Progress in Nonlinear Differential Equations and Their Applications
83

Gyula Csató
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Olivier Kneuss

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Volume 83



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Gyula Csató
Lausanne, Switzerland

Bernard Dacorogna
Lausanne, Switzerland

Olivier Kneuss
Lausanne, Switzerland

ISBN 978-0-8176-8312-2 e-ISBN 978-0-8176-8313-9
DOI 10.1007/978-0-8176-8313-9
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011941790

Mathematics Subject Classification: 15A75, 35FXX, 58AXX



Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.birkhauser-science.com)

# Preface

In the present book we study the pullback equation for differential forms

$$\varphi^{*}(g) = f,$$

namely, given two differential $k$-forms $f$ and $g$ we want to discuss the equivalence of such forms. This turns out to be a system of nonlinear first-order partial differential equations in the unknown map $\varphi$.

The problem that we study here is a particular case of the equivalence of tensors which has received considerable attention. However, the pullback equation for differential forms has quite different features than those for symmetric tensors, such as Riemannian metrics, which has also been studied a great deal. In more physical terms, the problem of equivalence of forms can also be seen as a problem of mass transportation.

This is an important problem in geometry and in analysis. It has been extensively studied, in the cases $k = 2$ and $k = n$, but much less when $3 \leq k \leq n-1$. The problem considered here of finding normal forms (Darboux theorem, Pfaff normal form) is a fundamental question in symplectic and contact geometry. With respect to the literature in geometry, the main emphasis of the book is on regularity and boundary conditions. Indeed, special attention has been given to getting optimal regularity; this is a particularly delicate point and requires estimates for elliptic equations and fine properties of Hölder spaces.

In the case of volume forms (i.e., $k = n$), our problem is clearly related to the widely studied subject of optimal mass transportation. However, our analysis is not in this framework. As stated before, the two main points of our analysis are that we provide optimal regularity in Hölder spaces and, at the same time, we are able to handle boundary conditions.

Our book will hopefully appeal to both geometers and analysts. In order to make the subject more easily attractive for the analysts, we have reduced as much as possible the notations of geometry. For example, we have restricted our attention to domains in $\mathbb{R}^n$, but it goes without saying that all results generalize to manifolds with or without boundary.

In Part I we gather some basic facts about exterior and differential forms that are used throughout Parts II and IV. Most of the results are standard, but they are presented so that the reader may be able to grasp the main results of the subject without getting too involved with the terminology and concepts of differential geometry.

Part II presents the classical Hodge decomposition following the proof of Morrey, but with some variants, notably in our way of deriving the Gaffney inequality. We also give applications to several versions of the Poincaré lemma that are constantly used in the other parts of the book. Part II can be of interest independently of the main subject of the book.

Part III discusses the case $k = n$. We have tried in this part to make it, as much as possible, independent of the machinery of differential forms. Indeed, Part III can essentially be read with no reference to the other parts of the work, except for the properties of Hölder spaces presented in Part V.

Part IV deals with the general case. Emphasis in this part is given to the symplectic case $k = 2$. We also briefly deal with the simpler cases $k = 0, 1, n-1$. The case $3 \leq k \leq n-2$ is much harder and we are able to obtain results only for forms having a special structure. The difficulty is already at the algebraic level.

In Part V we gather several basic properties of Hölder spaces that are used extensively throughout the book. Due to the nonlinearity of the pullback equation, Hölder spaces are much better adapted than Sobolev spaces. The literature on Hölder spaces is considerably smaller than the one on Sobolev spaces. Moreover, the results presented here cannot be found solely in a single reference. We hope that this part will be useful to mathematicians well beyond those who are primarily interested in the pullback equation.

**Acknowledgments** Several results of Part IV find their origins from joint works and discussions with S. Bandyopadhyay. During the preparation of the manuscript, we have benefited from many helpful comments by P. Bousquet, G. Cupini, W. Gangbo, N. Kamran, T. Ratiu, K.D. Semmler, D. Serre and D. Ye. The discussions with M. Troyanov have been particularly fruitful. We also thank H. Brézis for accepting, with enthusiasm, our book in the Birkhäuser series that he edits.

