_1| \le 1$ and $|\cos \nu A_1| \le 1$, we can write (4.13) and (4.14) as

$$\Phi_{1,1}(A_1, \lambda) = w_1^{1/4}(0) \cos \nu A_1 + \frac{1}{\nu} M(\nu)$$

$$\frac{d\Phi_{1,1}}{dt_1}(A_1, \lambda) = -\nu w_1^{1/4}(0) \sin \nu A_1 + M(\nu)$$

where $M(\nu)$ has the form indicated in (4.3). It then follows from (4.10) and (4.11) that the initial values of $\Phi_{1,2}(t_2)$ at $t_2 = 0$ are

$$\sigma_1 w_1^{1/4}(0) \cos \nu A_1 + \frac{1}{\nu} M(\nu)$$

and

$$-\frac{\nu}{\sigma_1} w_1^{1/4}(0) \sin \nu A_1 + M(\nu). \tag{4.17}$$

Turning to Lemma 2.1 again, applied now to $\Phi_{1,2}(t_2,\lambda)$ over the interval $0 \le t_2 \le A_2$, and using the initial values (4.17), we have

$$\Phi_{1,2}(A_2,\lambda) = w_1^{1/4}(0)\{\sigma_1 \cos\nu A_1 \cos\nu A_2 - \frac{1}{\sigma_1}\sin\nu A_1 \sin\nu A_2\} + \frac{1}{\nu}M(\nu) \tag{4.18}$$

$$\frac{d\Phi_{1,2}}{dt_2}(A_2,\lambda) = -w_1^{1/4}(0)\nu\{\sigma_1 \cos\nu A_1 \sin\nu A_2 + \frac{1}{\sigma_1}\sin\nu A_1 \cos\nu A_2\} + M(\nu) \tag{4.19}$$

where, by (2.11),

$$\nu \ge \int_0^{A_2} |Q_2(t_2)|dt_2. \tag{4.20}$$

Next we consider similarly ϕ_2 and its transforms $\Phi_{2,1}$ and $\Phi_{2,2}$. By (2.1) and (4.7), $\Phi_{2,1}(t_1,\lambda)$ has the initial values

$$0, \quad w_1^{-1/4}(0)$$

at $t_1 = 0$. Then Lemma 2.1 gives

$$\Phi_{2,1}(A_1,\lambda) = \frac{1}{\nu}w_1^{-1/4}(0)\sin\nu A_1 + \frac{1}{\nu^2}M(\nu)$$
$$\frac{d\Phi_{2,1}}{dt_1}(A_1,\lambda) = w_1^{-1/4}(0)\cos\nu A_1 + \frac{1}{\nu}M(\nu),$$

where (4.16) holds. It then follows from (4.10) and (4.11) that the initial values of $\Phi_{2,2}(t_2,\lambda)$ at $t_2 = 0$ are

$$\frac{\sigma_1}{\nu}w_1^{-1/4}(0)\sin\nu A_1 + \frac{1}{\nu^2}M(\nu)$$

and

$$\frac{1}{\sigma_1}w_1^{-1/4}(0)\cos\nu A_1 + \frac{1}{\nu}M(\nu). \tag{4.21}$$

Turning to Lemma 2.1 once again, applied now to $\Phi_{2,2}(t_2,\lambda)$ over the interval $0 \le t \le A_2$, and using the initial values (4.21), we have

$$\Phi_{2,2}(A_2,\lambda) = \frac{1}{\nu}w_1^{-1/4}(0)\{\sigma_1 \sin\nu A_1 \cos\nu A_2 + \frac{1}{\sigma_1}\cos\nu A_1 \sin\nu A_2\} + \frac{1}{\nu^2}M(\nu) \tag{4.22}$$

$$\frac{d\Phi_{2,2}}{dt_2}(A_2,\lambda) = w_1^{-1/4}(0)\{-\sigma_1 \sin\nu A_1 \sin\nu A_2 + \frac{1}{\sigma_1}\cos\nu A_1 \cos\nu A_2\} + \frac{1}{\nu}M(\nu) \tag{4.23}$$

where (4.20) holds.

