Giovambattista Amendola
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Thermodynamics of Materials with Memory

Theory and Applications

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Giovambattista Amendola
Dipartimento di Matematica
Applicata "U. Dini"
University of Pisa
via F. Bounarroti 1c
56127 Pisa
Italy
amendola@dma.unipi.it

Mauro Fabrizio
Dipartimento di Matematica
University of Bologna
Piazza di Porta S. Donato 5
40127 Bologna
Italy
mauro.fabrizio@unibo.it

John Murrough Golden
Dublin Institute of Technology
School of Mathematical Sciences
Kevin Street
Dublin 8
Ireland
murrough.golden@dit.ie

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Dedicated to Adele, Alessandra, and Marie

Preface

This book arose out of a conversation that took place in a bookshop in Berkeley, California, almost a decade ago. The original motivation was to provide a text on continuum thermodynamics that would allow a systematic derivation and discussion of free-energy functionals for materials with memory, including in particular explicit expressions for the minimum and related free energies, which were being developed at the time.

Progress was very slow, due to other commitments. The vision of what the book would explore broadened considerably over the years, in particular to include minimal states and a new single-integral free-energy functional that explicitly depends on the minimal state. Also, it was decided to include a detailed description of an alternative approach to the analysis of the integrodifferential equations describing the evolution of viscoelastic materials under varying loads, using minimal states and free-energy functionals depending on the minimal state. This is a novel approach to a well-known topic.

Our desire was to make the work as self-contained as possible, so chapters dealing with the general theory of continuum mechanics were included, with sections devoted to classical materials, specifically elastic bodies and fluids without explicit memory-dependence. These provided essential background to the more general and modern developments relating to materials with memory.

It was furthermore felt that certain other topics had not been covered previously in book form and should be included, in particular control theory and the Saint-Venant and inverse problems, as well as some discussion of nonsimple behavior, for materials with memory.

The book is divided into four parts. The mathematical presentation in the first three parts is largely accessible not only to applied mathematicians but also to mathematically oriented engineers and scientists. However, a higher standard is required for some of the chapters in the final part.

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Giovambattista Amendola
Mauro Fabrizio
Murrough Golden

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Part IV The Dynamical Equations for Materials with Memory

Introduction

In this work, we consider materials the constitutive equations of which contain a dependence upon the past history of kinetic variables. In particular, we deal with the constraints imposed upon these constitutive equations by the laws of thermodynamics. Such materials are often referred to as *materials with memory* or with hereditary effects.

The study of materials with memory arises from the pioneering articles of Boltzmann [20, 21] and Volterra [211, 212, 213], in which they sought an extension of the concept of an elastic material. The key assumption of the theory was that the stress at a time t depends upon the history of the deformation up to t. The hypothesis that the remote history has less influence than the recent history is already implicit in their work. This assumption, later termed *the fading memory principle* by Coleman and Noll [40], is imposed by means of a constitutive equation for the stress, of integral type, which in the linear case involves a suitable kernel (relaxation function) that is a positive, monotonic, decreasing function.

In the classical approach to materials with memory, the state is identified with the history of variables carrying information about the input processes. We show in this book how Noll's definition of state [188] is more convenient for application to such materials. Indeed, Noll takes the material response as the basis for the definition of state: if an arbitrary continuation of different given histories leads to the same response of the material, then the given histories are equivalent and the state is represented as the class of all such equivalent histories. We refer to this class as the *minimal state*.

The concept of a minimal state is developed and applied in [116] to the case of linear viscoelasticity with scalar relaxation functions given by a sum of exponentials. A subsequent paper [57] presents a treatment in three dimensions and in the more general context of thermodynamically compatible (tensor-valued) relaxation functions, taking into account weak regularity of histories and processes.

A generalization of minimal states to materials under nonisothermal conditions is discussed in Section 6.4 of the present work. A functional $\mathbf{I}^t$ is introduced, given by (6.4.2) with the crucial property expressed by (6.4.3). This quantity characterizes the minimal state. Special cases of it are used in a variety of contexts in later chapters. It

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is closely related to the response of the material after time t, where the input variable is null for a finite period after this time on the material element (i.e., a "small" neighborhood of a fixed and arbitrary point of the body) under consideration. This characterization of the state is an interesting alternative to the usual one based on knowledge of the deformation history.

It seems more appropriate to refer to materials with states characterized in this way as *materials with relaxation* rather than materials with fading memory.

For the usual definition of state, a fading-memory property of the response functional [36] is required, as opposed to the case in which the minimal state is adopted, where indeed the relaxation property of the response functional suffices. Obviously, whenever the stress-response functional is such that knowledge of the minimal state turns out to be equivalent to knowledge of the past history, the property of relaxation of the stress response implies fading memory of the related functional. In this sense, the class of materials with relaxation is larger than the one described by constitutive equations with fading memory.