The research of the third author has been, in part, subsidized by a grant of the Fonds National Suisse de la Recherche Scientifique.

Lausanne

*CSATÓ Gyula*
*DACOROGNA Bernard*
*KNEUSS Olivier*

# Contents

# Chapter 1
# Introduction

## 1.1 Statement of the Problem

The aim of this book is the study of the *pullback equation*

$$\varphi^*(g) = f. \tag{1.1}$$

More precisely, we want to find a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$; preferably we want this map to be a diffeomorphism that satisfies the above equation, where $f$ and $g$ are differential $k$-forms, $0 \leq k \leq n$. Most of the time we will require these two forms to be closed. Before going further, let us examine the exact meaning of (1.1). We write

$$g(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k}(x)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and similarly for $f$. The meaning of (1.1) is that

$$\sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} \circ \varphi \; d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_k} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where

$$d\varphi^i = \sum_{j=1}^{n} \frac{\partial \varphi^i}{\partial x_j} dx^j.$$

This turns out to be a *nonlinear* (if $2 \leq k \leq n$) homogeneous of degree $k$ (in the derivatives) first-order system of $\binom{n}{k}$ partial differential equations. Let us see the form that the equation takes when $k = 0, 1, 2, n$.

*Case:* $k = 0$. Equation (1.1) reads as

$$g(\varphi(x)) = f(x)$$

while

$$dg = 0 \;\Leftrightarrow\; \operatorname{grad} g = 0.$$

G. Csató et al., *The Pullback Equation for Differential Forms*, Progress in Nonlinear Differential Equations and Their Applications 83, DOI 10.1007/978-0-8176-8313-9_1,

We will be, only marginally, interested in this elementary case, which is trivial for closed forms. In any case, (1.1) is *not*, when $k=0$, a differential equation.

*Case:* $k=1$. The form $g$, and analogously for $f$, can be written as

$$g(x)=\sum_{i=1}^{n} g_i(x)\,dx^i.$$

Equation (1.1) then becomes

$$\sum_{i=1}^{n} g_i(\varphi(x))\,d\varphi^i=\sum_{i=1}^{n} f_i(x)\,dx^i$$

while

$$dg=0 \Leftrightarrow \operatorname{curl} g=0 \Leftrightarrow \frac{\partial g_i}{\partial x_j}-\frac{\partial g_j}{\partial x_i}=0, \quad 1\le i<j\le n.$$

Writing

$$d\varphi^i=\sum_{j=1}^{n}\frac{\partial \varphi^i}{\partial x_j}dx^j$$

and substituting into the equation, we find that (1.1) is equivalent to

$$\sum_{j=1}^{n} g_j(\varphi(x))\frac{\partial \varphi^j}{\partial x_i}(x)=f_i(x), \quad 1\le i\le n.$$

This is a system of $\binom{n}{1}=n$ first-order *linear* (in the first derivatives) partial differential equations.

*Case:* $k=2$. The form $g$, and analogously for $f$, can be written as

$$g=\sum_{1\le i<j\le n} g_{ij}(x)\,dx^i\wedge dx^j$$

while

$$dg=0 \Leftrightarrow \frac{\partial g_{ij}}{\partial x_k}-\frac{\partial g_{ik}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_i}=0, \quad 1\le i<j<k\le n.$$

The equation $\varphi^*(g)=f$ becomes

$$\sum_{1\le p<q\le n} g_{pq}(\varphi(x))\,d\varphi^p\wedge d\varphi^q=\sum_{1\le i<j\le n} f_{ij}(x)\,dx^i\wedge dx^j.$$

We get, as before, that (1.1) is equivalent, for every $1\le i<j\le n$, to

$$\sum_{1\le p<q\le n} g_{pq}(\varphi(x))\left(\frac{\partial \varphi^p}{\partial x_i}\frac{\partial \varphi^q}{\partial x_j}-\frac{\partial \varphi^p}{\partial x_j}\frac{\partial \varphi^q}{\partial x_i}\right)(x)=f_{ij}(x),$$

which is a *nonlinear* homogeneous of degree 2 (in the derivatives) system of $\binom{n}{2}=\frac{n(n-1)}{2}$ first-order partial differential equations.

