Algebra and Applications

F.E.A. Johnson

Syzygies and Homotopy Theory

Springer

Syzygies and Homotopy Theory

For further volumes:
www.springer.com/series/6253

Algebra and Applications

Volume 17

Algebra and Applications aims to publish well written and carefully refereed monographs with up-to-date information about progress in all fields of algebra, its classical impact on commutative and noncommutative algebraic and differential geometry, K-theory and algebraic topology, as well as applications in related domains, such as number theory, homotopy and (co)homology theory, physics and discrete mathematics.

Particular emphasis will be put on state-of-the-art topics such as rings of differential operators, Lie algebras and super-algebras, group rings and algebras, C^*-algebras, Kac-Moody theory, arithmetic algebraic geometry, Hopf algebras and quantum groups, as well as their applications. In addition, Algebra and Applications will also publish monographs dedicated to computational aspects of these topics as well as algebraic and geometric methods in computer science.

F.E.A. Johnson

Syzygies and Homotopy Theory

Springer

Prof. F.E.A. Johnson
Department of Mathematics
University College London
London, UK
feaj@math.ucl.ac.uk

ISSN 1572-5553 e-ISSN 2192-2950
Algebra and Applications
ISBN 978-1-4471-2293-7 e-ISBN 978-1-4471-2294-4
DOI 10.1007/978-1-4471-2294-4
Springer London Dordrecht Heidelberg New York

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Control Number: 2011942989

Mathematics Subject Classification (2000): 16E05, 20C07, 55P15

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

or

Du côté de Chez Swan

To the memory of
my parents

Preface

The underlying motivation for this book is the study of the algebraic homotopy theory of nonsimply connected spaces; in the first instance, the algebraic classification of certain finite dimensional geometric complexes with nontrivial fundamental group G; more specifically, directed towards two basic problems, the $\mathcal{D}(2)$ and $\mathcal{R}(2)$ problems explained below.

The author's earlier book [52] demonstrated the equivalence of these two problems and developed algebraic techniques which were effective enough to solve them for some *finite* fundamental groups ([52], Chap. 12). However the theory developed there breaks down at a number of crucial points when the fundamental group G becomes infinite. In order to consider these problems for general finitely presented fundamental groups the foundations must first be re-built ab initio; in large part the aim of the present monograph is to do precisely that.

The $\mathcal{R}(2)$–$\mathcal{D}(2)$ Problem Having specified the fundamental group, the types of complex we aim to study are, from the point of view of homotopy theory, the simplest finite dimensional complexes which can then be envisaged; namely n-dimensional complexes X with $n \geq 2$ which satisfy

$$\pi_r(\widetilde{X}) = 0 \quad \text{for } r < n, \tag{$*$}$$

where $\widetilde{X}$ is the universal cover of X. These restrictions alone are not sufficient to specify the next homotopy group $\pi_n(\widetilde{X})$; nor, however, is the choice of $\pi_n(\widetilde{X})$ entirely arbitrary. We shall explain in detail throughout the book how to parametrize the possible choices for $\pi_n(\widetilde{X})$ as a module over the group ring $\mathbf{Z}[G]$ and the extent to which an admissible choice determines the homotopy type of X.

Given a complex X as above we can construct the cellular chain complex

$$C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_2} C_0,$$

where $C_r = H_r(\widetilde{X}^r, \widetilde{X}^{r-1}; \mathbf{Z})$ is a free $\mathbf{Z}[G]$-module with basis the r-cells of X. By the Hurewicz theorem, the conditions (∗) above force

$$H_r(C_*) = \begin{cases} \mathbf{Z} & r = 0, \\ 0 & 1 \le r < n, \\ \pi_n(X) & r = n, \end{cases}$$

so that we may extend the above chain complex to an exact sequence

$$C_*(X) = (0 \to \pi_n(\widetilde{X}) \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to \mathbf{Z} \to 0).$$

By an *algebraic n-complex* over $\mathbf{Z}[G]$ we mean an exact sequence of $\mathbf{Z}[G]$-modules

$$A_* = (0 \to J \to A_n \xrightarrow{\partial_n} A_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0 \to \mathbf{Z} \to 0)$$

in which each A_r is finitely generated and free over $\mathbf{Z}[G]$. An algebraic n-complex A_* is said to be *geometrically realizable* when there exists a geometric n-complex X of type (∗) such that $C_*(X) \simeq A_*$. One may then ask the obvious question:

$\mathcal{R}(n)$: Is every algebraic n-complex geometrically realizable?

For $n \ge 3$ the $\mathcal{R}(n)$ problem is answered in the affirmative in Chap. 9. In fact, this is a special case of an older and much more general result of Wall [98]. The question that remains is genuinely problematic:

$\mathcal{R}(2)$: Is every algebraic 2-complex geometrically realizable?

Whilst important in its own right, the $\mathcal{R}(2)$-problem is also of interest via its relation to a notorious and more obviously geometrical problem in low dimensional topology. First make a definition; say that a 3-dimensional cell complex X is *cohomologically 2-dimensional* when $H_3(\widetilde{X}; \mathbf{Z}) = H^3(X; \mathcal{B}) = 0$ for all coefficient systems $\mathcal{B}$ on X. The problem may then be stated as follows:

$\mathcal{D}(2)$: Let X be a finite connected cell complex of geometrical dimension 3 which is cohomologically 2-dimensional. Is X is homotopy equivalent to a finite complex of geometrical dimension 2?

Both $\mathcal{D}(2)$ and $\mathcal{R}(2)$ problems are parametrized by the fundamental group under discussion; each finitely presented group G has its own $\mathcal{D}(2)$ problem and its own $\mathcal{R}(2)$ problem. Moreover, for a given fundamental group G the $\mathcal{D}(2)$ problem is entirely equivalent to the $\mathcal{R}(2)$ problem; to solve one is to solve the other. This equivalence was shown by the present author in [51, 52], subject to a mild condition on G which was subsequently shown to be unnecessary by Mannan [71].

This book is in two parts, Theory and Practice. In this Preface we give a brief outline of the theory; a summary of the practical aspects is given in the Conclusion.

The Method of Syzygies The basic model in the theory of modules is the theory of vector spaces over a field. However, the modules encountered in this book are defined over more general rings and in dealing with them it is useful to keep in mind how far one is being forced to deviate from the basic paradigm.