A significant advantage of the response-based definition of state relates to the physical features of the state itself. Indeed, the "future stress" $\mathbf{I}^t(\tau)$ can be determined through measurements and does not require knowledge of the past history at all.

For materials with memory, there are in general many different functional forms with the required properties for a free energy. Some of these are functions of the minimal state, while others do not have this property (see, e.g., [57]).

In Part III, these functional forms are explored for different categories of materials with memory. We note that for materials whose constitutive relation for the response functional has a linear memory term, all free energies associated with this material have memory terms that are quadratic functionals.

A new class of *single-integral-type* free energies, for certain categories of relaxation functions, is introduced in Section 9.1.3 as a quadratic form of the time derivative of the state variable $\mathbf{I}^t$ (see, e.g., [128, 129] for discussion and analysis of single-integral type free energies that are quadratic forms of histories). For exponentially decaying relaxation functions, it can be shown that the dissipation associated with such energies is bounded below by a time-decay coefficient multiplied by the purely memory-dependent part of the free energy. This property turns out to be crucial in the analysis of PDEs relating to linear viscoelastic materials, which is developed in Part IV.

An analogous property holds for a family of multiple-integral free-energy functionals that are the generalization of the previous single-integral-type free energy. We may refer to such a family as the *n-family*. For $n = 1$ one recovers the single-integral case.

In Chapters 10–14, explicit forms of the minimum free energy are derived both in the general nonisothermal case and, more specifically, for viscoelastic solids, fluids, and rigid heat conductors. Different forms of relaxation functions are also considered. The minimum free energy is always a function of the minimal state. Indeed, an explicit formula is derived in Section 11.2 for this quantity as a quadratic functional of minimal-state variables related to $\mathbf{I}^t$.

In Chapters 15 and 16, relaxation functions consisting of sums of decaying exponentials multiplying polynomials are considered. A family of free energies, including the minimum, maximum, and intermediate forms, are given explicitly. All of these are functions of state and derivable from an optimization procedure.

In Part IV, we observe that the new approach outlined above and the new free energies, in both cases adapted to the theory of viscoelasticity, have interesting applications to the PDEs governing the motion of a suitable class of viscoelastic bodies. In particular, the use of the new free energies given by quadratic forms of the minimal state variables yields results relating to well-posedness and stability for the IBVP. This formulation allows for initial data belonging to broader functional spaces than those usually considered in the literature, which are based on histories.

Indeed, the response-based definition of state is useful for both the study of IBVP on the one hand and the evolution of linear viscoelastic systems on the other hand.

Furthermore, an application of semigroup theory to this class of materials is presented. Here, besides having the system of equations in a more general form than for the classical approach, results on asymptotic stability are again obtained for initial data belonging to a space broader than the one usually employed when states and histories are identified.

The book is divided into four parts, Part I dealing with the general principles of continuum mechanics and with elastic materials and classical fluids, which of course provide the foundation for developments in later chapters. A general treatment of continuum thermodynamics is presented in Part II.

In Part III, materials that are described by constitutive equations with linear memory terms are discussed in some detail. The specific cases included are viscoelastic solids and fluids, together with rigid heat conductors. Also, as noted earlier, the derivation of explicit forms of free energies is considered in depth. Part IV deals with the application of results and ideas from Part III to the equations of motion of linear viscoelastic materials.

Notation conventions are described at the beginning of Appendix A. Relevant mathematical topics are summarized in Appendices A, B, and C.

Part I
Continuum Mechanics and Classical Materials

Chapter 1
Introduction to Continuum Mechanics

1.1 Introduction

In this initial chapter, we introduce various fundamentals: description of deformation, definition and interpretation of the strain and stress tensors, balance laws, and general restrictions on constitutive equations. These provide the foundation for later developments.

A number of excellent, indeed hardly to be bettered, presentations of these basic topics exist in the literature, notably in [209, 210, 127, 139] and [168]. Several formulations of standard arguments in this chapter and the next are based on those in [127, 168].

An introduction to some notation and results relating to finite-dimensional vector spaces required in this and later chapters is given in Section A.2.

1.2 Kinematics

1.2.1 Continuous Bodies. Deformations. Strain Tensors

We will consider bodies the mass of which is distributed continuously. Moreover, a given body will occupy different regions at different times, but none of these regions will be intrinsically associated with the body. Thus, formally, a *continuous body* $\mathcal{B}$ is a set of material points $\mathbf{X}$, $\mathbf{Y}$,... endowed with a structure defined by a class Φ of one-to-one mappings $\varphi : \mathcal{B} \to \mathcal{E}$, where $\mathcal{E}$ is the three-dimensional Euclidean space, such that:

(i) $\varphi(\mathcal{B})$ is a Kellogg regular region;*

* By a Kellogg regular region we mean a domain of the Euclidean space $\mathcal{E}$ bounded by a union of a finite number of surfaces of class C^1. A more formal definition of a subbody is given in [15, 169, 189] (see also [1]).