*Case:* $k = n$. In this case we always have $df = dg = 0$. By abuse of notations, if we identify volume forms and functions, we get that the equation $\varphi^*(g) = f$ becomes

$$g(\varphi(x)) \det \nabla\varphi(x) = f(x).$$

It is then a nonlinear homogeneous of degree $n$ (in the derivatives) first-order partial differential equation.smallskip

The main questions that we will discuss are the following.

1) *Local existence*. This is the easiest question. We will handle fairly completely the case of closed 2-forms, which is the case of the Darboux theorem. The cases of 1 and $(n-1)$-forms as well as the case of $n$-forms will also be dealt with. It will turn out that the case $3 \le k \le n-2$ is much more difficult and we will be able to handle only closed $k$-forms with special structure.
2) *Global existence*. This is a much more difficult problem. We will obtain results in the case of volume forms and of closed 2-forms.
3) *Regularity*. A special emphasis will be given on getting sharp regularity results. For this reason we will have to work with Hölder spaces $C^{r,\alpha}$, $0 < \alpha < 1$, not with spaces $C^r$. Apart from the linear problems considered in Part II, we will not deal with Sobolev spaces. In the present context the reason is that Hölder spaces form an algebra contrary to Sobolev spaces (with low exponents).

## 1.2 Exterior and Differential Forms

In Chapter 2 we have gathered some algebraic results about exterior forms that are used throughout the book.

### *1.2.1 Definitions and Basic Properties of Exterior Forms*

Let $1 \le k \le n$ be an integer. An exterior $k$-form will be denoted by

$$f = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

The set of exterior $k$-forms over $\mathbb{R}^n$ is a vector space and is denoted $\Lambda^k(\mathbb{R}^n)$ and its dimension is

$$\dim(\Lambda^k(\mathbb{R}^n)) = \binom{n}{k}.$$

If $k = 0$, we set

$$\Lambda^0(\mathbb{R}^n) = \mathbb{R}.$$

By abuse of notations, we will, when convenient and in order not to burden the notations, identify $k$-forms with vectors in $\mathbb{R}^{\binom{n}{k}}$.

(i) The *exterior product* of $f \in \Lambda^k(\mathbb{R}^n)$ with $g \in \Lambda^l(\mathbb{R}^n)$, denoted by $f \wedge g$, is defined as usual (cf. Definition 2.2) and it belongs to $\Lambda^{k+l}(\mathbb{R}^n)$. The *scalar product* between two $k$-forms $f$ and $g$ is denoted by

$$\langle g; f \rangle = \sum_{1 \leq i_1 < \cdots < i_k \leq n} g_{i_1 \cdots i_k} f_{i_1 \cdots i_k}.$$

The *Hodge star operator* (cf. Definition 2.9) associates to $f \in \Lambda^k(\mathbb{R}^n)$ a form $(*f) \in \Lambda^{n-k}(\mathbb{R}^n)$. We define (cf. Definition 2.11) the *interior product* of $f \in \Lambda^k(\mathbb{R}^n)$ with $g \in \Lambda^l(\mathbb{R}^n)$ by

$$g \lrcorner f = (-1)^{n(k-l)} * (g \wedge (*f)).$$

These definitions are linked through the following elementary facts (cf. Proposition 2.16). For every $f \in \Lambda^k(\mathbb{R}^n)$, $g \in \Lambda^{k+1}(\mathbb{R}^n)$ and $h \in \Lambda^1(\mathbb{R}^n)$,

$$|h|^2 f = h \lrcorner (h \wedge f) + h \wedge (h \lrcorner f),$$
$$\langle h \wedge f; g \rangle = \langle f; h \lrcorner g \rangle.$$

(ii) Let $A \in \mathbb{R}^{n \times n}$ be a matrix and let $f \in \Lambda^k(\mathbb{R}^n)$ be given by

$$f = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} e^{i_1} \wedge \cdots \wedge e^{i_k}.$$

We define (cf. Definition 2.17) the *pullback of f by A,* denoted $A^*(f)$, by

$$A^*(f) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} A^{i_1} \wedge \cdots \wedge A^{i_k} \in \Lambda^k(\mathbb{R}^n),$$

where $A^j$ is the $j$th row of $A$ and is identified by

$$A^j = \sum_{k=1}^{n} A^j_k e^k \in \Lambda^1(\mathbb{R}^n).$$

If $k = 0$, we then let

$$A^*(f) = f.$$

The present definition is consistent with the one given at the beginning of the chapter; just set $\varphi(x) = Ax$ in (1.1).

