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Lorenz J. Halbeisen

Combinatorial Set Theory

With a Gentle Introduction to Forcing

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With a Gentle Introduction to Forcing

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To Joringel,
Meredith, Andrin, and Salome

Preface

By the campanologist, the playing of tunes is considered to be a childish game; the proper use of bells is to work out mathematical permutations and combinations.
His passion finds its satisfaction in mathematical completeness and mechanical perfection.

DOROTHY L. SAYERS
The Nine Tailors, 1934

This book provides a self-contained introduction to *Axiomatic Set Theory* with main focus on *Infinitary Combinatorics* and the *Forcing Technique*. The book is intended to be used as a textbook in undergraduate and graduate courses of various levels, as well as for self-study. To make the book valuable for experienced researchers also, some historical background and the sources of the main results have been provided in the NOTES, and some topics for further studies are given in the section RELATED RESULTS—where those containing open problems are marked with an asterisk.

The axioms of Set Theory ZFC, consisting of the axioms of *Zermelo–Fraenkel Set Theory* (denoted ZF) and the *Axiom of Choice*, are the foundation of Mathematics in the sense that essentially all Mathematics can be formalised within ZFC. On the other hand, Set Theory can also be considered as a mathematical theory, like Group Theory, rather than the basis for building general mathematical theories. This approach allows us to drop or modify axioms of ZFC in order to get, for example, a Set Theory without the *Axiom of Choice* (see Chapter 4) or in which just a weak form of the *Axiom of Choice* holds (see Chapter 7). In addition, we are also allowed to extend the axiomatic system ZFC in order to get, for example, a Set Theory in which, in addition to the ZFC axioms, we also have *Martin's Axiom* (see Chapter 13), which is a very powerful axiom with many applications for *Infinitary Combinatorics* as well as other fields of Mathematics. However, this approach prevents us from using any kind of Set Theory which goes beyond ZFC, which is used, for example, to prove the existence of a countable model of ZFC (see the *Löwenheim–Skolem Theorem* in Chapter 15).

Most of the results presented in this book are combinatorial results, in particular the results in *Ramsey Theory* (introduced in Chapter 2 and further developed in

Chapter 11), or those results whose proofs have a combinatorial flavour. For example, we get results of the latter type if we work in Set Theory without the *Axiom of Choice*, since in the absence of the *Axiom of Choice*, the proofs must be constructive and therefore typically have a much more combinatorial flavour than proofs in ZFC (examples can be found in Chapters 4 & 7). On the other hand, there are also elegant combinatorial proofs using the *Axiom of Choice*. An example is the proof in Chapter 6, where it is shown that one can divide the solid unit ball into five parts, such that one can build two solid unit balls out of these five parts—another such paradoxical result is given in Chapter 17, where it is shown that it might be possible in ZF to decompose a square into more parts than there are points on the square.

Even though the ZFC axiomatic system is the foundation of Mathematics, by *Gödel's Incompleteness Theorem*—briefly discussed at the end of Chapter 3—no axiomatic system of Mathematics is complete in the sense that every statement can either be proved or disproved; in other words, there are always statements which are independent of the axiomatic system. The main tool to show that a certain statement is independent of the axioms of Set Theory is Cohen's *Forcing Technique*, which he originally developed in the early 1960s in order to show that there are models of ZF in which the *Axiom of Choice* fails (see Chapter 17) and that the *Continuum Hypothesis* is independent of ZFC (see Chapter 14). The *Forcing Technique* is introduced and discussed in great detail in Part II, and in Part III it is used to investigate combinatorial properties of the set of real numbers. This is done by comparing the *Cardinal Characteristics of the Continuum* introduced in Chapter 8.

The following table indicates which of the main topics appear in which chapter, where $***$ means that it is the main topic of that chapter, $**$ means that some new results in that topic are proved or at least that the topic is important for understanding certain proofs, and $*$ means that the topic appears somewhere in that chapter, but not in an essential way:

Chapter	1	2	3	4	5	6	7	8	9	10	11
Forcing Technique	*										
Axiom of Choice & ZF	*	*	***	***	***	**	***				
Ramsey Theory	*	***		*	**		*	**	***	***	***
Cardinal Characteristics	*	*						***	***	*	
	Part I										

Chapter	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
Forcing Technique	***	**	***	*	***	**	**	**	***	**	**	**	**	**	**	**
Axiom of Choice & ZF						***										
Ramsey Theory		**	*				*					*	*	**	***	
Cardinal Characteristics		**					**	**		**	**	**	**	**	**	***
	Part II								Part III							

For example *Ramsey's Theorem*, which is the nucleus of *Ramsey Theory*, is the main topic in Chapter 2, it is used in some proofs in Chapters 4 & 7, it is used as a choice principle in Chapter 5, it is related to two *Cardinal Characteristics* defined in Chapter 8, it is used to define what is called a Ramsey ultrafilter in Chapter 10, it is used in the proof of the *Hales–Jewett Theorem* in Chapter 11, and it is used to formulate a combinatorial feature of Mathias reals in Chapter 24. Furthermore, one can see that *Cardinal Characteristics* are our main tool in Part III in the investigation of combinatorial properties of various forcing notions, even in the cases when—in Chapters 25 & 26—the existence of Ramsey ultrafilters are investigated. Finally, in Chapter 27 we show how *Cardinal Characteristics* can be used to shed new light on a classical problem in Measure Theory. On the other hand, the *Cardinal Characteristics* are used to describe some combinatorial features of different forcing notions. In particular, it will be shown that the cardinal characteristic $\mathfrak{h}$ (introduced in Chapter 8 and investigated in Chapter 9) is closely related to Mathias forcing (introduced in Chapter 24), which is used in Chapter 25 to show that the existence of Ramsey ultrafilters is independent of ZFC.

I tried to write this book like a piece of music, not just writing note by note, but using various themes or voices—like *Ramsey's Theorem* and the cardinal characteristic $\mathfrak{h}$—again and again in different combinations. In this undertaking, I was inspired by the English art of bell ringing and tried to base the order of the themes on Zarlino's introduction to the art of counterpoint.

Acknowledgement. First of all, I would like to thank Andreas Blass for his valuable remarks and comments, as well as for his numerous corrections, which improved the quality of the book substantially. Furthermore, I would like to thank my spouse Stephanie Halbeisen, not only for reading Chapters 1 & 12, and parts of Chapters 5 & 13, but also for her patience during the last seven years. I would also like to thank Dandolo Flumini for reading Chapters 2, 3, 13, 14, 15, Ioanna Dimitriou for reading Chapters 16 & 17, and Gearóidín Diserens for reading Chapter 1 as well as the introductory comments of several chapters. Finally, I would like to thank Jörg Sixt, editor of Springer-Verlag, for making every effort to ensure that the book was published in the optimal style.