Linear algebra over a field is rendered tractable by the fact that every module over a field is free; that is, has a spanning set of linearly independent vectors. General module theory takes as its point of departure the observation that when a module M is not free we may at least make a first approximation to its being free by taking a surjective homomorphism $\varphi : F_0 \to M$ where F_0 is free to obtain an exact sequence

$$0 \to K_1 \to F_0 \xrightarrow{\varphi} M \to 0.$$

We find it instructive to regard the kernel K_1 as *a first derivative* of M. Setting aside temporarily the question of uniqueness one may repeat the construction and approximate K_1 in turn by a free module to obtain an exact sequence

$$0 \to K_2 \to F_1 \to K_1 \to 0.$$

Iterating we obtain a long exact sequence

$$\begin{array}{ccccccccccccc} \xrightarrow{\partial_{n+1}} F_n & \xrightarrow{\partial_n} & F_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots & \xrightarrow{\partial_3} F_2 & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 \to \mathrm{M} \to 0 \\ \searrow & K_n & \nearrow & & & \searrow & K_2 & \nearrow \searrow & K_1 & \nearrow \end{array}$$

Thus arises the notion of *free resolution*, made famous by the work of Hilbert on Invariant Theory [43]. The intermediate modules K_n are called the *syzygies of* M. Indeed, the etymology ($\sigma\upsilon\zeta\upsilon\gamma o\zeta$ = yoke) is determined by the conventional view that the K_n are connections in this sense. Nevertheless, we prefer to regard them as objects in their own right, as *derivatives of* M. Before doing this, however, we must first answer the question we have avoided; to what extent are they unique?

At one level the most simple minded considerations show that they *cannot possibly* be unique; given an exact sequence

$$0 \to K_1 \to F_0 \xrightarrow{\varphi} M \to 0$$

then by stabilizing the middle term thus $0 \to K_1 \oplus \Lambda \to F_0 \oplus \Lambda \xrightarrow{\varphi} M \to 0$ it is clear that if K_1 is to be considered as a first derivative of M then $K_1 \oplus \Lambda$ must also be so considered. So much must have been apparent to Hilbert. Even so, it is clear that the pioneers of the subject considered that the syzygies *ought*, somehow, to be unique. In the original context of Invariant Theory [28] this can be made to work if the resolution is, in some sense, minimal. In our context, as we shall see, the notion of 'uniqueness via minimality' fails badly. However there is indeed a sense in which the syzygies are uniquely specified, and it is to this we now turn.

Stable Modules and Schanuel's Lemma According to legend, in the autumn of 1958, during a lecture of Kaplansky at the University of Chicago, Stephen Schanuel, then still an undergraduate, observed that if we are given exact sequences of modules over a ring Λ

$$0 \to K \to \Lambda^n \stackrel{\varphi}{\to} M \to 0;$$

$$0 \to K' \to \Lambda^m \stackrel{\varphi}{\to} M \to 0$$

then $K \oplus \Lambda^m \cong K' \oplus \Lambda^n$. In fact, Schanuel proved slightly more than this; however it suggests that given Λ-modules K, K' we should write:

$$K \sim K' \iff K \oplus \Lambda^m \cong K' \oplus \Lambda^n \quad \text{for some positive integers } m, n.$$

When this happens we say that K, K' are *stably equivalent*. The relation '$\sim$' is an equivalence relation on Λ modules and, applied to the above exact sequences, Schanuel's Lemma shows that $K \sim K'$; it is in this sense that syzygies are unique.

Schanuel's Lemma explains neatly why the attempt to force uniqueness of the syzygy modules by minimising the resolution is, in general, doomed to failure. Thus suppose that m is the minimum number of generators of the Λ-module M and suppose given exact sequences

$$0 \to K \to \Lambda^m \stackrel{\varphi}{\to} M \to 0;$$

$$0 \to K' \to \Lambda^m \stackrel{\varphi}{\to} M \to 0.$$

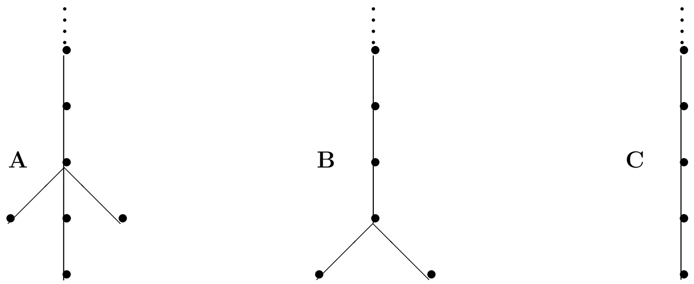
Schanuel's Lemma then tells us that $K \oplus \Lambda^m \cong K' \oplus \Lambda^m$. We are left to solve the following:

Cancellation Problem Does $K \oplus \Lambda^m \cong K' \oplus \Lambda^m$ imply that $K \cong K'$?

In dealing with modules over integral group rings the expected answer is 'No'; as we shall see, cancellation is the exception not the rule. The failure of cancellation may be starkly portrayed by representing the stable module $[K]$ as a graph.

When M is a finitely generated Λ-module, the stable module $[M]$ has the structure of a directed graph in which the vertices are the isomorphism classes of modules $N \in [M]$ and where we draw an edge $N_1 \to N_2$ when $N_2 \cong N_1 \oplus \Lambda$. We will show, in Chap. 1, that $[M]$ is a 'tree with roots that do not extend infinitely downwards'. This graphical method of representing stable modules is due to Dyer and Sieradski [24].

The extent to which cancellation fails in $[M]$ is captured by the amount of branching. We illustrate the point with some examples; **A** below represents a tree with a single root and no branching above level two; **B** represents a tree with two roots but with no branching above level one; **C** represents a tree with a single root and no branching whatsoever. Cancellation holds in **C** but fails in both **A** and **B**.



A significant difference between finite and infinite groups is the extent of our knowledge of the branching behaviour in stable modules over $\mathbf{Z}[G]$. When G is finite, the Swan-Jacobinski Theorem [46, 93] imposes severe restrictions on the type of branching that may occur; for example, the odd syzygies $\Omega_{2n+1}(\mathbf{Z})$ can behave only like **B** and **C** with possibly multiple roots but with no branching above level one; the even syzygies $\Omega_{2n}(\mathbf{Z})$ may resemble any of the three types but nothing worse. By contrast, when G is infinite very little is known in detail about the levels at which a stable module over $\mathbf{Z}[G]$ may branch.[1] We explore this question for some familiar infinite groups starting with the most basic case, namely the stable class of 0.