(iii) We next define the notion of *rank* (also called rank of order 1 in Chapter 2) of $f \in \Lambda^k(\mathbb{R}^n)$. We first associate to the linear map

$$g \in \Lambda^1(\mathbb{R}^n) \to g \lrcorner f \in \Lambda^{k-1}(\mathbb{R}^n)$$

a matrix $\overline{f} \in \mathbb{R}^{\binom{n}{k-1} \times n}$ such that, by abuse of notations,

$$g \lrcorner f = \overline{f}\, g \quad \text{for every } g \in \Lambda^1 (\mathbb{R}^n) .$$

In this case, we have

$$g \lrcorner f$$
$$= \sum_{1 \leq j_1 < \cdots < j_{k-1} \leq n} \left( \sum_{\gamma=1}^{k} (-1)^{\gamma-1} \sum_{j_{\gamma-1} < i < j_\gamma} f_{j_1 \cdots j_{\gamma-1} i j_\gamma \cdots j_{k-1}}\, g_i \right) e^{j_1} \wedge \cdots \wedge e^{j_{k-1}} .$$

More explicitly, using the lexicographical order for the columns (index below) and the rows (index above) of the matrix $\overline{f}$, we have

$$(\overline{f})_i^{j_1 \cdots j_{k-1}} = f_{i\, j_1 \cdots j_{k-1}}$$

for $1 \leq i \leq n$ and $1 \leq j_1 < \cdots < j_{k-1} \leq n$. The rank of the $k$-form $f$ is then the rank of the $\binom{n}{k-1} \times n$ matrix $\overline{f}$ (or similarly the rank of the map $g \to g \lrcorner f$). We then write (in Chapter 2, we write $\operatorname{rank}_1 [f]$, but in the remaining part of the book we write only $\operatorname{rank} [f]$)

$$\operatorname{rank} [f] = \operatorname{rank} \left( \overline{f} \right) .$$

Note that only when $k = 2$ or $k = n$, the matrix $\overline{f}$ is a square matrix. We will get our best results precisely in these cases and when the matrix $\overline{f}$ is invertible.

We then have the following elementary result (cf. Proposition 2.37).

**Proposition 1.1.** *Let* $f \in \Lambda^k (\mathbb{R}^n)$, $f \neq 0$.

*(i) If* $k = 1$, *then the rank of* $f$ *is always* 1.

*(ii) If* $k = 2$, *then the rank of* $f$ *is even. The forms*

$$\omega_m = \sum_{i=1}^{m} e^{2i-1} \wedge e^{2i}$$

*are such that* $\operatorname{rank} [\omega_m] = 2m$. *Moreover,* $\operatorname{rank} [f] = 2m$ *if and only if*

$$f^m \neq 0 \quad \text{and} \quad f^{m+1} = 0,$$

*where* $f^m = \underbrace{f \wedge \cdots \wedge f}_{m \text{ times}}$.

*(iii) If* $3 \leq k \leq n$, *then*

$$\operatorname{rank} [f] \in \{k, k+2, \ldots, n\}$$

*and any of the values in* $\{k, k+2, \ldots, n\}$ *can be achieved by the rank of a k-form. In particular, if* $k = n-1$, *then* $\operatorname{rank} [f] = n-1$, *whereas if* $k = n$, *then* $\operatorname{rank} [f] = n$.