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(ii) if $\varphi, \psi \in \Phi$, then the function $\lambda = \varphi \circ \psi^{-1} : \psi(\mathcal{B}) \to \varphi(\mathcal{B}) \in C^1(\psi(\mathcal{B}))$ is called a *deformation* (of class C^1) of $\mathcal{B}$ from $\psi(\mathcal{B})$ to $\varphi(\mathcal{B})$;

(iii) if $\varphi \in \Phi$ and $\lambda : \varphi(\mathcal{B}) \to \mathcal{E}$ is a deformation of class C^1, then the mapping $\lambda \circ \varphi$ is also in Φ.

The functions φ are referred to as localizations of $\mathcal{B}$, and they determine the possible configurations of the body in the space $\mathcal{E}$. A localization provides at any material point $\mathbf{X} \in \mathcal{B}$ the corresponding geometric point $\mathbf{x} = \varphi(\mathbf{X}) \in \mathcal{E}$.

The hypotheses (i)–(iii) introduce a unique structure of a differential variety on $\mathcal{B}$.†

The set Φ of all possible localizations of $\mathcal{B}$ allows us to locate $\mathcal{B}$ in $\mathcal{E}$, as well as to define the internal constraints of material systems. We consider as an example a rigid body for which the class Φ must be defined so that for each pair $\varphi_1, \varphi_2 \in \Phi$ we have

$$d\,(\varphi_1(\mathbf{X}), \varphi_1(\mathbf{Y})) = d\,(\varphi_2(\mathbf{X}), \varphi_2(\mathbf{Y}))$$

for all $\mathbf{X}, \mathbf{Y} \in \mathcal{B}$, where d is the metric of the Euclidean space $\mathcal{E}$.

Moreover, for any continuous body $\mathcal{B}$, it is possible to determine a class $\mathcal{S}$ of subbodies $A, B, C, \ldots$ of $\mathcal{B}$, characterized by the following properties:

(a) $\mathcal{B} \in \mathcal{S}$;

(b) any element $A \in \mathcal{S}$ is such that $\varphi(A)$ is a Kellogg regular region of $\mathcal{E}$, for any $\varphi \in \Phi$.‡

On the class $\mathcal{S}$ of subbodies it is possible to define a measure that allows us to give a definition of the density and of the mass.

Definition 1.2.1. The mass is a measure $M : \mathcal{S} \to \mathbb{R}^+$ absolutely continuous with respect to the ordinary volume measure; that is, for each $\varphi \in \Phi$ there is an integrable function $\hat{\rho}_\varphi : \varphi(\mathcal{B}) \to \mathbb{R}^+$, the density of mass, such that the mass relative to A is

$$M(A) = \int_{\varphi(A)} \hat{\rho}_\varphi(\mathbf{x}) dv,$$

for all $A \in \mathcal{S}$.

A motion of $\mathcal{B}$ with respect to a fixed observer O is a sufficiently regular function§

$$\tilde{\chi} : \mathcal{B} \times I \to \mathcal{E}, \tag{1.2.1}$$

where $I \subset \mathbb{R}$ is a time interval.

† In other words, the body $\mathcal{B}$ does not identify itself with a particular configuration, but with the set of all possible configurations it can assume and hence with a differential variety.

‡ The given definition for a subbody is independent of the chosen localization φ. In fact, if ψ is another localization, then the transformation $\lambda = \varphi \circ \psi^{-1} : \psi(\mathcal{B}) \to \varphi(\mathcal{B})$ possesses an inverse of class C^1. Therefore, if $\varphi(A)$ is a regular region, then $\psi(A)$ will be a regular region of $\mathcal{E}$.

§ With respect to each context the condition of being sufficiently regular may have various senses. For our purposes the function χ is assumed to be twice continuously differentiable in the domain of existence.

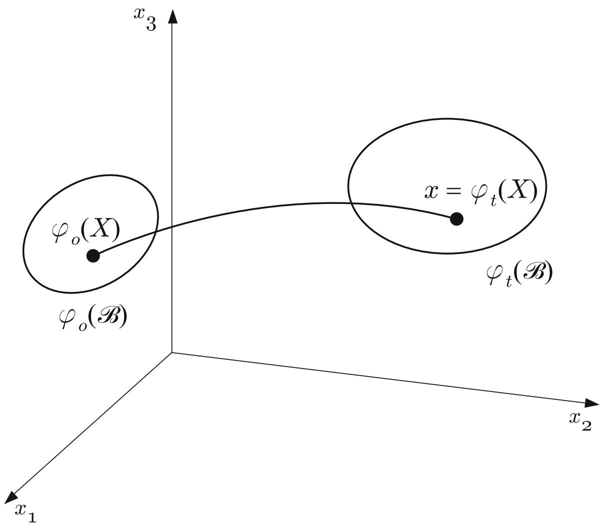

Fig. 1.2.1 The deformation of a body from $\varphi_0(\mathcal{B})$ to $\varphi_t(\mathcal{B})$.