Iterated Fibre Squares and Stably Free Modules In passing from finite groups to infinite groups the first point of difference is the increased incidence of non-cancellation. For finite Φ non-cancellation over $\mathbf{Z}[\Phi]$ is comparatively rare. By the theorem of Swan and Jacobinski, it can only occur when the real group ring

$$\mathbf{R}[\Phi] \cong \prod_{i=1}^{m} M_{d_i}(\mathcal{D}_i)$$

fails the *Eichler condition*; that is when for some i, $d_i = 1$ and $\mathcal{D}_i = \mathbf{H}$ is the division ring of Hamiltonian quaternions. However, the proof of the Swan-Jacobinski theorem does not survive the passage to infinite groups and so we are forced to fall back on other methods.

The approach which has proved profitable is the method of iterated fibre squares which was used by Swan in [94] to consider the *extent* to which non-cancellation fails in finite groups which fail the Eichler condition. We elaborate the necessary theory of fibre squares in Chap. 3. As a working method it proceeds like this; take a convenient finite group Φ and establish the cancellation properties of $\mathbf{Z}[\Phi]$ from first principles by using the method of fibre squares. Now generalize the statement, replacing $\mathbf{Z}[\Phi]$ by $R[\Phi]$; on taking $R = \mathbf{Z}[G]$ where G is infinite one hopes to analyze the cancellation properties of $R[\Phi] \cong \mathbf{Z}[G \times \Phi]$. Some successful attempts are exhibited in Chaps. 10 through 12.

[1] Although over more general rings, for example the coordinate rings of spheres, the pattern of branching away from the main stem may be very complicated.

The Derived Module Category We have set ourselves the task of classifying algebraic complexes and, in particular, algebraic 2-complexes. To see the relevance of syzygies for this, suppose given a Λ-module M and write $\Omega_n(M)$ for the stable class any nth-syzygy of M; then we may portray an algebraic 2-complex formally as

$$\begin{array}{ccccccccccccc} 0 & \to & \Omega_3(\mathbf{Z}) & \to & F_2 & & \xrightarrow{\partial_2} & & F_1 & & \xrightarrow{\partial_1} & & F_0 \to \mathbf{Z} \to 0 \\ & & & & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & & & & & \Omega_2(\mathbf{Z}) & & & & \Omega_1(\mathbf{Z}) & & \end{array}$$

showing, in particular, that when X is a connected geometric 2-complex with $\pi_1(X) = G$ the $\mathbf{Z}[G]$-module $\pi_2(\widetilde{X})$ is constrained to lie in the third syzygy $\Omega_3(\mathbf{Z})$.

The Ω_n formalism was first introduced by Heller in the context of modular representations of finite groups [39]. In that restricted setting it is relatively easy, with suitable interpretations, to regard the correspondence $M \mapsto \Omega_n(M)$ as a functor. In more general contexts attempting to make Ω_n functorial involves additional technical complications.

The first question to be answered is '*In what category is* $\Omega_n(M)$ *supposed to live*?' As a first approximation we take the quotient of the category $\mathcal{M}\mathrm{od}_\Lambda$ of Λ-modules obtained by ignoring morphisms which factorize through a free module; more precisely, we equate morphisms whose difference factorizes through a free module; that is if $f, g : M \to N$ are Λ-homomorphisms we write '$f \approx g$' when $f - g$ can be written as a composite $f - g = \xi \circ \eta$ as below where F is a free module:

$$\begin{array}{ccccc} M & & \xrightarrow{f-g} & & N \\ & {\scriptstyle\eta}\searrow & & \nearrow{\scriptstyle\xi} & \\ & & F & & \end{array}$$

The quotient category $\mathcal{D}\mathrm{er}(\Lambda) = \mathcal{M}\mathrm{od}_\Lambda / \approx$ is called the *derived module category*. It is too crude an approximation, if only on the basis of size for, as we have imposed no size restrictions, our modules can be arbitrarily large. We can attempt to restrict all definitions to apply only to finitely generated modules; thus if N is a module we say that its stable class $[N]$ is finitely generated when N is finitely generated; in that case, *any* module in $[N]$ is also finitely generated. In the original context of modular representation theory, such size restriction causes no difficulty. In our more general context however, the difficulty arises that if M is finitely generated then $\Omega_n(M)$ need not be. To restrict attention to rings where this behaviour does not occur would exclude the integral group rings $\mathbf{Z}[G]$ of many interesting groups [53] (See Appendix D).

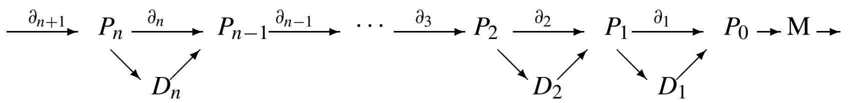
However, under a mild restriction on the ring,[2] if M is countably generated so also is $\Omega_n(M)$; then restricting all definitions to apply only to countably generated modules yields a derived module category $\mathcal{D}\mathrm{er}_\infty(\Lambda)$ of realistic size.

[2]Weak coherence. See Chap. 1.

There is, however, a complication more subtle than mere size. Recall that any projective module is a direct summand of a free module. Thus the above condition '$f \approx g$' is equivalent to the requirement that $f - g$ factors through a projective. This has the eventual consequence for modules K, K' over Λ that

$$K \cong_{\mathcal{D}\mathrm{er}} K' \iff K \oplus P \cong_{\Lambda} K' \oplus P'$$

for some projective modules P, P'; that is, isomorphism classes in $\mathcal{D}$er correspond not to stability classes of modules but, in MacLane's terminology, to *projective equivalence classes*[3] ([68], p. 101). Moreover, this applies even when all modules under consideration are finitely generated. In the original context of modular representation theory all projective modules are free, there is no distinction between stability and projective equivalence and Ω_n defines a functor on the derived module category. However, in general, to obtain functoriality one must consider not Ω_n but rather its analogue using the appropriate notion of *generalized syzygy*; disregarding finiteness restrictions and taking the successive kernels in a projective resolution $\mathcal{P}$



the correspondence $M \mapsto D_n$ gives a functor $D_n : \mathcal{D}\mathrm{er}_\infty \to \mathcal{D}\mathrm{er}_\infty$. As classes of modules $\Omega_n(M) \subset D_n(M)$ and we may regard $\Omega_n(M)$ as a sort of *polarization state* of $D_n(M)$. We note that for most computational purposes we may legitimately revert to $\Omega_n(M)$ as $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_n(M), N) \equiv \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(D_n(M), N)$.