*Remark 1.2 (cf. Propositions 2.24 and 2.33).* The rank is an invariant for the pull-back equation. More precisely, if there exists $A \in \mathrm{GL}(n)$ (i.e., $A$ is an invertible $n \times n$ matrix) such that

$$A^*(g) = f,$$

then

$$\operatorname{rank}[g] = \operatorname{rank}[f].$$

Conversely, when $k = 1, 2, n-1, n,$ if $\operatorname{rank}[g] = \operatorname{rank}[f]$, then there exists $A \in \mathrm{GL}(n)$ such that

$$A^*(g) = f.$$

However, the converse is not true, in general, if $3 \leq k \leq n-2$. For example (cf. Example 2.36), when $k = 3$, the forms

$$f = e^1 \wedge e^2 \wedge e^3 + e^4 \wedge e^5 \wedge e^6,$$
$$g = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^2 \wedge e^4 \wedge e^6 + e^3 \wedge e^5 \wedge e^6$$

have both rank $= 6$, but there is no $A \in \mathrm{GL}(6)$ so that

$$A^*(g) = f.$$

Similarly and more strikingly (cf. Example 2.35), when $k = 4$ and

$$f = e^1 \wedge e^2 \wedge e^3 \wedge e^4 + e^1 \wedge e^2 \wedge e^5 \wedge e^6 + e^3 \wedge e^4 \wedge e^5 \wedge e^6,$$

there is no $A \in \mathrm{GL}(6)$ such that

$$A^*(f) = -f.$$

### 1.2.2 Divisibility

We then discuss the notion of *divisibility* for exterior forms. Given two integers $1 \leq l \leq k \leq n$, a $k$-form $f$ and a $l$-form $g$, we want to know if we can find a $(k-l)$-form $u$ so that

$$f = g \wedge u.$$

This is an important question in the theory of Grassmann algebras. A well-known result is the so called Cartan lemma (cf. Theorem 2.42).

**Theorem 1.3 (Cartan lemma).** *Let* $1 \leq k \leq n$ *and* $f \in \Lambda^k(\mathbb{R}^n)$ *with* $f \neq 0$. *Let* $1 \leq l \leq k$ *and* $g_1, \ldots, g_l \in \Lambda^1(\mathbb{R}^n)$ *be such that*

$$g_1 \wedge \cdots \wedge g_l \neq 0.$$

*Then there exists* $u \in \Lambda^{k-l}(\mathbb{R}^n)$ *verifying*

$$f = g_1 \wedge \cdots \wedge g_l \wedge u$$

*if and only if*

$$f \wedge g_1 = \cdots = f \wedge g_l = 0.$$

*Remark 1.4.* In the same spirit, the following facts can easily be proved (cf. Proposition 2.43):

(i) The form $f \in \Lambda^k(\mathbb{R}^n)$ is totally divisible, meaning that there exist $f_1, \cdots, f_k \in \Lambda^1(\mathbb{R}^n)$ such that

$$f = f_1 \wedge \cdots \wedge f_k$$

if and only if

$$\operatorname{rank}[f] = k.$$

(ii) If $k$ is odd and if $f \in \Lambda^k(\mathbb{R}^n)$ with $\operatorname{rank}[f] = k+2$, then there exist $u \in \Lambda^1(\mathbb{R}^n)$ and $g \in \Lambda^{k-1}(\mathbb{R}^n)$ such that

$$f = g \wedge u.$$

Our main result (cf. Theorem 2.45 for a more general statement) will be the following theorem obtained by Dacorogna–Kneuss [31]. It generalizes the Cartan lemma.

**Theorem 1.5.** *Let $0 \leq l \leq k \leq n$ be integers. Let $g \in \Lambda^l(\mathbb{R}^n)$ and $f \in \Lambda^k(\mathbb{R}^n)$. The following statements are then equivalent:*

*(i) There exists $u \in \Lambda^{k-l}(\mathbb{R}^n)$ verifying*

$$f = g \wedge u.$$

*(ii) For every $h \in \Lambda^{n-k}(\mathbb{R}^n)$, the following implication holds:*

$$[h \wedge g = 0] \quad \Rightarrow \quad [h \wedge f = 0].$$

### *1.2.3 Differential Forms*

In Chapter 3 we have gathered the main notations concerning differential forms.

**Definition 1.6.** Let $\Omega \subset \mathbb{R}^n$ be open and $f \in C^1\left(\Omega; \Lambda^k\right)$, namely

$$f = \sum_{1 \leq i_1 < \cdots < i_k \leq n} f_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

(i) The *exterior derivative* of $f$ denoted $df$ belongs to $C^0\left(\Omega; \Lambda^{k+1}\right)$ and is defined by

$$df = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sum_{m=1}^{n} \frac{\partial f_{i_1 \cdots i_k}}{\partial x_m} dx^m \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

If $k = n$, then $df = 0$.