In what follows we will identify the body $\mathcal{B}$ with one of its particular configurations, namely the reference configuration $\varphi_0(\mathcal{B})$ (see Figure 1.2.1). Moreover, the function $\tilde{\chi}$ is such that for each $t \in I$, the new function $\tilde{\chi}_t : \varphi_0(\mathcal{B}) \to \varphi_t(\mathcal{B})$, which represents the deformation of the body $\mathcal{B}$ from $\varphi_0(\mathcal{B})$ to $\varphi_t(\mathcal{B})$, has an inverse, that is, there exists a function

$$\tilde{\chi}_t^{-1} : \varphi_t(\mathcal{B}) \to \varphi_0(\mathcal{B}). \tag{1.2.2}$$

Hence $\tilde{\chi}_t$ is assumed to be one-to-one. This hypothesis expresses the requirement that the body not penetrate itself. Thus, two distinct points of the configuration $\varphi_0(\mathcal{B})$ must be distinct in all other configurations.

It is possible to write the transformations (1.2.1) and (1.2.2) in the following forms:

$$\begin{aligned} \mathbf{x} &= \tilde{\chi}(\mathbf{X}, t), \\ \mathbf{X} &= \tilde{\chi}^{-1}(\mathbf{x}, t). \end{aligned} \tag{1.2.3}$$

The function defined by $(1.2.3)_1$ represents the position occupied by the particle $\mathbf{X}$ at the instant t, while relation $(1.2.3)_2$ locates the particle $\mathbf{X}$ that occupies the point $\mathbf{x}$ at the instant t. The variables $(\mathbf{X}, t)$ are the *Lagrangian* or *material coordinates*, while $(\mathbf{x}, t)$ are the *Eulerian* or *spatial coordinates*. The relations (1.2.3) demonstrate that it is possible to express any physical quantity $\mathcal{F}$ in terms of material or spatial coordinates by

$$\tilde{\mathcal{F}}(\mathbf{X}, t) = \tilde{\mathcal{F}}(\tilde{\chi}^{-1}(\mathbf{x}, t), t) = \hat{\mathcal{F}}(\mathbf{x}, t). \tag{1.2.4}$$

Definition 1.2.2. The *Lagrangian description* is the description of motion in terms of the variables $(\mathbf{X}, t)$, while the *Eulerian description* is that referring to the variables $(\mathbf{x}, t)$.

As an example we consider the velocity of a particle $\mathbf{X}$ at the instant t, defined as

$$\tilde{\mathbf{v}}(\mathbf{X}, t) = \frac{\partial \tilde{\chi}}{\partial t}(\mathbf{X}, t);$$

on the basis of relation $(1.2.3)_2$ it is possible to express such a quantity in terms of the Eulerian variables as

$$\hat{\mathbf{v}}(\mathbf{x}, t) = \tilde{\mathbf{v}}(\tilde{\chi}^{-1}(\mathbf{x}, t), t). \tag{1.2.5}$$

Remark 1.2.3. The time derivative of a quantity $\mathcal{F}$ has different expressions, depending on the description. In fact, by direct differentiation with respect to t of (1.2.4), we obtain

$$\frac{\partial \tilde{\mathcal{F}}}{\partial t} = \frac{\partial \hat{\mathcal{F}}}{\partial t} + \nabla_{\mathbf{x}} \hat{\mathcal{F}} \cdot \mathbf{v}, \tag{1.2.6}$$

where $\nabla_{\mathbf{x}}$ is the spatial gradient operator. The partial derivative on the left is taken holding $\mathbf{X}$ fixed, while in that on the right, $\mathbf{x}$ is fixed.

The derivative $\frac{\partial \tilde{\mathcal{F}}}{\partial t}$ is the *material derivative* (or total derivative), denoted by

$$\frac{d\hat{\mathcal{F}}}{dt} = \frac{\partial \tilde{\mathcal{F}}}{\partial t}. \tag{1.2.7}$$

If we choose as $\mathcal{F}$ the velocity $\mathbf{v}$, then, by virtue of (1.2.6), we have that the acceleration is given by

$$\mathbf{a} = \frac{\partial}{\partial t}\tilde{\mathbf{v}}(\mathbf{X}, t) = \frac{\partial \hat{\mathbf{v}}}{\partial t}(\mathbf{x}, t) + \nabla_{\mathbf{x}} \hat{\mathbf{v}}(\mathbf{x}, t)\, \mathbf{v}.$$

Definition 1.2.4. The material gradient of deformation is the tensor

$$\mathbf{F}(\mathbf{X}, t) = \nabla_{\mathbf{X}} \tilde{\chi}(\mathbf{X}, t), \quad \text{that is,} \ F_{ij} = \frac{\partial \tilde{\chi}_i}{\partial X_j}, \tag{1.2.8}$$

where $\nabla_{\mathbf{X}}$ is the material gradient operator . The velocity gradient is the tensor

$$\mathbf{L}(\mathbf{X}, t) = \mathbf{L}(\tilde{\chi}(\mathbf{X}, t), t) = \nabla_{\mathbf{x}} \hat{\mathbf{v}}(\mathbf{x}, t). \tag{1.2.9}$$