Eliminating Injectives In the late 1940s the introduction of Eilenberg-Maclane cohomology as the *derived functors* of Hom completely transformed module theory. The indeterminate nature of syzygies was replaced by the definiteness of computable invariants. In the aftermath the syzygetic method, insofar as it was still pursued, was regarded as an unwelcome reminder of a more primitive past. For us now, however, its rehabilitation via the derived module category raises the question of relating syzygies directly to cohomology.

Here we encounter a difficulty which is inherent in the cohomological method itself. In the standard treatments it is shown that one may compute the derived functor of $\mathrm{Hom}(-,-)$ *either* by taking a projective resolution in the first variable *or*, *equally*, by taking an injective co-resolution in the second. Moreover, this symmetry is not a point of esoteric scholarship, or at least, not merely so. With each variable one has a long exact sequence obtained by systematic appeal to the properties of the appropriate type of module. Which leads us back to the two sorts of modules themselves.

---

[3]For countably generated modules it is technically more convenient to replace the relation of projective equivalence by the equivalent notion of *hyperstable equivalence*, which is to say that $K \oplus \Lambda^\infty \cong_\Lambda K' \oplus \Lambda^\infty$. But again, see Chap. 1.

Projective modules, as direct summands of free modules, were in common use[4] before the name was ever applied to them; however the history and nature of injective modules is entirely different. Whereas projective modules are unavoidable, injective modules are a deliberate contrivance, only introduced to have arrow-theoretic properties dual to those of projectives [6]. Whereas projective modules are natural, injective modules are formal. Whereas projective modules are constructible (and we shall show how to construct some of them) injective modules are essentially non-constructible. One needs a theorem to show they exist. Except in the most elementary cases, where the point is irrelevant, they are not describable by any *effective* process. In our context this last point is the most pressing; injectives are so different from the objects with which we must deal that, arguments of formal simplicity notwithstanding, the need to dispense with them becomes insistent.[5]

The elimination of magic from homological algebra, in this case the avoidance of injective modules, forces us in every case to use projective resolutions. Whilst dispensing with the dualising services of injectives it is nevertheless essential to employ some form of homological duality which, however weak, can be confined entirely within the 'projective quotient' category. In fact, this requirement has a precedent as does the remedy; in the cohomology of lattices over finite groups the dual arrow theoretic properties of projectives are possessed by projectives themselves. Thus one may dispense with injectives entirely and describe the theory solely in terms of projectives. This is *Tate cohomology*, a point to which we will return. Our solution is comparable but not quite so convenient.

Corepresentability of Cohomology The appropriate notion, which we shall use systematically, is that of 'coprojectivity'; a module M is said to be *coprojective* when $\mathrm{Ext}^1(M, \Lambda) = 0$. To see how coprojectivity works take an exact sequence $\mathcal{E} = (0 \to K \stackrel{i}{\to} F \stackrel{\varphi}{\to} M \to 0)$ where F is free so that K is a first syzygy of M; if $\alpha : K \to N$ is a Λ-homomorphism one may form the pushout diagram

$$\begin{array}{c} \mathcal{E} \\ \downarrow \\ \alpha_*(\mathcal{E}) \end{array} \quad c = \begin{pmatrix} 0 \to K \stackrel{i}{\to} & F \stackrel{\varphi}{\to} & M \to 0 \\ \downarrow \alpha & \downarrow \nu & \downarrow \mathrm{Id} \\ 0 \to N \to & \varinjlim(\alpha, i) \to & M \to 0 \end{pmatrix}$$

from which we obtain the *connecting homomorphism* $\delta : \mathrm{Hom}_\Lambda(K, N) \to \mathrm{Ext}^1(M, N)$ by means of $\delta([\mathcal{E}]) = [\alpha_*(\mathcal{E})]$. When M is coprojective (and not otherwise) δ descends to give a natural equivalence $\delta : \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(K, -) \to \mathrm{Ext}^1(M, -)$ so that we may write

$$\mathrm{Ext}^1(M, -) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_1(M), -).$$

[4] For example in Wedderburn theory.

[5] The disadvantages, *for any practical purpose*, of an object about which one has to think hard before even being able to admit its existence ought to be obvious. Doubtless some will regret this as yet another instance of a depressing but universal trend; in Weber's succinct phrase 'The elimination of Magic from the World' ([99], p. 105).

In other-words, when M is coprojective, $\Omega_1(M)$ is a *corepresenting object* for $\mathrm{Ext}^1(M,-)$[6] considered as a functor on the derived module category. More generally, in higher dimensions there is a corresponding corepresentation theorem

$$H^n(M,-) \cong \mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_n(M),-)$$

which holds provided that $H^n(M,\Lambda)=0$. That is, we have replaced the *derived functor* H^n by the *derived object* Ω_n. Corepresenting cohomology in this way is the first step towards geometrizing extension theory so as to be able to apply it to the question of realizing algebraic complexes. Moreover, the groups $\mathrm{Hom}_{\mathcal{D}\mathrm{er}}(\Omega_n(M),N)$ are then natural generalizations of the Tate cohomology groups defined for modules over finite groups.

Homotopy Classification and the Swan Homomorphism The problem of classifying algebraic complexes up to homotopy equivalence may be compared with the simpler Yoneda theory of module extensions up to congruence [68, 101]. For a specified fundamental group G let $\mathbf{Alg}_n(\mathbf{Z})$ denote the set of homotopy types of algebraic n-complexes of the form

$$A_* = (0 \to J \to A_n \to A_{n-1} \to \cdots \to A_0 \to \mathbf{Z} \to 0).$$

The stabilization $\Sigma_+(A_*)$ is obtained by adding $\Lambda = \mathbf{Z}[G]$ to the final two terms thus

$$\Sigma_+(A_*) = (0 \to J \oplus \Lambda \to A_n \oplus \Lambda \to A_{n-1} \to \cdots \to A_0 \to \mathbf{Z} \to 0)$$

and $\mathbf{Alg}_n(\mathbf{Z})$ also acquires a tree structure by drawing arrows $A_* \to \Sigma_+(A_*)$. Moreover the correspondence $A_* \mapsto J$ defines a mapping of trees, 'algebraic π_n',

$$\pi_n : \mathbf{Alg}_n(\mathbf{Z}) \to \Omega_{n+1}(\mathbf{Z}).$$

In his unpublished paper [12] Browning described the fibres $\pi_2 : \mathbf{Alg}_2(\mathbf{Z}) \to \Omega_3(\mathbf{Z})$ for those finite groups G which satisfy the Eichler condition. In [52], generalizing a criterion of Swan [91], we showed, still within the confines of finite groups, how to circumvent dependence on the Eichler condition and gave a rather different description of the fibres of π_2. Here we show how to extend the description of [52] to a much wider class of rings.[7]

A significant difficulty lies in being able to generalize the Swan mapping. In the original version [91] the homomorphism property of the Swan mapping is an easy consequence of special circumstances; in the wider context it is less obvious. Again

[6]Notice that the blank space would normally have to be co-resolved by means of injectives; the coprojectivity hypothesis removes this necessity.