Winterthur, October 2011 *Lorenz Halbeisen*

Contents

Chapter 1
The Setting

For one cannot order or compose anything, or understand the nature of the composite, unless one knows first the things that must be ordered or combined, their nature, and their cause.

GIOSEFFO ZARLINO
Le Istitutioni Harmoniche, 1558

Combinatorics with all its various aspects is a broad field of Mathematics which has many applications in areas like Topology, Group Theory and even Analysis. A reason for its wide range of applications might be that Combinatorics is rather a way of thinking than a homogeneous theory, and consequently Combinatorics is quite difficult to define. Nevertheless, let us start with a definition of Combinatorics which will be suitable for our purpose:

Combinatorics is the branch of Mathematics which studies collections of objects that satisfy certain criteria, and is in particular concerned with deciding how large or how small such collections might be.

Below we give a few examples which should illustrate some aspects of infinitary Combinatorics. At the same time, we present the main topics of this book, which are the *Axiom of Choice*, *Ramsey Theory*, *cardinal characteristics of the continuum*, and *forcing*.

Let us start with an example from Graph Theory: A *graph* is a set of *vertices*, where some pairs of vertices are connected by an *edge*. Connected pairs of vertices are called *neighbours*. A graph is *infinite* if it has an infinite number of vertices. A *tree* is a *cycle-free* (*i.e.*, one cannot walk in proper cycles along edges), *connected* (*i.e.*, any two vertices are connected by a path of edges) graph, where one of its vertices is designated as the *root*. A tree is *finitely branching* if every vertex has only a finite number of neighbours. Furthermore, a *branch* through a tree is a maximal edge-path beginning at the root, in which no edge appears twice.

Now we are ready to state König's Lemma, which is often used implicitly in fields like Combinatorics, Topology, and many other branches of Mathematics.

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König's Lemma. *Every infinite, finitely branching tree contains an infinite branch.*

At first glance, this result looks straightforward and one would construct an infinite branch as follows: Let v_0 be the root. Since the tree is infinite but finitely branching, there must be a neighbour of v_0 from which we reach infinitely many vertices without going back to v_0. Let v_1 be such a neighbour of v_0. Again, since we reach infinitely many vertices from v_1 (without going back to v_1) and the tree is finitely branching, there must be a neighbour of v_1, say v_2, from which we reach infinitely many vertices without going back to v_2. Proceeding in this way, we finally get the infinite branch $(v_0, v_1, v_2, \ldots)$.

Let us now have a closer look at this proof: Firstly, in order to prove that the set of neighbours of v_0 from which we reach infinitely many vertices without going back to v_0 is not empty, we need an infinite version of the so-called Pigeon-Hole Principle. The Pigeon-Hole Principle can be seen as the fundamental principle of Combinatorics.

Pigeon-Hole Principle. *If $n+1$ pigeons roost in n holes, then at least two pigeons must share a hole. More prosaically: If m objects are coloured with n colours and $m > n$, then at least two objects have the same colour.*

An infinite version of the Pigeon-Hole Principle reads as follows:

Infinite Pigeon-Hole Principle. *If infinitely many objects are coloured with finitely many colours, then infinitely many objects have the same colour.*

Using the Infinite Pigeon-Hole Principle we are now sure that the set of neighbours of v_0 from which we reach infinitely many vertices without going back to v_0 is not empty. However, the next problem we face is which element we should choose from that non-empty set. If the vertices are ordered in some way, then we can choose the first element with respect to that order, but otherwise, we would need some kind of choice function which selects *infinitely often* (and this is the crucial point!) one vertex from a given non-empty set of vertices. Such a choice function is guaranteed by the Axiom of Choice, denoted AC, which is discussed in Chapter 5.

Axiom of Choice. *For every family $\mathscr{F}$ of non-empty sets, there is a function f —called choice function—which selects one element from each member of $\mathscr{F}$ (i.e., for each $x \in \mathscr{F}$, $f(x) \in x$); or equivalently, every Cartesian product of non-empty sets is non-empty.*

The Axiom of Choice is one of the main topics of this book: In Chapter 3, the axioms of Zermelo–Fraenkel Set Theory (*i.e.*, the usual axioms of Set Theory except AC) are introduced. In Chapter 4 we shall introduce the reader to Zermelo–Fraenkel Set Theory and show how combinatorics can, to some extent, replace the Axiom of Choice. Subsequently, the Axiom of Choice (and some of its weaker forms) is introduced in Chapter 5. From then on, we always work in Zermelo–Fraenkel Set Theory *with* the Axiom of Choice—even in the case as in Chapters 7 & 17 when we construct models of Set Theory in which AC fails.

Now, let us turn back to König's Lemma. In order to prove König's Lemma we do not need full AC, since it would be enough if every family of non-empty *finite* sets had a choice function—the family would consist of all subsets of neighbours of vertices. However, as we will see later, even this weaker form of AC is a proper axiom and is independent of the other axioms of Set Theory (*cf.* PROPOSITION 7.7). Thus, depending on the axioms of Set Theory we start with, AC—as well as some weakened forms of it—may fail, and consequently, König's Lemma may become unprovable. On the other hand, as we will see in Chapter 5, König's Lemma may be used as a non-trivial choice principle.

Thus, this first example shows that—with respect to our definition of Combinatorics given above—some "objects satisfying certain criteria," may, but need not, exist.

The next example can be seen as a problem in infinitary *Extremal Combinatorics*. The word "extremal" describes the nature of problems dealt with in this field and refers to the second part of our definition of Combinatorics, namely "how large or how small collections satisfying certain criteria might be."

If the objects considered are infinite, then the answer, how large or how small certain sets are, depends again on the underlying axioms of Set Theory, as the next example shows.

Reaping Families. *A family $\mathscr{R}$ of infinite subsets of the natural numbers $\mathbb{N}$ is said to be reaping if for every colouring of $\mathbb{N}$ with two colours there exists a monochromatic set in the family $\mathscr{R}$.*

For example, the set of all infinite subsets of $\mathbb{N}$ is such a family. The *reaping number* $\mathfrak{r}$—a so-called *cardinal characteristic of the continuum*—is the smallest cardinality (*i.e.*, size) of a reaping family. In general, a *cardinal characteristic of the continuum* is typically defined as the smallest cardinality of a subset of a given set S which has certain combinatorial properties, where S is of the same cardinality as the continuum $\mathbb{R}$.

Consider the cardinal characteristic $\mathfrak{r}$ (*i.e.*, the size of the smallest reaping family). Since $\mathfrak{r}$ is a well-defined cardinality we can ask: How large is $\mathfrak{r}$? Can it be countable? Is it always equal to the cardinality of the continuum?