Remark 1.2.5. If we set $\dot{\mathbf{F}} = \frac{\partial \mathbf{F}}{\partial t}$, then

$$\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}. \tag{1.2.10}$$

In fact, we have

$$\dot{\mathbf{F}} = \nabla_{\mathbf{X}} \tilde{\mathbf{v}} = \nabla_{\mathbf{x}} \hat{\mathbf{v}} \nabla_{\mathbf{X}} \tilde{\chi}. \tag{1.2.11}$$

Remark 1.2.6. The requirement that the body not penetrate itself is expressed by the assumption that

$$\det(\mathbf{F}) = \det(\nabla_{\mathbf{X}}\tilde{\chi}) \neq 0.$$

Further, a deformation with $\det(\nabla_{\mathbf{X}}\tilde{\chi}) < 0$ cannot be reached by a continuous process of deformation starting from the reference configuration, that is, by a continuous one-parameter family $\tilde{\chi}_\sigma$ $(0 \leq \sigma \leq 1)$ of deformations with $\tilde{\chi}_0$ the identity, $\tilde{\chi}_1 = \tilde{\chi}$, and $\det(\nabla_{\mathbf{X}}\tilde{\chi}_\sigma)$ never zero. Indeed, since $\det(\nabla_{\mathbf{X}}\tilde{\chi}_\sigma)$ is strictly positive at $\sigma = 0$, it must be strictly positive for all σ. Thus, we require that

$$\det \mathbf{F} > \mathbf{0}. \tag{1.2.12}$$

The above discussion motivates the following definition.

Definition 1.2.7. By a *deformation* of $\mathcal{B}$ we mean a smooth, one-to-one mapping $\tilde{\chi}$, that maps $\mathcal{B}$ onto a closed region in $\mathcal{E}$ and satisfies (1.2.12). The vector

$$\mathbf{u}(\mathbf{X}, t) = \tilde{\chi}(\mathbf{X}, t) - \mathbf{X}$$

represents the *displacement* of $\mathbf{X}$. A deformation with $\mathbf{F}$ constant is called *homogeneous* .

The geometric significance of the tensor $\mathbf{F}$ becomes clear on observing that

$$\tilde{\chi}(\mathbf{X}', t) - \tilde{\chi}(\mathbf{X}, t) = \nabla_{\mathbf{X}}\tilde{\chi}(\mathbf{X}, t)(\mathbf{X}' - \mathbf{X}) + \mathbf{o}\left(\left|\mathbf{X}' - \mathbf{X}\right|\right),$$

for all $\mathbf{X}'$ in a neighborhood of $\mathbf{X}$, so that we can write

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}. \tag{1.2.13}$$

Thus, the tensor $\mathbf{F}$ transforms the small quantity $d\mathbf{X}$ of the configuration $\varphi_0(\mathcal{B})$ into the small displacement $d\mathbf{x}$ of the configuration $\varphi_t(\mathcal{B})$ (see Figure 1.2.2). Let

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \tag{1.2.14}$$

be the polar decomposition of $\mathbf{F}$ at a given point, where $\mathbf{R}$ represents the rotation tensor, $\mathbf{U}$ is the right stretch tensor, and $\mathbf{V}$ is the left stretch tensor for the deformation $\tilde{\chi}$. Thus, $\mathbf{R}(P)$ measures the local rigid rotation of points near P, while $\mathbf{U}(P)$ and $\mathbf{V}(P)$ measure local stretching from P. The tensors $\mathbf{U}(P)$, $\mathbf{V}(P)$ are symmetric. Since $\mathbf{U} = \sqrt{\mathbf{F}^T\mathbf{F}}$ and $\mathbf{V} = \sqrt{\mathbf{F}\mathbf{F}^T}$ involve the square roots of $\mathbf{F}^T\mathbf{F}$ and $\mathbf{F}\mathbf{F}^T$, their computation is often difficult. For this reason we introduce the *right and left Cauchy–Green strain tensors* $\mathbf{C}$ and $\mathbf{B}$, defined by

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \quad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T, \tag{1.2.15}$$

and note that

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^T, \quad \mathbf{B} = \mathbf{R}\mathbf{C}\mathbf{R}^T.$$