To complete the proof of (4.4), we use (4.7), (4.18), (4.22) and (4.23) to revert to the values of ϕ_1 and ϕ_2' at $x = a$. Thus, with σ_2 as in (4.2),

$$\begin{aligned}\phi_1(a,\lambda) &= w_2^{-1/4}(a)\Phi_{1,2}(A_2,\lambda)\\ &= \sigma_1\sigma_2 \cos\nu A_1 \cos\nu A_2 - \frac{\sigma_2}{\sigma_1}\sin\nu A_1 \sin\nu A_2 + \frac{1}{\nu}M(\nu)\end{aligned} \tag{4.24}$$

and

$$\phi_2'(a,\lambda) = (w_2^{-1/4})'(a)\Phi_{2,2}(A_2,\lambda) + w_2^{1/4}(a)\frac{d\Phi_{2,2}}{dt_2}(A_2,\lambda)$$
$$= \frac{1}{\sigma_1\sigma_2}\cos\nu A_1 \cos\nu A_2 - \frac{\sigma_1}{\sigma_2}\sin\nu A_1 \sin\nu A_2 + \frac{1}{\nu}M(\nu). \quad (4.25)$$

Now (4.4) follows since (4.6) accommodates both (4.16) and (4.20), and the lemma is proved.

Let us now write (4.4) as

$$D(\lambda) = \frac{1}{2}(\sigma_1 + \frac{1}{\sigma_1})(\sigma_2 + \frac{1}{\sigma_2})\cos\nu I + \frac{1}{2}(\sigma_1 - \frac{1}{\sigma_1})(\sigma_2 - \frac{1}{\sigma_2})\cos\nu J + \frac{1}{\nu}M(\nu), \quad (4.26)$$

where

$$I = A_1 + A_2 = \int_0^a w^{1/2}(x)dx, \quad J = |A_1 - A_2|. \quad (4.27)$$

Then the equations $D(\lambda) = \pm 2$ for the periodic and semi-periodic (respectively) eigenvalues become

$$\cos\nu I = f_\pm(\nu) + \frac{1}{\nu}M(\nu), \quad (4.28)$$

where

$$f_\pm(\nu) = (\sigma_1 + \frac{1}{\sigma_1})^{-1}(\sigma_2 + \frac{1}{\sigma_2})^{-1}\{-(\sigma_1 - \frac{1}{\sigma_1})(\sigma_2 - \frac{1}{\sigma_2})\cos\nu J \pm 4\}. \quad (4.29)$$

In the next section, we discuss the solutions of (4.28), and we are interested in identifying situations where the main condition (3.4) in Theorem 3.1 is satisfied, and the $\frac{1}{\nu}$ term in (4.28) does not materially affect the analysis for large ν.

We note in passing that a similar type of equation to (4.28) and (4.29) (with $\sigma_1\sigma_2 = 1$) appears in [11, (1.5)] for a different step-function example, and the difficulty of analysing the equation is commented upon. Nevertheless, even without the restriction to the step-function relation $\sigma_1\sigma_2 = 1$, we shall extract sufficient information for our purposes from (4.28) and (4.29).

5. Lower bounds concerning the spectral gaps

We first note that the solutions of (4.28)–(4.29) are known from other sources, but not sufficiently accurately for our purposes. Thus [7, Theorem 1], [16], for $\nu = \sqrt{\lambda_{2m+1}}$ or $\sqrt{\lambda_{2m+2}}$, we have

$$|\nu - 2(m+1)\pi I^{-1}| \le I^{-1}(\omega_1 + \omega_2) + o(1) \quad (m \to \infty), \quad (5.1)$$

where

$$\omega_j = \tan^{-1}(\frac{1}{2}|\sigma_j - \frac{1}{\sigma_j}|) \quad (0 < \omega_j < \frac{1}{2}\pi) \quad (5.2)$$

and I is as in (4.27). For $\nu = \sqrt{\mu_{2m}}$ or $\sqrt{\mu_{2m+1}}$, we replace $2(m+1)$ by $2m+1$ in (5.1). However, (5.1) only requires w_1 and w_2 to be differentiable once. Under our assumptions of twice differentiability, we can go further in the following proposition which identifies a situation where (3.4) is satisfied.