Let us just show that a reaping family can never be countable: Let $\mathscr{A} = \{A_i : i \in \mathbb{N}\}$ be any countable family of infinite subsets of $\mathbb{N}$. For each $i \in \mathbb{N}$, pick n_i and m_i from the set A_i in such a way that, at the end, for all i we have $n_i < m_i < n_{i+1}$. Now we colour all n_i's blue and all the other numbers red. For this colouring, there is no monochromatic set in $\mathscr{A}$, and hence, $\mathscr{A}$ cannot be a reaping family. The Continuum Hypothesis, denoted CH, states that every subset of the continuum $\mathbb{R}$ is either countable or of cardinality $\mathfrak{c}$, where $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$. Thus, if we assume CH, then any reaping family is of cardinality $\mathfrak{c}$. The same holds if we assume Martin's Axiom which will be introduced in Chapter 13.

On the other hand, with the *forcing technique*—invented by Paul Cohen in the early 1960s—one can show that the axioms of Set Theory do not decide whether or not the cardinals $\mathfrak{r}$ and $\mathfrak{c}$ are equal. The forcing technique is introduced in Part II and a model in which $\mathfrak{r} < \mathfrak{c}$ is given in Chapter 18.

Thus, the second example shows that—depending on the additional axioms of Set Theory we start with—we can get different answers when we try to "decide how large or how small certain collections might be."

Many more cardinal characteristics like $\mathfrak{hom}$ and $\mathfrak{par}$ (see below) are introduced in Chapter 8. Possible (*i.e.*, consistent) relations between these cardinals are investigated in Part II and more systematically in Part III—where the cardinal characteristics are also used to distinguish the combinatorial features of certain forcing notions.

Another field of Combinatorics is the so-called Ramsey Theory, and since many results in this work rely on Ramsey-type theorems, let us give a brief description of Ramsey Theory.

Loosely speaking, *Ramsey Theory* (which can be seen as a part of extremal Combinatorics) is the branch of Combinatorics which deals with structures preserved under partitions, or colourings. Typically, one looks at the following kind of question: If a particular object (*e.g.*, algebraic, geometric or combinatorial) is arbitrarily coloured with finitely many colours, what kinds of monochromatic structure can we find?

For example, VAN DER WAERDEN'S THEOREM, which will be proved in Chapter 11, tells us that *for any positive integers r and n, there is a positive integer N such that for every r-colouring of the set* $\{0, 1, \dots, N\}$ *we find always a monochromatic* (*non-constant*) *arithmetic progression of length n.*

Even though VAN DER WAERDEN'S THEOREM is one of the earliest results in Ramsey Theory, the most famous result in Ramsey Theory is surely RAMSEY'S THEOREM (which will be discussed in detail in the next chapter):

RAMSEY'S THEOREM. *Let n be any positive integer. If we colour all n-element subsets of $\mathbb{N}$ with finitely many colours, then there exists an infinite subset of $\mathbb{N}$ all of whose n-element subsets have the same colour.*

There is also a finite version of RAMSEY'S THEOREM which gives an answer to problems like the following:

How many people must be invited to a party in order to make sure that three of them mutually shook hands on a previous occasion or three of them mutually did not shake hands on a previous occasion?

It is quite easy to show that at least six people must be invited. On the other hand, if we ask how many people must get invited such that there are five people who all mutually shook hands or did not shake hands on a previous occasion, then the precise number is not known—but it is conjectured that it is sufficient to invite 43 people.

As we shall see later, RAMSEY'S THEOREM has many—sometimes unexpected—applications. For example, if we work in Set Theory without AC, then RAMSEY'S THEOREM can help to construct a choice function, as we will see in Chapter 4. Sometimes we get Ramsey-type (or anti-Ramsey-type) results even for partitions into infinitely many classes (*i.e.*, using infinitely many colours). For example, one can show that there is a colouring of the points in the Euclidean plane with countably many colours, such that no two points of any "copy of the rationals" have the same colour. This result can be seen as an anti-Ramsey-type theorem (since we are far away from "monochromatic structures"), and it shows that Ramsey-type theorems cannot be generalised arbitrarily. However, concerning RAMSEY'S THEOREM, we can ask for a "nice" family $\mathscr{F}$ of infinite subsets of $\mathbb{N}$, such that for every colouring of the n-element subsets of $\mathbb{N}$ with finitely many colours, there exists a homogeneous set in the family $\mathscr{F}$, where an infinite set $x \subseteq \mathbb{N}$ is called *homogeneous* if all n-element subsets of x have the same colour. Now, "nice" could mean "as small as possible" but also "being an ultrafilter." In the former case, this leads to the *homogeneous number* $\mathfrak{hom}$, which is the smallest cardinality of a family $\mathscr{F}$ which contains a homogeneous set for every 2-colouring of the 2-element subsets of $\mathbb{N}$. One can show that $\mathfrak{hom}$ is uncountable and—like for the reaping number—that the axioms of Set Theory do not decide whether or not $\mathfrak{hom}$ is equal to $\mathfrak{c}$ (see Chapter 18). The latter case, where "nice" means "being an ultrafilter," leads to so-called *Ramsey ultrafilters*. It is not difficult to show that Ramsey ultrafilters exist if one assumes CH or Martin's Axiom (see Chapter 10), but on the other hand, the axioms of Set Theory alone do not imply the existence of Ramsey ultrafilters (see PROPOSITION 25.11). A somewhat anti-Ramsey-type question would be to ask how many 2-colourings of the 2-element subsets of $\mathbb{N}$ we need to make sure that no single infinite subset of $\mathbb{N}$ is almost homogeneous for all these colourings, where a set H is called *almost homogeneous* if there is a finite set K such that $H \setminus K$ is homogeneous. This question leads to the *partition number* $\mathfrak{par}$. Again, $\mathfrak{par}$ is uncountable and the axioms of Set Theory do not decide whether or not $\mathfrak{par}$ is equal to $\mathfrak{c}$ (see for example Chapter 18).

RAMSEY'S THEOREM, as well as Ramsey Theory in general, play an important role throughout this book. Especially in all chapters of Part I, except for Chapter 3, we shall meet—sometimes unexpectedly—RAMSEY'S THEOREM in one form or other.

NOTES

Gioseffo Zarlino. All citations of Zarlino (1517–1590) are taken from Part III of his book entitled *Le Istitutioni Harmoniche* (*cf.* [1]). This section of Zarlino's *Institutioni* is concerned primarily with the art of counterpoint, which is, according to Zarlino, *the concordance or agreement born of a body with diverse parts, its various melodic lines accommodated to the total composition, arranged so that voices are separated by commensurable, harmonious intervals*. The word "counterpoint" presumably originated at the beginning of the 14th century and was derived from "punctus contra punctum," *i.e.*, point against point or note against note. Zarlino himself was an Italian music theorist and composer. While he composed a number of masses, motets and madrigals, his principal claim to fame is as a music theorist: For example, Zarlino was ahead of his time in proposing that the octave should be divided into twelve equal semitones—for the lute, that is to say, he advocated a practice in the 16th century which was universally adopted three centuries later. He also advocated equal temperament for keyboard instruments and just intonation for unaccompanied vocal music and strings—a system which has been successfully practised up to the present day. Furthermore, Zarlino arranged the modes in a different order of succession, beginning with the Ionian mode instead of the Dorian mode. This arrangement seems almost to have been dictated by a prophetic anticipation of the change which was to lead to the abandonment of the modes in favour of a newer tonality, for his series begins with a form which corresponds exactly with our modern major mode and ends with the prototype of the descending minor scale of modern music. (For the terminology of music theory we refer the interested reader to Benson [2].)