[7]We note that a very special case of our classification theorem, for algebraic n-complexes over the group rings of n-dimensional Poincaré Duality groups ($n \geq 4$), was given by Dyer in [23].

take an exact sequence $\mathcal{E} = (0 \to J \stackrel{i}{\to} F \stackrel{\varphi}{\to} M \to 0)$ where F is free; if $\alpha : J \to J$ is a Λ-homomorphism one may again form the pushout diagram

$$\begin{array}{ccc} J & \stackrel{i}{\to} & F \\ \downarrow \alpha & & \downarrow \nu \\ J & \to & \varinjlim(\alpha, i) \end{array}$$

It turns out (*Swan's projectivity criterion*) that $\varinjlim(\alpha, i)$ is projective precisely when α is an isomorphism in $\mathcal{D}\mathrm{er}$. When M and J are finitely generated one obtains a mapping

$$S : \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J) \to \widetilde{K_0}(\Lambda)$$

to the reduced projective class group of Λ. This is the generalized Swan mapping and is, nontrivially, a homomorphism. This result was first shown in [56]. Moreover, despite the apparent dependence upon J, when M is coprojective it depends only upon M and is independent of the sequence $\mathcal{E}$ used to produce it. More generally, if

$$0 \to J \to A_n \to A_{n-1} \to \cdots \to A_0 \to \mathbf{Z} \to 0$$

is an algebraic n-complex and $H^{n+1}(M, \Lambda) = 0$ the same mapping $S : \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J) \to \widetilde{K_0}(\Lambda)$ again reappears independently of the sequence used to produce it. By contrast, however, the natural mapping $\nu_J : \mathrm{Aut}_\Lambda(J) \to \mathrm{Aut}_{\mathcal{D}\mathrm{er}}(J)$ is heavily dependent on J. The detailed homotopy classification of algebraic n-complexes over M requires a knowledge of the cosets $\mathrm{Ker}(S)/\mathrm{Im}(\nu_J)$ as J runs through $\Omega_{n+1}(M)$.

Imposing the coprojectivity condition or its higher dimensional analogues does, of course, restrict the range of applicability of the theory. In practice it is not too serious; for example, the classification of algebraic 2-complexes over $\mathbf{Z}[G]$ requires us to impose the condition

$$H^3(\mathbf{Z}, \mathbf{Z}[G]) = 0.$$

This condition is satisfied in many familiar cases; in particular, when G is a virtual duality group of virtual dimension n it is satisfied whenever $n \neq 3$.

Parametrizing the First Syzygy In applying the classification theorem to our original problem one needs specific information about the syzygies $\Omega_n(\mathbf{Z})$. In practice, this is a matter of severe computational difficulty. At the time of writing, the only finite fundamental groups for which there are complete descriptions for *all* $\Omega_n(\mathbf{Z})$ are certain groups of periodic cohomology. For infinite fundamental groups the situation is far worse.

In the first instance we are content to study $\Omega_1(\mathbf{Z})$. Here we find that the branching properties at the minimal level are intimately related to the existence of stably free modules; that is, to the stable class of the zero module. When G is infinite and $\mathrm{Ext}^1(\mathbf{Z}, \mathbf{Z}[G]) = 0$ we show that the stably free modules describe a lower bound for the branching behaviour in $\Omega_1(\mathbf{Z})$ and give a complete description of the minimal level $\Omega_1^{\min}(\mathbf{Z})$. This is done in Chap. 13.

Finally, in the most familiar case where $\mathrm{Ext}^1(\mathbf{Z}, \mathbf{Z}[G]) \neq 0$, namely when $G \cong F_n \times C_m$, we give a complete description of all the *odd* syzygies $\Omega_{2n+1}(\mathbf{Z})$. By way of illustration we conclude the book with Edwards' solution [25, 26] of the $\mathcal{R}(2)$ problem for the groups $C_\infty \times C_m$.

Acknowledgments The author wishes to express his thanks to his colleagues Dr. R.M. Hill and Dr. M.L. Roberts; the former for his insights into cyclotomic fields; the latter for some helpful discussions on free ideal rings.

The author has had the advantage of being able to rehearse the theory presented here over a number of years to the captive audience of his students; in alphabetical order: Tim Edwards, Susanne Gollek, Jodie Humphreys, Pouya Kamali, Daniel Laydon, Wajid Mannan, Dominique Miranda, Jamil Nadim, Seamus O'Shea, Jonathan Remez, Isidoros Strouthos. The theory has gained thereby, not only in clarity from their perceptive comments but also, as will be seen in the text, in substance from some original and significant contributions.

London, England F.E.A. Johnson

Contents

Part I
Theory

Chapter 1
Preliminaries

Many of the arguments in this book are formulated in terms of modules over the group ring $\mathbf{Z}[G]$ where G is a specified fundamental group. Thus, in part, this book is concerned with the general theory of modules and so, by association, with the general theory of rings. Given the pathology of which the subject is capable there is a tendency, frequently indulged in the literature, to present Ring Theory as a menagerie of wild beasts with strange and terrifying properties. Regardless of appearances that is not our aim here. The rings we consider are comparatively well behaved. However, in order to explain quite how well behaved we are forced to discuss a small amount of pathology if only to say what delinquencies we need not tolerate.