Proposition 5.1. *Suppose that* $\sigma_1 \neq \sigma_2$ *and that* $\sigma_1\sigma_2 \neq 1$. *Then*

$$\left.\begin{array}{c}\sqrt{\lambda_{2m+2}} - \sqrt{\lambda_{2m+1}} \\ \sqrt{\mu_{2m+1}} - \sqrt{\mu_{2m}}\end{array}\right\} \geq 2\alpha/I + O(m^{-1}) \tag{5.3}$$

as $m \to \infty$, *where*

$$\cos\alpha = (\sigma_1 + \frac{1}{\sigma_1})^{-1}(\sigma_2 + \frac{1}{\sigma_2})^{-1}\{|(\sigma_1 - \frac{1}{\sigma_1})(\sigma_2 - \frac{1}{\sigma_2})| + 4\} \tag{5.4}$$

and $0 < \alpha < \pi/2$.

Proof. We first check that α is well defined by (5.4). Thus we need to check that

$$|(\sigma_1 - \frac{1}{\sigma_1})(\sigma_2 - \frac{1}{\sigma_2})| + 4 < (\sigma_1 + \frac{1}{\sigma_1})(\sigma_2 + \frac{1}{\sigma_2}). \tag{5.5}$$

There are two cases to consider:

(i) $\sigma_1 > 1$ and $\sigma_2 < 1$ (or vice versa)
(ii) $\sigma_1 > 1$ and $\sigma_2 > 1$ (or $\sigma_1 < 1$ and $\sigma_2 < 1$).

In case (i), (5.5) simplifies to $\sigma_1\sigma_2 + 1/\sigma_1\sigma_2 > 2$, which is true when $\sigma_1\sigma_2 \neq 1$. In case (ii), (5.5) simplifies to $\sigma_1/\sigma_2 + \sigma_2/\sigma_1 > 2$, which is true when $\sigma_1 \neq \sigma_2$.

To obtain the upper inequality in (5.3) we first note from (5.1) and (5.2) that $\sqrt{\lambda_{2m+1}}$ and $\sqrt{\lambda_{2m+2}}$ lie in the open interval $((2m+1)\pi/I, (2m+3)\pi/I)$. We therefore consider ν to lie in this interval and let ν_1 and ν_2 be the solutions of

$$\cos\nu I = \cos\alpha + \frac{1}{\nu}M(\nu),$$

where $M(\nu)$ is as in (4.28). Thus

$$\nu_1, \ \ \nu_2 = 2(m+1)\pi/I \pm \alpha/I + O(m^{-1}). \tag{5.6}$$

In (ν_1, ν_2) we have

$$\cos\nu I > \cos\alpha + \frac{1}{\nu}M(\nu) \geq f_+(\nu) + \frac{1}{\nu}M(\nu)$$

by (4.29) and (5.4), and hence $D(\lambda) > 2$ in (ν_1, ν_2). Hence $\sqrt{\lambda_{2m+1}} \leq \nu_1$ and $\sqrt{\lambda_{2m+2}} \geq \nu_2$, and (5.3) follows from (5.6). The proof for $\sqrt{\mu_{2m+1}} - \sqrt{\mu_{2m}}$ is similar , completing the proof of the proposition. □

There are two, more specialised situations where (5.3) can be improved to an asymptotic formula. The cases are described by I and J in (4.27), and we present them in the following subsections.