Zarlino's most notable student was the music theorist and composer Vincenzo Galilei, the father of Galileo Galilei.

König's Lemma and Ramsey's Theorem. A proof of König's Lemma can be found in König's book on Graph Theory [3, VI, §2, Satz 6], where he called the result *Unendlichkeitslemma*. As a first application of the *Unendlichkeitslemma* he proved the following theorem of de la Vallée Poussin: *If E is a subset of the open unit interval* $(0,1)$ *which is closed in* $\mathbb{R}$ *and I is a set of open intervals covering E, then there is a natural number n, such that if one partitions* $(0,1)$ *into* 2^n *intervals of length* 2^{-n}*, each of these intervals containing a point of E is contained in an interval of I.* Using the *Unendlichkeitslemma*, König also showed that VAN DER WAERDEN'S THEOREM is equivalent to the following statement: *If the positive integers are finitely coloured, then there are arbitrarily long monochromatic arithmetic progressions.* In a similar way we will use König's Lemma to derive the FINITE RAMSEY THEOREM from RAMSEY'S THEOREM (*cf.* COROLLARY 2.3).

At first glance, König's Lemma and RAMSEY'S THEOREM seem to be quite unrelated statements. In fact, König's Lemma is a proper (but rather weak) choice principle, whereas RAMSEY'S THEOREM is a very powerful combinatorial tool. However, as we shall see in Chapter 5, RAMSEY'S THEOREM can also be considered as a proper choice principle which turns out to be even stronger than König's Lemma (see THEOREM 5.17).

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Part I
Topics in Combinatorial Set Theory

Chapter 2
Overture: Ramsey's Theorem

Musicians in the past, as well as the best of the moderns, believed that a counterpoint or other musical composition should begin on a perfect consonance, that is, a unison, fifth, octave, or compound of one of these.

GIOSEFFO ZARLINO
Le Istitutioni Harmoniche, 1558

The Nucleus of Ramsey Theory

Most of this text is concerned with sets of subsets of the natural numbers, so, let us start there: The set $\{0, 1, 2, \ldots\}$ of **natural numbers** (or of non-negative integers) is denoted by ω. It is convenient to consider a natural number n as an n-element subset of ω, namely as the set of all numbers smaller than n, so, $n = \{k \in \omega : k < n\}$. In particular, $0 = \emptyset$, where $\emptyset$ is the **empty set**. For any $n \in \omega$ and any set S, let $[S]^n$ denote the set of all n-element subsets of S (*e.g.*, $[S]^0 = \{\emptyset\}$). Further, the set of all finite subsets of a set S is denoted by $[S]^{<\omega}$.

For a finite set S let $|S|$ denote the number of elements in S, also called the **cardinality** of S.

A set S is called **countable** if there is an enumeration of S, *i.e.*, if $S = \emptyset$ or $S = \{x_i : i \in \omega\}$. In particular, every finite set is countable. However, when we say that a set is countable we usually mean that it is a countably infinite set. For any set S, $[S]^\omega$ denotes the set of all countably infinite subsets of S, in particular, since every infinite subset of ω is countable, $[\omega]^\omega$ is the set of *all* infinite subsets of ω.

Let S be an arbitrary non-empty set. A binary relation "$\sim$" on S is an **equivalence relation** if it is

- *reflexive* (*i.e.*, for all $x \in S$: $x \sim x$),
- *symmetric* (*i.e.*, for all $x, y \in S$: $x \sim y \leftrightarrow y \sim x$), and
- *transitive* (*i.e.*, for all $x, y, z \in S$: $x \sim y \wedge y \sim z \rightarrow x \sim z$).

The **equivalence class** of an element $x \in S$, denoted $[x]\tilde{}$, is the set $\{y \in S : x \sim y\}$. We would like to recall the fact that, since "$\sim$" is an equivalence relation, for any

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$x, y \in S$ we have *either* $[x]^{\tilde{}} = [y]^{\tilde{}}$ *or* $[x]^{\tilde{}} \cap [y]^{\tilde{}} = \emptyset$. A set $A \subseteq S$ is a set of **representatives** if for each equivalence class $[x]^{\tilde{}}$ we have $|A \cap [x]^{\tilde{}}| = 1$; in other words, A has exactly one element in common with each equivalence class. It is worth mentioning that in general, the existence of a set of representatives relies on the Axiom of Choice (see Chapter 5).

For sets A and B, let ${}^A B$ denote the set of all functions $f : A \to B$. For $f \in {}^A B$ and $S \subseteq A$ let $f[S] := \{f(x) : x \in S\}$ and let $f|_S \in {}^S B$ (the restriction of f to S) be such that for all $x \in S$, $f(x) = f|_S(x)$.

Further, for sets A and B, let the set-theoretic difference of A and B be the set $A \setminus B := \{a \in A : a \notin B\}$.

For some positive $n \in \omega$, let us colour all n-element subsets of ω with three colours, say red, blue, and yellow. In other words, each n-element set of natural numbers $\{k_1, \dots, k_n\}$ is coloured either red, or blue, or yellow. Now one can ask whether there is an infinite subset H of ω such that all its n-element subsets have the same colour (*i.e.*, $[H]^n$ is **monochromatic**). Such a set we would call **homogeneous** (for the given colouring). In the terminology above, this question reads as follows: Given any colouring (*i.e.*, function) $\pi : [\omega]^n \to 3$, where $3 = \{0, 1, 2\}$, does there exist a set $H \in [\omega]^\omega$ such that $\pi|_{[H]^n}$ is constant? Alternatively, one can define an equivalence relation "$\sim$" on $[\omega]^n$ by stipulating $x \sim y$ *iff* $\pi(x) = \pi(y)$ and ask whether there exists a set $H \in [\omega]^\omega$ such that $[H]^n$ is included in one equivalence class. The answer to this question is given by RAMSEY'S THEOREM 2.1 below, but before we state and prove this theorem, let us say a few words about its background.