In components, we have

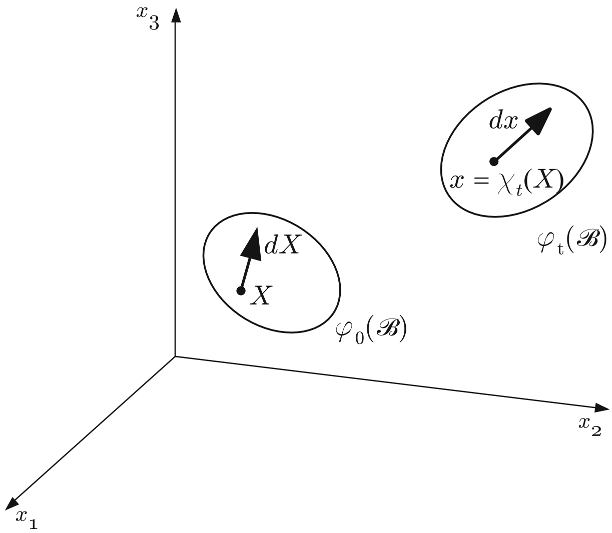

Fig. 1.2.2 The quantities $d\mathbf{X}$ and $d\mathbf{x}$ related by (1.2.13).

$$C_{ij} = \sum_{m=1}^{3} \frac{\partial \tilde{\chi}_m}{\partial X_i} \frac{\partial \tilde{\chi}_m}{\partial X_j}, \qquad B_{ij} = \sum_{m=1}^{3} \frac{\partial \tilde{\chi}_i}{\partial X_m} \frac{\partial \tilde{\chi}_j}{\partial X_m}.$$

Since $\mathbf{Cu} \cdot \mathbf{v} = \mathbf{Fu} \cdot \mathbf{Fv}$ for all $\mathbf{u}, \mathbf{v} \in V$ and $\mathbf{Cu} \cdot \mathbf{u} = \mathbf{Fu} \cdot \mathbf{Fu} > 0$ for all $\mathbf{u} \in V \backslash \{\mathbf{0}\}$, it follows that $\mathbf{C}$ is a symmetric and positive definite tensor (Section A.2.1).

In view of the relation (1.2.12), it follows that $\mathbf{F}$ admits an inverse denoted by $\mathbf{F}^{-1}$, the *spatial gradient of deformation*, given by

$$\mathbf{F}^{-1} = \nabla_{\mathbf{x}} \mathbf{X}, \quad \text{or} \quad F_{ij}^{-1} = \frac{\partial \tilde{\chi}_i^{-1}}{\partial x_j}.$$

With this we can introduce the *right and left Cauchy strain tensors*, $\mathbf{c}$ and $\mathbf{b}$, defined by

$$\mathbf{c} = \left(\mathbf{F}^{-1}\right)^T \mathbf{F}^{-1}, \qquad \mathbf{b} = \mathbf{F}^{-1} \left(\mathbf{F}^{-1}\right)^T, \tag{1.2.16}$$

or, in components,

$$c_{ij} = \sum_{m=1}^{3} \frac{\partial \tilde{\chi}_m^{-1}}{\partial x_i} \frac{\partial \tilde{\chi}_m^{-1}}{\partial x_j}, \qquad b_{ij} = \sum_{m=1}^{3} \frac{\partial \tilde{\chi}_i^{-1}}{\partial x_m} \frac{\partial \tilde{\chi}_j^{-1}}{\partial x_m}.$$

If $d\mathbf{X}$ and $\delta\mathbf{X}$ are two displacement elements related to the point $\mathbf{X}$ that at the instant t are transformed into two displacements $d\mathbf{x}$ and $\delta\mathbf{x}$, respectively, related to the point $\mathbf{x} = \tilde{\chi}(\mathbf{X}, t)$, so that

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}, \quad \delta\mathbf{x} = \mathbf{F}\delta\mathbf{X}, \tag{1.2.17}$$

then

$$d\mathbf{x} \cdot \delta\mathbf{x} = d\mathbf{X} \cdot \mathbf{F}^T\mathbf{F}\delta\mathbf{X} = d\mathbf{X} \cdot \mathbf{C}\delta\mathbf{X}. \tag{1.2.18}$$

If the continuous body is rigid, then from the relation (1.2.18) we get necessarily $\mathbf{C} = \mathbf{1}$, the unit second-order tensor. When the body is not rigid we can determine the elongation of the element $d\mathbf{X}$, associated with the tensor $\mathbf{C}$, by

$$|d\mathbf{x}|^2 = d\mathbf{x} \cdot d\mathbf{x} = d\mathbf{X} \cdot \mathbf{C}d\mathbf{X}, \tag{1.2.19}$$

so that the relative elongation is

$$|d\mathbf{x}|^2 - |d\mathbf{X}|^2 = 2\mathbf{E}d\mathbf{X} \cdot d\mathbf{X} = 2\mathbf{e}d\mathbf{x} \cdot d\mathbf{x},$$

where

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}), \quad \mathbf{e} = \frac{1}{2}(\mathbf{1} - \mathbf{c}), \tag{1.2.20}$$

are *Green's strain tensor* and *Almansi's strain tensor*, respectively. Obviously, for a rigid deformation of the body, we have $\mathbf{E} = \mathbf{0}$ and $\mathbf{e} = \mathbf{0}$. Thus, the tensor $\mathbf{E}$ appears as a measure of Lagrangian deformation, while the tensor $\mathbf{e}$ represents a measure of Eulerian deformation .