5.1. The case $I = 2J\ \ (\neq 0)$

Proposition 5.2. *Let $I = 2J$. Then there are numbers $\psi_\pm$ and $\omega_\pm$ with*

$$\psi_- \in (\pi/2, \pi], \quad \omega_- \in [0, \pi/2), \quad \psi_+ \in (\pi/2, \pi), \ \ \omega_+ \in (0, \pi/2)$$

such that, as $m \to \infty$,

$$\begin{aligned}
\sqrt{\lambda_{4m-1}} &= 2(2m\pi - \omega_-)/I + O(m^{-1}) \\
\sqrt{\lambda_{4m}} &= 2(2m\pi + \omega_-)/I + O(m^{-1}) \\
\sqrt{\lambda_{4m+1}} &= 2(2m\pi + \psi_-)/I + O(m^{-1}) \\
\sqrt{\lambda_{4m+2}} &= 2\{2(m+1)\pi - \psi_-\}/I + O(m^{-1}) \\
\sqrt{\mu_{4m}} &= 2(2m\pi + \omega_+)/I + O(m^{-1}) \\
\sqrt{\mu_{4m+1}} &= 2(2m\pi + \psi_+)/I + O(m^{-1}) \\
\sqrt{\mu_{4m+2}} &= 2\{2(m+1)\pi - \psi_+\}/I + O(m^{-1}) \\
\sqrt{\mu_{4m+3}} &= 2\{2(m+1)\pi - \omega_+\}/I + O(m^{-1}).
\end{aligned}$$

Further, $\psi_- < \pi$ if $\sigma_1 \neq \sigma_2$ and $\omega_- > 0$ if $\sigma_1\sigma_2 \neq 1$.

Proof. Defining $\theta = \nu J$ and using $I = 2J$, we write (4.28)-(4.29) as

$$(\sigma_1 + \frac{1}{\sigma_1})(\sigma_2 + \frac{1}{\sigma_2})(2\cos^2\theta - 1) + (\sigma_1 - \frac{1}{\sigma_1})(\sigma_2 - \frac{1}{\sigma_2})\cos\theta \mp 4 = \frac{1}{\nu}M(\nu). \quad (5.7)$$

The left-hand side here is a quadratic $p_-(c)$ (or $p_+(c)$) in $\cos\theta$ ($c = \cos\theta$). It is easy to check that $p_\pm(0) < 0$ and

$$p_\pm(1) = 2(\sqrt{\sigma_1\sigma_2} \pm \frac{1}{\sqrt{\sigma_1\sigma_2}})^2 \geq 0$$

$$p_\pm(-1) = 2(\sqrt{\sigma_1/\sigma_2} \pm \sqrt{\sigma_2/\sigma_1})^2 \geq 0,$$

where the inequalities are strict for p_+ while, for p_-, the first inequality is strict if $\sigma_1\sigma_2 \neq 1$ and the second if $\sigma_1 \neq \sigma_2$. Thus each $p_\pm(c)$ has two distinct zeros $c_\pm$ and $d_\pm$ with

$$-1 < c_+ < 0, \quad 0 < d_+ < 1$$

$$-1 \leq c_- < 0, \quad 0 < d_- \leq 1$$

and $c_- > -1$ if $\sigma_1 \neq \sigma_2$ and $d_- < 1$ if $\sigma_1\sigma_2 \neq 1$. □

Now define $\psi_\pm$ and $\omega_\pm$ by

$$\left.\begin{aligned} \cos\psi_\pm = c_\pm \quad (\pi/2 < \psi_+ < \pi, \quad \pi/2 < \psi_- \leq \pi) \\ \cos\omega_\pm = d_\pm \quad (0 < \omega_+ < \pi/2 \quad 0 \leq \omega_- < \pi/2) \end{aligned}\right\}. \quad (5.8)$$

Then, for any integer $m > 0$, consider θ in the ranges

$$\left.\begin{aligned} 2m\pi - \omega_- < \theta < 2m\pi + \omega_- \\ 2m\pi + \psi_- < \theta < 2(m+1)\pi - \psi_- \end{aligned}\right\} \quad (5.9)$$

$$\left.\begin{aligned} 2m\pi + \omega_+ < \theta < 2m\pi + \psi_+ \\ 2(m+1)\pi - \psi_+ < \theta < 2(m+1)\pi - \omega_+ \end{aligned}\right\}. \quad (5.10)$$