Ramsey proved his theorem in order to investigate a problem in formal logic, namely the problem of finding a regular procedure to determine the truth or falsity of a given logical formula in the language of *First-Order Logic*, which is also the language of Set Theory (*cf.* Chapter 3). However, RAMSEY'S THEOREM is a purely combinatorial statement and was the nucleus—but not the earliest result—of a whole combinatorial theory, the so-called *Ramsey Theory*. We would also like to mention that Ramsey's original theorem, which will be discussed later, is somewhat stronger than the theorem stated below but is, like König's Lemma, not provable without assuming some form of the Axiom of Choice (see PROPOSITION 7.8).

THEOREM 2.1 (RAMSEY'S THEOREM). *For any number $n \in \omega$, for any positive number $r \in \omega$, for any $S \in [\omega]^\omega$, and for any colouring $\pi : [S]^n \to r$, there is always an $H \in [S]^\omega$ such that H is homogeneous for π, i.e., the set $[H]^n$ is monochromatic.*

Before we prove RAMSEY'S THEOREM, let us consider a few examples: In the first example we colour the set of prime numbers $\mathbb{P}$ with two colours. A **Wieferich prime** is a prime number p such that p^2 divides $2^{p-1} - 1$, denoted $p^2 \mid 2^{p-1} - 1$. Recall that by FERMAT'S LITTLE THEOREM we have $p \mid 2^{p-1} - 1$ for any prime p. Now, define the 2-colouring π_1 of $\mathbb{P}$ by stipulating

$$\pi_1(p) = \begin{cases} 0 & \text{if } p \text{ is a Wieferich prime,} \\ 1 & \text{otherwise.} \end{cases}$$

Let $H_0 = \{p \in \mathbb{P} : p^2 \mid 2^{p-1} - 1\}$ and $H_1 = \mathbb{P} \setminus H_0$. The only numbers which are known to belong to H_0 are 1093 and 3511. On the other hand, it is not known whether H_1 is infinite. However, by the Infinite Pigeon-Hole Principle we know that at least one of the two sets H_0 and H_1 is infinite, which gives us a homogeneous set for π_1.

As a second example, define the 2-colouring π_2 of the set of 2-element subsets of $\{7l : l \in \omega\}$ by stipulating

$$\pi_2\big(\{n, m\}\big) = \begin{cases} 0 & \text{if } n^m + m^n + 1 \text{ is prime,} \\ 1 & \text{otherwise.} \end{cases}$$

An easy calculation modulo 3 shows that the set $H = \{42k + 14 : k \in \omega\} \subseteq \{7l : l \in \omega\}$ is homogeneous for π_2; in fact, for all $\{n, m\} \in [H]^2$ we have $3 \mid (n^m + m^n + 1)$.

Before we give a third example, we prove the following special case of RAMSEY'S THEOREM.

PROPOSITION 2.2. *For any positive number $r \in \omega$, for any $S \in [\omega]^\omega$, and for any colouring $\pi : [S]^2 \to r$, there is always an $H \in [S]^\omega$ such that $[H]^2$ is monochromatic.*

Proof. The proof is in fact just a consequence of the Infinite Pigeon-Hole Principle; firstly, the Infinite Pigeon-Hole Principle is used to construct homogeneous sets for certain 2-colourings τ and then it is used to show the existence of a homogeneous set for π.

Let $S_0 = S$ and let $a_0 = \min(S_0)$. Define the r-colouring $\tau_0 : S_0 \setminus \{a_0\} \to r$ by stipulating $\tau_0(b) := \pi(\{a_0, b\})$. By the Infinite Pigeon-Hole Principle there is an infinite set $S_1 \subseteq S_0 \setminus \{a_0\}$ such that $\tau_0|_{S_1}$ is constant (*i.e.*, $\tau_0|_{S_1}$ is a constant function) and let $\rho_0 := \tau_0(b)$, where b is any member of S_1. Now, let $a_1 = \min(S_1)$ and define the r-colouring $\tau_1 : S_1 \setminus \{a_1\} \to r$ by stipulating $\tau_1(b) := \pi(\{a_1, b\})$. Again we find an infinite set $S_2 \subseteq S_1 \setminus \{a_1\}$ such that $\tau_1|_{S_2}$ is constant and let $\rho_1 := \tau_1(b)$, where b is any member of S_2. Proceeding this way we finally get infinite sequences $a_0 < a_1 < \ldots < a_n < \ldots$ and $\rho_0, \rho_1, \ldots$. Notice that by construction, for all $n \in \omega$ and all $k > n$ we have $\pi(\{a_n, a_k\}) = \tau_n(a_k) = \rho_n$. Define the r-colouring $\tau : \{a_n : n \in \omega\} \to r$ by stipulating $\tau(a_n) := \rho_n$. Again by the Infinite Pigeon-Hole Principle there is an infinite set $H \subseteq \{a_n : n \in \omega\}$ such that $\tau|_H$ is constant, which implies that H is homogeneous for π, *i.e.*, $[H]^2$ is monochromatic. ⊣

As a third example, consider the 17-colouring π_3 of the set of 9-element subsets of $\mathbb{P}$ defined by stipulating

$$\pi_3\big(\{p_1, \ldots, p_9\}\big) = c \quad \iff \quad p_1 \cdot p_2 \cdot \ldots \cdot p_9 \equiv c \mod 17.$$

For $0 \leq k \leq 16$ let $P_k = \{p \in \mathbb{P} : p \equiv k \mod 17\}$. Then, by Dirichlet's theorem on primes in arithmetic progression, P_k is infinite whenever $\gcd(k, 17) = 1$, *i.e.*, for all positive numbers $k \leq 16$. Thus, by an easy calculation modulo 17 we find for $1 \leq k \leq 16$, that P_k is homogeneous for π_3.

Now we give a complete proof of RAMSEY'S THEOREM 2.1:

Proof of Ramsey's Theorem. The proof is by induction on n. For $n = 2$ we get PROPOSITION 2.2. So, we assume that the statement is true for $n \geq 2$ and prove it for $n + 1$. Let $\pi : [\omega]^{n+1} \to r$ be any r-colouring of $[\omega]^{n+1}$. For each integer $a \in \omega$ let π_a be the r-colouring of $[\omega \setminus \{a\}]^n$ defined as follows:

$$\pi_a(x) = \pi\big(x \cup \{a\}\big).$$

By induction hypothesis, for each $S' \in [\omega]^\omega$ and for each $a \in S'$ there is an $H_a^{S'} \in [S' \setminus \{a\}]^\omega$ such that $H_a^{S'}$ is homogeneous for π_a. Construct now an infinite sequence $a_0 < a_1 < \ldots < a_i < \ldots$ of natural numbers and an infinite sequence $S_0 \supseteq S_1 \supseteq \ldots \supseteq S_i \supseteq \ldots$ of infinite subsets of ω as follows: Let $S_0 = S$ and $a_0 = \min(S)$, and in general let

$$S_{i+1} = H_{a_i}^{S_i}, \quad \text{and} \quad a_{i+1} = \min\{a \in S_{i+1} : a > a_i\}.$$