In terms of the displacement vector $\mathbf{u}(\mathbf{X}, t) = \tilde{\chi}(\mathbf{X}, t) - \mathbf{X}$ or $\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \tilde{\chi}^{-1}(\mathbf{x}, t)$, the gradients of deformation are

$$\mathbf{F} = \nabla_{\mathbf{X}}\mathbf{u} + \mathbf{1}, \quad \mathbf{F}^{-1} = \mathbf{1} - \nabla_{\mathbf{x}}\mathbf{u},$$

and hence, from (1.2.20), the strain tensors are

$$\begin{aligned} \mathbf{E} &= \frac{1}{2}\left[\nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T + (\nabla_{\mathbf{X}}\mathbf{u})^T \nabla_{\mathbf{X}}\mathbf{u}\right], \\ \mathbf{e} &= \frac{1}{2}\left[\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^T - (\nabla_{\mathbf{x}}\mathbf{u})^T \nabla_{\mathbf{x}}\mathbf{u}\right]. \end{aligned} \tag{1.2.21}$$

The relations (1.2.21) are known as the *strain–displacement (or geometrical) relations*.

Remark 1.2.8. **(Geometric significance of the strain tensors)** The components E_{11}, E_{22}, and E_{33} of the strain tensor $\mathbf{E}$ characterize the relative elongations in the directions of $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, respectively, while the components E_{ij}, with $i \neq j$, represent a measure of the variation of angles due to the process of deformation.

To see this, we first note that the relation (1.2.19) can be written in the form

$$\frac{|d\mathbf{x}|^2}{|d\mathbf{X}|^2} = \mathbf{N} \cdot \mathbf{C}\mathbf{N},$$

where $\mathbf{N} = \frac{d\mathbf{X}}{|d\mathbf{X}|}$. If we set $\Lambda_{(\mathbf{N})} = \frac{|d\mathbf{x}|}{|d\mathbf{X}|}$, then we have

$$\Lambda_{(\mathbf{N})} = (\mathbf{N} \cdot \mathbf{C}\mathbf{N})^{\frac{1}{2}} = \sqrt{\mathbf{N} \cdot (\mathbf{1} + 2\mathbf{E})\,\mathbf{N}}.$$

We further introduce the unit elongation $E_{(\mathbf{N})}$ in the direction of unit vector $\mathbf{N}$, by

$$E_{(\mathbf{N})} = \Lambda_{(\mathbf{N})} - 1 = \frac{|d\mathbf{x}| - |d\mathbf{X}|}{|d\mathbf{X}|},$$

so that when $\mathbf{N} = \mathbf{i}_1$, for example, then

$$E_{(\mathbf{i}_1)} = \sqrt{1 + 2E_{11}} - 1,$$

and hence E_{11} appears as a measure for the elongation in the direction of $\mathbf{i}_1$.

Let us further consider the vectors $d\mathbf{X}_1 = dX_1\mathbf{i}_1$ and $d\mathbf{X}_2 = dX_2\mathbf{i}_2$ and let $d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1$ and $d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$ be the corresponding vectors in the current configuration. Obviously, we have $d\mathbf{X}_1 \cdot d\mathbf{X}_2 = 0$, that is, the angle Θ_{12} between these vectors is $\frac{\pi}{2}$. On the other hand, the corresponding angle θ_{12} between the vectors $d\mathbf{x}_1$ and $d\mathbf{x}_2$ is given by

$$\cos\theta_{12} = \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{|d\mathbf{x}_1|\,|d\mathbf{x}_2|} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}} = \frac{2E_{12}}{\sqrt{(1 + 2E_{11})\,(1 + 2E_{22})}},$$

and hence E_{12} appears as a measure of the variation of the angle Θ_{12} due to the deformation.

We now recall that given a tensor $\mathbf{S} \in \mathrm{Lin}(\mathbb{R}^3)$, the determinant of $\mathbf{S} - \lambda\mathbf{1}$ admits the representation (the Cayley–Hamilton theorem)

$$\det(\mathbf{S} - \lambda\mathbf{1}) = -\lambda^3 + I_1(\mathbf{S})\lambda^2 - I_2(\mathbf{S})\lambda + I_3(\mathbf{S})$$

for every $\lambda \in \mathbb{R}$, where

$$\begin{aligned} I_1(\mathbf{S}) &= tr\mathbf{S} = S_{11} + S_{22} + S_{33}, \\ I_2(\mathbf{S}) &= \frac{1}{2}\left[(tr\mathbf{S})^2 - tr\left(\mathbf{S}^2\right)\right], \\ I_3(\mathbf{S}) &= \det\mathbf{S}. \end{aligned} \tag{1.2.22}$$