1.1 Restrictions on Rings and Modules

The rings we encounter are typically, though not exclusively, integral group rings. In principle we would prefer simply to say that the rings we meet will have properties which are no worse than the worst behaviour one can expect from $\mathbf{Z}[G]$ where G is a finitely presented group; but of course we must be more precise than that. The first restriction we impose is the *invariant basis number property* (= IBN); that is, for positive integers a, b:

$$\Lambda^a \cong \Lambda^b \quad \Longrightarrow \quad a = b. \tag{IBN}$$

Although this condition is a definite restriction it is too weak for many purposes and there are two progressively stronger notions which are more useful; the first is the *surjective rank property* (= SR):

$$\text{If } \varphi : \Lambda^N \to \Lambda^n \text{ is a surjective } \Lambda\text{-homomorphism then, } n \le N. \tag{SR}$$

Finally we have the so-called *weak finiteness* property (= WF).

$$\text{If } \varphi : \Lambda^a \to \Lambda^a \text{ is a surjective } \Lambda\text{-homomorphism then } \varphi \text{ is bijective.} \tag{WF}$$

F.E.A. Johnson, *Syzygies and Homotopy Theory*, Algebra and Applications 17,
DOI 10.1007/978-1-4471-2294-4_1, © Springer-Verlag London Limited 2012

It is straightforward to see that WF $\Longrightarrow$ SR $\Longrightarrow$ IBN. In [15] Cohn shows that if there exists a ring homomorphism $\Lambda \to \mathbf{F}$ to a field then Λ has the SR property. Thus if A is a commutative ring then *any group ring* $A[G]$ satisfies SR. Furthermore, in addition to possessing the SR property, for any group G the integral group ring $\mathbf{Z}[G]$ also satisfies WF. The main details of a proof of this last were outlined in a paper of Montgomery [75].

For reasons explained below, we also impose the following very mild restriction:

Weak Coherence If M is a countably generated Λ-module and $N \subset M$ is a Λ-submodule then N is also countably generated.

We denote by $\mathcal{M}\mathrm{od}_\Lambda$ the category of right Λ-modules and by $\mathcal{M}\mathrm{od}_\infty$ the full subcategory of countably generated modules; $\mathcal{M}\mathrm{od}_\infty$ is then equivalent to a small category. The force of imposing the weak coherence condition is that $\mathcal{M}\mathrm{od}_\infty$ becomes an abelian category in the formal sense of [74].

There is a stronger notion; let $\mathcal{M}\mathrm{od}_{fp}(= \mathcal{M}\mathrm{od}_{fp}(\Lambda))$ denote the category of *finitely presented* right Λ-modules; Λ is said to be *coherent* when $\mathcal{M}\mathrm{od}_{fp}$ is an abelian category. Ideally one would like to impose this stronger condition. However, to do so would exclude too many significant examples.

Clearly every countable ring is weakly coherent. Hence, the integral group ring $\mathbf{Z}[G]$ of any countable group G is weakly coherent. By contrast, coherence is a far less common property. Admittedly, if G is finite then $\mathbf{Z}[G]$ is coherent; however, there are many finitely presented infinite groups G where $\mathbf{Z}[G]$ fails to be coherent, even some which satisfy otherwise strong geometrical finiteness conditions. For example, if G contains a direct product of two nonabelian free groups then $\mathbf{Z}[G]$ fails to be coherent. The topic is considered further in Appendix D.

Finally, we need to mention duality. We set out with the intention of always working with right modules. Over general rings, this is not possible if one wants also to deal with duality, for if M is a right Λ-module then the dual module $\mathrm{Hom}_\Lambda(M, \Lambda)$ is naturally a left module via the action

$$\begin{aligned} \bullet : \Lambda \times \mathrm{Hom}_\Lambda(M, \Lambda) &\to \mathrm{Hom}_\Lambda(M, \Lambda) \\ (\lambda \bullet f)(x) &= \lambda f(x) \end{aligned}$$

In general there is no way around this; there exist rings in which the category of left modules is not equivalent to the category of right modules. However, in the case of group rings $\Lambda = \mathbf{Z}[G]$ we can circumvent this difficulty by the familiar device of converting left modules back to right modules

$$\begin{aligned} * : \mathrm{Hom}_\Lambda(M, \Lambda) \times \Lambda &\to \mathrm{Hom}_\Lambda(M, \Lambda) \\ f * \lambda &= \overline{\lambda} \bullet f \end{aligned}$$

via the canonical (anti)-involution $\overline{g} = g^{-1}$. More generally one may do this whenever the ring Λ has a distinguished (anti)-involution. With this convention the dual module $\mathrm{Hom}_\Lambda(M, \Lambda)$ so equipped as a right module is denoted by M^*.

1.2 Stable Modules and Tree Structures

Let Λ be a ring with the surjective rank property SR of Sect. 1.1. We denote by '$\sim$' the stability relation on Λ modules; that is

$$M_1 \sim M_2 \iff M_1 \oplus \Lambda^{n_1} \cong M_2 \oplus \Lambda^{n_2}$$

for some integers $n_1, n_2 \geq 0$; the relation '$\sim$' is an equivalence on isomorphism classes of Λ-modules. For any Λ-module M, we denote by $[M]$ the corresponding *stable module*; that is, the set of isomorphism classes of modules N such that $N \sim M$. One sees easily that:

$$M \text{ is finitely generated if and only if each } N \in [M] \text{ is finitely generated.} \tag{1.1}$$

When M is a *nonzero* finitely generated Λ-module we define the Λ-rank of M by

$$\mathrm{rk}_\Lambda(M) = \min\{a \in \mathbf{Z}_+ \text{ for which there is a surjective } \Lambda\text{-homomorphism } \varphi : \Lambda^a \to M\}.$$

Proposition 1.2 *If $N \in [M]$ then for each integer $a > 0$, $N \oplus \Lambda^a \ncong N$.*

Proof Put $\mu = \mathrm{rk}_\Lambda(N)$ and let $\varphi : \Lambda^\mu \to N$ be a surjective homomorphism. If $N \cong N \oplus \Lambda^a$ for some $a \geq 1$ then for all $k \geq 1$, $N \cong N \oplus \Lambda^{ka}$. Choose $k \geq 1$ such that $\mu < ka$. Let $h_k : N \to N \oplus \Lambda^{ka}$ be an isomorphism and let $\pi_k : N \oplus \Lambda^{ka} \to \Lambda^{ka}$ be the projection. Then $\pi_k \circ h_k \circ \mu : \Lambda^\mu \to \Lambda^{ka}$ is a surjective homomorphism and $\mu < ka$. This is a contradiction, hence $N \ncong N \oplus \Lambda^a$ when $a \geq 1$. □

We define a function $g : [M] \times [M] \to \mathbf{Z}$, the 'gap function' as follows

$$g(N_1, N_2) = g \iff N_1 \oplus \Lambda^{a+g} \cong N_2 \oplus \Lambda^a,$$

where both a and $a + g$ are positive integers.We must first show that:

Proposition 1.3 *g is a well defined function.*

Proof Suppose that $N_1 \oplus \Lambda^p \cong N_2 \oplus \Lambda^q$ and also that $N_1 \oplus \Lambda^r \cong N_2 \oplus \Lambda^s$. We will show

$$p - q = r - s. \tag{*}$$

To see this, observe that $N_1 \oplus \Lambda^{p+r} \cong N_2 \oplus \Lambda^{q+r}$ and that $N_1 \oplus \Lambda^{p+r} \cong N_2 \oplus \Lambda^{p+s}$. Thus

$$N_2 \oplus \Lambda^{q+r} \cong N_2 \oplus \Lambda^{p+s}.$$

Suppose that $q + r \neq p + s$. Then without loss of generality we may suppose that $p + s < q + r$. Putting $N_3 = N_2 \oplus \Lambda^{p+s}$ and $\alpha = q + r - (p + s)$ we see that

$N_3 \oplus \Lambda^\alpha \cong N_3$ where $\alpha > 0$. This contradicts Proposition 1.2 above. Hence $q + r = p + s$ and so $p - q = r - s$ as claimed. □

It is straightforward to check that

$$g(N, N \oplus \Lambda^b) = b, \tag{1.4}$$

$$g(N_2, N_1) = -g(N_1, N_2), \tag{1.5}$$

$$g(N_1, N_3) = g(N_1, N_2) + g(N_2, N_3). \tag{1.6}$$

Lemma 1.7 *Let Λ be a ring with the surjective rank property and let M be a finitely generated Λ-module; if $K \in [M]$ is such that $0 \le g(K, M)$ then $g(K, M) \le \mathrm{rk}_\Lambda(M)$.*

Proof Put $m = \mathrm{rk}_\Lambda(M)$ and let $\varphi : \Lambda^m \to M$ be a surjective Λ-homomorphism. Suppose that $K \in [M]$ is such that $0 \le g(K, M) = k$ and let $h : M \oplus \Lambda^a \to K \oplus \Lambda^{a+k}$ be an isomorphism. If $\pi : K \oplus \Lambda^{a+k} \to \Lambda^{a+k}$ is the projection then $\pi \circ h \circ (\varphi \oplus \mathrm{Id}) : \Lambda^{m+a} \to \Lambda^{a+k}$ is also a surjective homomorphism. Hence by the surjective rank property for Λ, $a + k \le a + m$ and so $k \le m$ as claimed. □

We say that a module $M_0 \in [M]$ is a *root module* for $[M]$ when $0 \le g(M_0, K)$ for all $K \in [M]$. We show:

Theorem 1.8 *Let Λ be a ring with the surjective rank property and let M be a finitely generated Λ-module; then $[M]$ contains a root module.*

Proof If $K \in [M]$, either $g(K, M) < 0$ or, by above, $0 \le g(K, M)$ and $g(K, M) \le \mathrm{rk}_\Lambda(M)$. Either way

$$g(K, M) \le \mathrm{rk}_\Lambda(M),$$

and the mapping $K \mapsto g(K, M)$ gives a function $[M] \to \mathbf{Z}$ which is bounded above by $\mathrm{rk}_\Lambda(M)$. Thus there exists $M_0 \in [M]$ which maximises this function; that is,

$$g(M_0, M) = \max\{g(K, M) : K \in [M]\}.$$

We claim that for all $K \in [M]$, $0 \le g(M_0, K)$. Otherwise, if there exists $K \in [M]$ such that $g(M_0, K) < 0$ then $g(K, M_0) < 0$ and so

$$g(K, N) = g(K, M_0) + g(M_0, N) > g(M_0, N)$$

which contradicts the choice of M_0. Thus $0 \le g(M_0, K)$ for all $K \in [M]$, and M_0 is a root module as claimed. □

If M_0 is a root module for $[M]$ we may define a height function $h : [M] \to \mathbf{N}$ by

$$h(L) = g(M_0, L).$$

Whilst ostensibly the height function depends upon M_0, in fact it is intrinsic to the stable module $[M]$; to see this, suppose that M_0 and M_0' are both root modules for $[M]$ and consider the respective height functions $h(L) = g(M_0, L)$ and $h'(L) = g(M_0', L)$. From (1.6) above $g(M_0, L) = g(M_0, M_0') + g(M_0', L)$ so that

$$h(L) = g(M_0, M_0') + h'(L).$$

However $g(M_0, M_0') = h(M_0') \geq 0$ whilst $g(M_0, M_0') = -g(M_0', M_0) = -h'(M_0) \leq 0$. Thus $g(M_0, M_0') = 0$ and so

$$h(L) = h'(L).$$

When the ring Λ has the surjective rank property and M is a finitely generated Λ-module we may speak unequivocally of *the height function* $h : [M] \to \mathbf{N}$ on the stable module $[M]$.

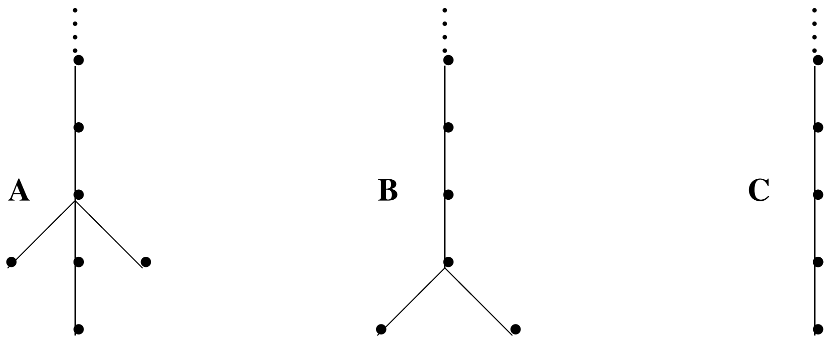
When M is a finitely generated Λ-module, the stable module $[M]$ has the structure of a graph in which the vertices are the isomorphism classes of modules $N \in [M]$ and where we draw an edge $N_1 \to N_2$ when $N_2 \cong N_1 \oplus \Lambda$. Recall that a graph is said to be a *tree* when it contains no nontrivial loop. Since each module $N \in [M]$ has a unique arrow which exits the vertex represented by N, namely the arrow $N \to N \oplus \Lambda$, it follows that the only way of having a non trivial loop in $[M]$ would be if $N \cong N \oplus \Lambda^a$ for some $a > 0$. However, this possibility is precluded by Proposition 1.2, so that we have:

Proposition 1.9 *Let Λ be a ring having the surjective rank property; if M is a finitely generated module over Λ then $[M]$ is an infinite (directed) tree.*

Without attempting any more precise characterization of the (directed) tree structures which may arise in this way, it is evident that they are good deal more specialised than indicated by the statement of Proposition 1.9. For example, we have already observed that a unique arrow exits any vertex. Furthermore, the existence of root modules and the associated existence of a height function $h : [M] \to \mathbf{N}$ implies that $[M]$ may be represented as a 'tree with roots'. In particular if we regard the integers $\mathbf{Z}$ as a directed tree in the obvious way, namely:

$$\mathbf{Z} = (\cdots \to -(\mathbf{n}+\mathbf{1}) \to -\mathbf{n} \to \cdots \to -\mathbf{1} \to \mathbf{0} \to \mathbf{1} \to \cdots \to \mathbf{n} \to (\mathbf{n}+\mathbf{1}) \to \cdots)$$

then it is an easy deduction from the height function, as constructed on $[M]$, that $\mathbf{Z}$ does not imbed in $[M]$. We may paraphrase this by saying that the roots of $[M]$ *do not extend infinitely downwards*. To illustrate the point consider again the tree diagrams noted in the Introduction; **A** below represents a tree with a single root and no branching above level two; **B** represents a tree with two roots but with no branching above level one; **C** represents a tree with a single root and no branching whatsoever.



These examples all actually arise; denoting the quaternion group of order $4n$ by $Q(4n)$ then **A** represents the stable class of 0 over the integral group ring $\mathbf{Z}[Q(24)]$ whilst **B** represents the stable class $\Omega_3(\mathbf{Z})$ over $\mathbf{Z}[Q(32)]$. Any stable module in which cancellation holds is represented by **C**; for example (as we shall see in Chap. 15) the stable class $\Omega_3(\mathbf{Z})$ over the group ring $\mathbf{Z}[C_\infty \times C_m]$ for any integer $m \geq 2$.

1.3 Stably Free Modules and Gabel's Theorem

The most basic cancellation problem arises when one considers [0], the stable class of the zero module; evidently a module S belongs to [0] when, for some integers a, $b \geq 1$

$$S \oplus \Lambda^a \cong \Lambda^b.$$

Any such module S is finitely generated. More generally, one says that a module S is *stably free* when $S \oplus \Lambda^a$ is a free module of unspecified rank, finite or infinite. Clearly any free module is stably free; the issue is whether a stably free module is necessarily free. In fact, nothing new is gained by allowing infinitely generated stably free modules as shown by the following observation of Gabel [32, 65, 67].

Theorem 1.10 *Let S be a stably free Λ module*; *if S is not finitely generated then S is free*.

Proof Let F_X denote the free Λ module on the set X. The hypotheses may be expressed as follows:

(i) S is not finitely generated;

(ii) for some set X and some *finite set* Y there is a Λ-isomorphism $h : F_X \xrightarrow{\simeq} S \oplus F_Y$.

Note that X is necessarily infinite. Now let $\pi : S \oplus F_Y \to F_Y$ be the projection; putting

$$\widehat{h} = \pi \circ h : F_X \to F_Y$$

then $\widehat{h}$ is surjective. Moreover, h induces an isomorphism $h : \mathrm{Ker}(\widehat{h}) \xrightarrow{\simeq} S$ so that it is enough to show that $\mathrm{Ker}(\widehat{h})$ is free.

As F_Y is free we may choose a right inverse $s : F_Y \to F_X$ for $\widehat{h}$. For each $y \in Y$ there exists a finite subset $\sigma(y) \subset X$ such that $s(y)$ is a linear combination in the elements of $\sigma(y)$. Put $Z = \bigcup_{y \in Y} \sigma(y)$ and $\overline{Z} = X - Z$. Then Z is finite so that $\overline{Z}$ is infinite.

Now $\pi \circ h : F_Z \to F_Y$ is also surjective so that F_X is an internal sum (not necessarily direct) $F_X = \mathrm{Ker}(\widehat{h}) + F_Z$. However $F_Z/(\mathrm{Ker}(\widehat{h}) \cap F_Z) \cong F_Y$ so from the exact sequence

$$0 \to \mathrm{Ker}(\widehat{h}) \cap F_Z \to F_Z \to F_Z/(\mathrm{Ker}(\widehat{h}) \cap F_Z) \to 0$$

we see that

$$(\mathrm{Ker}(\widehat{h}) \cap F_Z) \oplus F_Y \cong F_Z. \tag{1.11}$$

From the exact sequence $0 \to \mathrm{Ker}(\widehat{h}) \cap F_Z \to \mathrm{Ker}(\widehat{h}) \to F_X/F_Z \to 0$ and the isomorphism $F_X/F_Z \cong F_{\overline{Z}}$ we see that

$$\mathrm{Ker}(\widehat{h}) \cong (\mathrm{Ker}(\widehat{h}) \cap F_Z) \oplus F_{\overline{Z}}. \tag{1.12}$$

As $\overline{Z}$ is infinite we may write it as a disjoint union $\overline{Z} = Y_1 \sqcup W$ where $Y_1 \subset \overline{Z}$ is a finite subset such that $|Y_1| = |Y|$. In particular we may write

$$F_{\overline{Z}} \cong F_Y \oplus F_W$$

for some infinite subset $W \subset X$ so from (1.12) we get

$$\mathrm{Ker}(\widehat{h}) \cong (\mathrm{Ker}(\widehat{h}) \cap F_Z) \oplus F_Y \oplus F_W. \tag{1.13}$$

From (1.11) and (1.13) we see that

$$\mathrm{Ker}(\widehat{h}) \cong F_Z \oplus F_W \cong F_{Z \sqcup W} \tag{1.14}$$

so that $\mathrm{Ker}(\widehat{h})$ is free as required. □

Gabel's Theorem confines the problem of stably free modules to the realm of finitely generated modules. Even so, the subject admits a certain amount of pathology; to avoid this we must impose the strongest of the restrictions of Sect. 1.1.

Given a stably free module S such that $S \oplus \Lambda^a \cong \Lambda^b$ one is tempted to make a definition of the *rank* $\mathrm{rk}(S)$ of S by

$$\mathrm{rk}(S) = b - a.$$