It is clear that for each $i \in \omega$, the set $[\{a_m : m > i\}]^n$ is monochromatic for π_{a_i}; let $\tau(a_i)$ be its colour (*i.e.*, τ is a colouring of $\{a_i : i \in \omega\}$ with at most r colours). By the Infinite Pigeon-Hole Principle there is an $H \subseteq \{a_i : i \in \omega\}$ such that τ is constant on H, which implies that $\pi|_{[H]^{n+1}}$ is constant, too. Indeed, for any $x_0 < \ldots < x_n$ in H we have $\pi(\{x_0, \ldots, x_n\}) = \pi_{x_0}(\{x_1, \ldots, x_n\}) = \tau(x_0)$, which completes the proof. ⊣

Corollaries of Ramsey's Theorem

In finite Combinatorics, the most important consequence of RAMSEY'S THEOREM 2.1 is its finite version:

COROLLARY 2.3 (FINITE RAMSEY THEOREM). *For all $m, n, r \in \omega$, where $r \geq 1$ and $n \leq m$, there exists an $N \in \omega$, where $N \geq m$, such that for every colouring of $[N]^n$ with r colours, there exists a set $H \in [N]^m$, all of whose n-element subsets have the same colour.*

Proof. Assume towards a contradiction that the FINITE RAMSEY THEOREM fails. So, there are $m, n, r \in \omega$, where $r \geq 1$ and $n \leq m$, such that for all $N \in \omega$ with $N \geq m$ there is a colouring $\pi_N : [N]^n \to r$ such that no $H \in [N]^m$ is homogeneous, *i.e.*, $[H]^n$ is not monochromatic. We shall construct an r-colouring π of $[\omega]^n$ such that no infinite subset of ω is homogeneous for π, contradicting RAMSEY'S THEOREM. The r-colouring π will be induced by an infinite branch through a finitely branching tree, where the infinite branch is obtained by König's Lemma. Thus, we first need an infinite, finitely branching tree. For this, consider the following graph G: The vertex set of G consists of $\emptyset$ and all colourings $\pi_N : [N]^n \to r$, where $N \geq m$, such that no $H \in [N]^m$ is homogeneous for π_N. There is an edge between $\emptyset$ and each r-colouring π_m of $[m]^n$, and there is an edge between the colourings π_N and π_{N+1}

iff $\pi_N \equiv \pi_{N+1}|_N$ (*i.e.*, for all $x \in [N]^n$, $\pi_{N+1}(x) = \pi_N(x)$). In particular, there is no edge between two different r-colouring of $[N]^n$. By our assumption, the graph G is infinite. Further, by construction, it is cycle-free, connected, finitely branching, and has a root, namely $\emptyset$. In other words, G is an infinite, finitely branching tree and therefore, by König's Lemma, contains an infinite branch of r-colourings, say $(\emptyset, \pi_m, \pi_{m+1}, \ldots, \pi_{m+i}, \ldots)$, where for all $i, j \in \omega$, the colouring π_{m+i+j} is an extension of the colouring π_{m+i}.

At this point we would like to mention that since for any $N \in \omega$ the set of all r-colouring of $[N]^n$ can be ordered, for example lexicographically, we do not need any non-trivial form of the Axiom of Choice to construct an infinite branch.

Now, the infinite branch $(\emptyset, \pi_m, \pi_{m+1}, \ldots)$ induces an r-colouring π of $[\omega]^n$ such that no m-element subset of ω is homogeneous. In particular, there is no infinite set $H \in [\omega]^\omega$ such that $\pi|_{[H]^n}$ is constant, which is a contradiction to RAMSEY'S THEOREM 2.1 and completes the proof. ⊣

The following corollary is a geometrical consequence of the FINITE RAMSEY THEOREM 2.3:

COROLLARY 2.4. *For every positive integer n there exists an $N \in \omega$ with the following property: If P is a set of N points in the Euclidean plane without three collinear points, then P contains n points which form the vertices of a convex n-gon.*

Proof. By the FINITE RAMSEY THEOREM 2.3, let N be such that for every 2-colouring of $[N]^3$ there is a set $H \in [N]^n$ such that $[H]^3$ is monochromatic. Now let N points in the plane be given, and number them from 1 to N in an arbitrary but fixed way. Colour a triple (i, j, k), where $i < j < k$, red, if travelling from i to j to k is in clockwise direction; otherwise, colour it blue. By the choice of N, there are n ordered points so that every triple has the same colour (*i.e.*, orientation) from which one verifies easily (*e.g.*, by considering the convex hull of the n points) that these points form the vertices of a convex n-gon. ⊣

The following theorem—discovered more than a decade before RAMSEY'S THEOREM—is perhaps the earliest result in Ramsey Theory:

COROLLARY 2.5 (SCHUR'S THEOREM). *If the positive integers are finitely coloured (i.e., coloured with finitely many colours), then there are three distinct positive integers x, y, z of the same colour, with $x + y = z$.*

Proof. Let r be a positive integer and let π be any r-colouring of $\omega \setminus \{0\}$. Let $N \in \omega$ be such that for every r-colouring of $[N]^2$ there is a homogeneous 3-element subset of N. Define the colouring $\pi^* : [N]^2 \to r$ by stipulating $\pi^*(i, j) = \pi(|i - j|)$, where $|i - j|$ is the modulus or absolute value of the difference $i - j$. Since N contains a homogeneous 3-element subset (for π^*), there is a triple $0 \le i < j < k < N$ such that $\pi^*(i, j) = \pi^*(j, k) = \pi^*(i, k)$, which implies that the numbers $x = j - i$, $y = k - j$, and $z = k - i$, have the same colour, and in addition we have $x + y = z$. ⊣

The next result is a purely number-theoretical result and follows quite easily from RAMSEY'S THEOREM. However, somewhat surprisingly, it is unprovable in Number Theory, or more precisely, in *Peano Arithmetic* (which will be discussed in Chapter 3). Before we can state the corollary, we have to introduce the following notion: A non-empty set $S \subseteq \omega$ is called *large* if S has more than $\min(S)$ elements. Further, for $n, m \in \omega$ let $[n, m] := \{i \in \omega : n \leq i \leq m\}$.

COROLLARY 2.6. *For all $n, k, r \in \omega$ with $r \geq 1$, there is an $m \in \omega$ such that for any r-colouring of $[[n, m]]^k$, there exists a large homogeneous set.*

Proof. Let $n, k, r \in \omega$, where $r \geq 1$, be some arbitrary but fixed numbers. Let $\pi : [\omega \setminus n]^k \to r$ be any r-colouring of the k-element subsets of $\{i \in \omega : i \geq n\}$. By RAMSEY'S THEOREM 2.1 there exists an infinite homogeneous set $H \in [\omega \setminus n]^\omega$. Let $a = \min(H)$ and let S denote the least $a + 1$ elements of H. Then S is large and $[S]^k$ is monochromatic.