We call $I_1(\mathbf{S})$, $I_2(\mathbf{S})$, $I_3(\mathbf{S})$ the *principal invariants* of $\mathbf{S}$ and observe that they are invariant under changes of reference frames. We also note that any other invariant of $\mathbf{S}$ is a function of its principal invariants. When $\mathbf{S}$ is symmetric, the principal invariants are completely characterized by the spectrum $\{\lambda_1, \lambda_2, \lambda_3\}$ of $\mathbf{S}$. Indeed,

$$\begin{aligned} I_1(\mathbf{S}) &= \lambda_1 + \lambda_2 + \lambda_3, \\ I_2(\mathbf{S}) &= \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \\ I_3(\mathbf{S}) &= \lambda_1\lambda_2\lambda_3. \end{aligned}$$

By substituting $\mathbf{S}$ by $\mathbf{C}$, $\mathbf{c}$, $\mathbf{E}$, or $\mathbf{e}$ in the above relations we can determine expressions for the principal invariants of these tensors and relationships between them. Thus, from (1.2.20) and (1.2.22), we obtain

$$I_1(\mathbf{C}) = 3 + 2I_1(\mathbf{E}), \quad I_2(\mathbf{C}) = 3 + 4I_1(\mathbf{E}) + 4I_2(\mathbf{E}),$$
$$I_3(\mathbf{C}) = 1 + 2I_1(\mathbf{E}) + 4I_2(\mathbf{E}) + 8I_3(\mathbf{E}),$$
$$I_1(\mathbf{c}) = 3 - 2I_1(\mathbf{e}), \quad I_2(\mathbf{c}) = 3 - 4I_1(\mathbf{e}) + 4I_2(\mathbf{e}),$$
$$I_3(\mathbf{c}) = 1 - 2I_1(\mathbf{e}) + 4I_2(\mathbf{e}) - 8I_3(\mathbf{e}).$$

Moreover, we observe that the relations (1.2.15), $(1.2.16)_1$, and $(1.2.22)_3$ give

$$I_3(\mathbf{C}) = (\det \mathbf{F})^2, \quad I_3(\mathbf{c}) = \frac{1}{(\det \mathbf{F})^2},$$

and hence

$$I_3(\mathbf{C}) I_3(\mathbf{c}) = 1.$$

Definition 1.2.9. The stretching $\mathbf{D}$ (or velocity of deformation) is

$$\mathbf{D} = \frac{1}{2}\left(\mathbf{L} + \mathbf{L}^T\right) = \frac{1}{2}\left[\nabla_{\mathbf{x}}\hat{\mathbf{v}} + (\nabla_{\mathbf{x}}\hat{\mathbf{v}})^T\right], \tag{1.2.23}$$

where $\mathbf{L}$ is defined by (1.2.9), while the spin $\mathbf{\Omega}$ is

$$\mathbf{\Omega} = \frac{1}{2}\left(\mathbf{L} - \mathbf{L}^T\right) = \frac{1}{2}\left[\nabla_{\mathbf{x}}\hat{\mathbf{v}} - (\nabla_{\mathbf{x}}\hat{\mathbf{v}})^T\right]. \tag{1.2.24}$$

Thus, the stretching and the spin represent the symmetric and skew parts of the spatial gradient of velocity, respectively. Moreover, we have

$$\mathbf{L} = \mathbf{D} + \mathbf{\Omega}. \tag{1.2.25}$$

Note that

$$\begin{aligned} \frac{d}{dt}|d\mathbf{x}|^2 &= \frac{d}{dt}(d\mathbf{x} \cdot d\mathbf{x}) = 2\frac{d}{dt}(d\mathbf{x}) \cdot d\mathbf{x} \\ &= 2\frac{d}{dt}(\mathbf{F}d\mathbf{X}) \cdot d\mathbf{x} = 2\frac{d}{dt}(\mathbf{F})\, d\mathbf{X} \cdot d\mathbf{x}, \end{aligned}$$

and hence, in view of relation (1.2.10),

$$\begin{aligned} \frac{d}{dt}|d\mathbf{x}|^2 &= 2\mathbf{LF}d\mathbf{X} \cdot d\mathbf{x} = 2\mathbf{L}d\mathbf{x} \cdot d\mathbf{x} = 2d\mathbf{x} \cdot \mathbf{L}^T d\mathbf{x} \\ &= 2d\mathbf{x} \cdot \left(\frac{\mathbf{L} + \mathbf{L}^T}{2}\right) d\mathbf{x} = 2d\mathbf{x} \cdot \mathbf{D}d\mathbf{x}. \end{aligned} \tag{1.2.26}$$

Thus, the stretching $\mathbf{D}$ is a measure of the variation per unit time of the arc $(d\mathbf{x})^2$. Therefore, when $\mathbf{D} = \mathbf{0}$ then there is no change in $|d\mathbf{x}|^2$ over time.

Theorem 1.2.10. *A necessary and sufficient condition for a motion to be locally rigid is* $\mathbf{D} = \mathbf{0}$.

Proof. From Taylor's formula, the velocity in a neighborhood of the point $\mathbf{x}_0$ is