The existence of a finite number m with the required properties now follows—using König's Lemma—in the very same way as the FINITE RAMSEY THEOREM followed from RAMSEY'S THEOREM (see the proof of the FINITE RAMSEY THEOREM 2.3). ⊣

Generalisations of Ramsey's Theorem

Even though Ramsey's theorems are very powerful combinatorial results, they can still be generalised. The following result will be used later in Chapter 7 in order to prove that the Prime Ideal Theorem—introduced in Chapter 5—holds in the ordered Mostowski permutation model (but it will not be used anywhere else in this book).

In order to illustrate the next theorem, as well as to show that it is optimal to some extent, we consider the following two examples: Firstly, define the 2-colouring π_1 of $[\omega]^2 \times [\omega]^3 \times [\omega]^1$ by stipulating

$$\pi_1\big(\{x_1, x_2\}, \{y_1, y_2, y_3\}, \{z_1\}\big) = \begin{cases} 1 & \text{if } 2^{x_1 \cdot x_2} + 13^{y_1 \cdot y_2 \cdot y_3} + 17^{z_1} - 3 \text{ is prime,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $H_1 = \{3 \cdot k : k \in \omega\}$, $H_2 = \{2 \cdot k : k \in \omega\}$, and $H_3 = \{6 \cdot k : k \in \omega\}$. Then an easy calculation modulo 7 shows that $[H_1]^2 \times [H_2]^3 \times [H_3]^1$ is an infinite monochromatic set.

Secondly, define the 2-colouring π_2 of $[\omega]^1 \times [\omega]^1$ by stipulating

$$\pi_2\big(\{x\}, \{y\}\big) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that whenever H_1 and H_2 are *infinite* subsets of ω, then $[H_1]^1 \times [H_2]^1$ is not monochromatic; on the other hand, we easily find arbitrarily large *finite* sets $M_1, M_2 \subseteq \omega$ such that $[M_1]^1 \times [M_2]^1$ is monochromatic.

Thus, if $[\omega]^{n_1} \times \ldots \times [\omega]^{n_l}$ is coloured with r colours, then, in general, we cannot expect to find infinite subsets of ω, say $H_1, \ldots, H_l$, such that $[H_1]^{n_1} \times \ldots \times [H_l]^{n_l}$ is monochromatic; but we always find arbitrarily large finite subsets of ω:

THEOREM 2.7. *Let $r, l, n_1, \dots, n_l \in \omega$ with $r \geq 1$ be given. For every $m \in \omega$ with $m \geq \max\{n_1, \dots, n_l\}$ there is some $N \in \omega$ such that whenever $[N]^{n_1} \times \dots \times [N]^{n_l}$ is coloured with r colours, then there are $M_1, \dots, M_l \in [N]^m$ such that $[M_1]^{n_1} \times \dots \times [M_l]^{n_l}$ is monochromatic.*

Proof. The proof is by induction on l and the induction step uses a so-called *product-argument*. For $l = 1$ the statement is equivalent to the FINITE RAMSEY THEOREM 2.3. So, assume that the statement is true for $l \geq 1$ and let us prove it for $l + 1$. By induction hypothesis, for every $r \geq 1$ there is an N_l (depending on r) such that for *every* r-colouring of $[N_l]^{n_1} \times \dots \times [N_l]^{n_l}$ there are $M_1, \dots, M_l \in [N_l]^m$ such that $[M_1]^{n_1} \times \dots \times [M_l]^{n_l}$ is monochromatic. Now, the crucial idea in order to apply the FINITE RAMSEY THEOREM is to consider the coloured l-tuples in $([N_l]^m)^l$ as new colours. More precisely, let u_l be the number of different l-tuples in $([N_l]^m)^l$ and let $r_l := u_l \cdot r$. Notice that each colour in r_l corresponds to a pair $\langle t, c \rangle$, where t is an l-tuple in $([N_l]^m)^l$ and c is one of r colours. Notice also that r_l is very large compared to r. Now, by the FINITE RAMSEY THEOREM 2.3, there is a number $N_{l+1} \in \omega$ such that whenever $[N_{l+1}]^{n_{l+1}}$ is coloured with r_l colours, then there exists an $M_{l+1} \in [N_{l+1}]^m$ such that $[M_{l+1}]^{n_{l+1}}$ is monochromatic. Let $N = \max\{N_l, N_{l+1}\}$ and let π be any r-colouring of $[N_l]^{n_1} \times \dots \times [N_l]^{n_l} \times [N]^{n_{l+1}}$. For every $F \in [N]^{n_{l+1}}$ let π^F be the r-colouring of $[N_l]^{n_1} \times \dots \times [N_l]^{n_l}$ defined by stipulating

$$\pi^F(X) = \pi\big(\langle X, F \rangle\big).$$

By the definition of N, for every $F \in [N]^{n_{l+1}}$ there is a lexicographically first l-tuple $(M_1^F, \dots, M_l^F) \in ([N_l]^m)^l$ such that $[M_1^F]^{n_1} \times \dots \times [M_l^F]^{n_l}$ is monochromatic for π^F. By definition of r_l we can define an r_l-colouring π_{l+1} on $[N]^{n_{l+1}}$ as follows: Every set $F \in [N]^{n_{l+1}}$ is coloured according to the l-tuple $t = (M_1^F, \dots, M_l^F)$ (which can be encoded as one of u_l numbers) and the colour $c = \pi^F(X)$, where X is any element of the set $[M_1^F]^{n_1} \times \dots \times [M_l^F]^{n_l}$; because $[M_1^F]^{n_1} \times \dots \times [M_l^F]^{n_l}$ is monochromatic for π^F, c is well-defined and one of r colours. In other words, for every $F \in [N]^{n_{l+1}}$, $\pi_{l+1}(F)$ correspond to a pair $\langle t, c \rangle$, where $t \in ([N_l]^m)^l$ and c is one of r colours. Finally, by definition of N, there is a set $M_{l+1} \in [N]^m$ such that $[M_{l+1}]^{n_{l+1}}$ is monochromatic for π_{l+1}, which implies that for all $F, F_1, F_2 \in [M_{l+1}]^{n_{l+1}}$ we get that

- $[M_1^F]^{n_1} \times \dots \times [M_l^F]^{n_l}$ is monochromatic for π^F,
- $(M_1^{F_1}, \dots, M_l^{F_1}) = (M_1^{F_2}, \dots, M_l^{F_2})$,
- and restricted to the set $[M_1^F]^{n_1} \times \dots \times [M_l^F]^{n_l}$, the colourings $\pi_l^{F_1}$ and $\pi_l^{F_2}$ are identical.

Hence, there are $M_1, \dots, M_{l+1} \in [N]^m$ such that $\pi|_{[M_1]^{n_1} \times \dots \times [M_{l+1}]^{n_{l+1}}}$ is constant, which completes the proof. ⊣

A very strong generalisation of RAMSEY'S THEOREM in terms of partitions is the PARTITION RAMSEY THEOREM 11.4. However, since the proof of this generalisation is quite involved, we postpone the discussion of that result until Chapter 11 and consider now some other possible generalisations of RAMSEY'S THEOREM: Firstly one could finitely colour all finite subsets of ω, secondly one could colour $[\omega]^n$ with infinitely many colours, and finally, one could finitely colour all the infinite subsets of ω. However, below we shall see that none of these generalisations works, but first, let us consider Ramsey's original theorem, which is—at least in the absence of the Axiom of Choice—also a generalisation of RAMSEY'S THEOREM.

Ramsey's Original Theorem. The theorem which Ramsey proved originally is somewhat stronger than what we proved above. In our terminology, it states as follows:

RAMSEY'S ORIGINAL THEOREM. *For any infinite set A, for any number $n \in \omega$, for any positive number $r \in \omega$, and for any colouring $\pi : [A]^n \to r$, there is an infinite set $H \subseteq A$ such that $[H]^n$ is monochromatic.*

Notice that the difference is just that the infinite set A is not necessarily a subset of ω, and therefore, it does not necessarily contain a countable infinite subset. However, this difference is crucial, since one can show that, like König's Lemma, this statement is not provable without assuming some form of the Axiom of Choice (AC). On the other hand, if one has AC, then every infinite set has a countably infinite subset, and so RAMSEY'S THEOREM implies the original version. Ramsey was aware of this fact and stated explicitly that he is assuming the *axiom of selections* (*i.e.*, AC). Even though we do not need full AC in order to prove RAMSEY'S ORIGINAL THEOREM, there is no way to avoid some non-trivial kind of choice, since there are models of Set Theory in which RAMSEY'S ORIGINAL THEOREM fails (*cf.* PROPOSITION 7.8). Consequently, RAMSEY'S ORIGINAL THEOREM can be used as a choice principle, which will be discussed in Chapter 5.

***Finite Colourings of* $[\omega]^{<\omega}$.** Assume we have coloured all the finite subsets of ω with two colours, say red and blue. Can we be sure that there is an infinite subset of ω such that all its finite subsets have the same colour? The answer to this question is negative and it is not hard to find a counterexample (*e.g.*, colour a set $x \in [\omega]^{<\omega}$ blue, if $|x|$ is even; otherwise, colour it red).

Thus, let us ask for slightly less. Is there at least an infinite subset of ω such that for each $n \in \omega$, all its n-element subsets have the same colour? The answer to this question is also negative: Colour a non-empty set $x \in [\omega]^{<\omega}$ red, if x has more than $\min(x)$ elements (*i.e.*, x is large); otherwise, colour it blue. Now, let I be an infinite subset of ω and let $n = \min(I)$. We leave it as an exercise to the reader to verify that $[I]^{n+1}$ is dichromatic.

The picture changes if we are asking just for an almost homogeneous sets: An infinite set $H \subseteq \omega$ is called **almost homogeneous** for a colouring $\pi : [\omega]^n \to r$ (where $n \in \omega$ and r is a positive integer), if there is a finite set $K \subseteq \omega$ such that $H \setminus K$ is homogeneous for π. Now, for a positive integer r consider any colouring

$\bar{\pi} : [\omega]^{<\omega} \to r$. Then, for each $n \in \omega$, $\bar{\pi}|_{[\omega]^n}$ is a colouring $\pi_n : [\omega]^n \to r$. Is there an infinite set $H \subseteq \omega$ which is almost homogeneous for all π_n's simultaneously? The answer to this question is affirmative and is given by the following result.

PROPOSITION 2.8. *Let $\{r_k : k \in \omega\}$ and $\{n_k : k \in \omega\}$ be two (possibly finite) sets of positive integers, and for each $k \in \omega$ let $\pi_k : [\omega]^{n_k} \to r_k$ be a colouring. Then there exists an infinite set $H \subseteq \omega$ which is almost homogeneous for each π_k ($k \in \omega$).*

Proof. A first attempt to construct the required almost homogeneous set would be to start with an $I_0 \in [\omega]^\omega$ which is homogeneous for π_0, then take an $I_1 \in [I_0]^\omega$ which is homogeneous for π_1, *et cetera*, and finally take the intersection of all the I_k's. Even though this attempt fails—since it is very likely that we end up with the empty set—it is the right direction. In fact, if the intersection of the I_k's would be non-empty, it would be homogeneous for all π_k's, which is more than what is required. In order to end up with an infinite set we just have to modify the above approach—the trick, which is used almost always when the word "almost" is involved, is called *diagonalisation*.

The proof is by induction on k: By RAMSEY'S THEOREM 2.1 there exists an $H_0 \in [\omega]^\omega$ which is homogeneous for π_0. Assume we have already constructed $H_k \in [\omega]^\omega$ (for some $k \geq 0$) such that H_k is homogeneous for π_k. Let $a_k = \min(H_k)$ and let $S_k = H_k \setminus \{a_k\}$. Then, again by RAMSEY'S THEOREM 2.1, there exists an $H_{k+1} \in [S_k]^\omega$ such that H_{k+1} is homogeneous for π_{k+1}. Let $H = \{a_k : k \in \omega\}$. Then, by construction, for every $k \in \omega$ we see that $H \setminus \{a_0, \dots, a_{k-1}\}$ is homogeneous for π_k, which implies that H is almost homogeneous for all π_k's simultaneously. ⊣

Now we could ask what is the least number of 2-colourings of 2-element subsets of ω we need in order to make sure that no single infinite subset of ω is almost homogeneous for all colourings simultaneously? By PROPOSITION 2.8 we know that countably many colourings are not sufficient, but as we will see later, the axioms of Set Theory do not decide how large this number is (*cf.* Chapter 18).

The dual question would be as follows: How large must a family of infinite subsets of ω be, in order to make sure that for each 2-colouring of the 2-element subsets of ω we find a set in the family which is homogeneous for this colouring? Again, the axioms of Set Theory do not decide how large this number is (*cf.* Chapter 18).

Going to the Infinite. There are two parameters involved in a colouring $\pi : [\omega]^n \to r$, namely n and r. Let first consider the case when $n = 2$ and $r = \omega$. In this case, we obviously cannot hope for any infinite homogeneous or almost homogeneous set. However, there are still infinite subsets of ω which are homogeneous in a broader sense which leads to the CANONICAL RAMSEY THEOREM. Even though the CANONICAL RAMSEY THEOREM is a proper generalisation of RAMSEY'S THEOREM, we will not discuss it here (but see RELATED RESULT 0).

In the case when $n = \omega$ and $r = 2$ we cannot hope for an infinite homogeneous set, as the following example illustrates (compare this result with Chapter 5 | RELATED RESULT 38):