(ii) The *interior derivative* or *codifferential* of $f$ denoted $\delta f$ belongs to $C^0\left(\Omega;\Lambda^{k-1}\right)$ and is defined by

$$\delta f = (-1)^{n(k-1)} * (d(*f)).$$

*Remark 1.7.* (i) If $k = 0$, then the operator $d$ can be identified with the gradient operator, while $\delta f = 0$ for any $f$.

(ii) If $k = 1$, then the operator $d$ can be identified with the curl operator and the operator $\delta$ is the divergence operator.

We next gather some well-known properties of the operators $d$ and $\delta$ (cf. Theorems 3.5 and 3.7).

**Theorem 1.8.** *Let* $f \in C^2(\Omega;\Lambda^k)$. *Then*

$$ddf = 0, \quad \delta\delta f = 0 \quad \textit{and} \quad d\delta f + \delta df = \Delta f.$$

We also need the following definition. In the sequel we will denote the exterior unit normal of $\partial\Omega$ by $\nu$.

**Definition 1.9.** The *tangential component* of a $k$-form $f$ on $\partial\Omega$ is the $(k+1)$-form

$$\nu \wedge f \in \Lambda^{k+1}.$$

The *normal component* of a $k$-form $f$ on $\partial\Omega$ is the $(k-1)$-form

$$\nu \lrcorner f \in \Lambda^{k-1}.$$

We easily deduce the following properties (cf. Theorem 3.23).

**Proposition 1.10.** *Let* $0 \le k \le n$ *and* $f \in C^1\left(\overline{\Omega};\Lambda^k\right)$; *then*

$$\begin{aligned} \nu \wedge f = 0 \text{ on } \partial\Omega \ &\Rightarrow \ \nu \wedge df = 0 \text{ on } \partial\Omega, \\ \nu \lrcorner f = 0 \text{ on } \partial\Omega \ &\Rightarrow \ \nu \lrcorner \delta f = 0 \text{ on } \partial\Omega. \end{aligned}$$

We will constantly use the integration by parts formula (cf. Theorem 3.28).

**Theorem 1.11.** *Let* $1 \le k \le n$, $f \in C^1\left(\overline{\Omega};\Lambda^{k-1}\right)$ *and* $g \in C^1\left(\overline{\Omega};\Lambda^k\right)$. *Then*

$$\int_\Omega \langle df; g\rangle + \int_\Omega \langle f; \delta g\rangle = \int_{\partial\Omega} \langle \nu \wedge f; g\rangle = \int_{\partial\Omega} \langle f; \nu \lrcorner g\rangle.$$

We will adopt the following notations.

**Notation 1.12.** *Let* $\Omega \subset \mathbb{R}^n$ *be open,* $r \ge 0$ *be an integer and* $0 \le \alpha \le 1 \le p \le \infty$. *Spaces with vanishing tangential or normal component will be denoted in the following way:*

$$
\begin{aligned}
C_T^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right) &= \{f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right) : \nu \wedge f = 0 \quad on\ \partial\Omega\},\\
C_N^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right) &= \{f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^k\right) : \nu \lrcorner f = 0 \quad on\ \partial\Omega\},\\
W_T^{r+1,p}\left(\Omega;\Lambda^k\right) &= \{f \in W^{r+1,p}\left(\Omega;\Lambda^k\right) : \nu \wedge f = 0 \quad on\ \partial\Omega\},\\
W_N^{r+1,p}\left(\Omega;\Lambda^k\right) &= \{f \in W^{r+1,p}\left(\Omega;\Lambda^k\right) : \nu \lrcorner f = 0 \quad on\ \partial\Omega\}.
\end{aligned}
$$

*The different sets of* harmonic fields *will be denoted by*

$$
\begin{aligned}
\mathscr{H}\left(\Omega;\Lambda^k\right) &= \{f \in W^{1,2}\left(\Omega;\Lambda^k\right) : df = 0 \text{ and } \delta f = 0 \text{ in } \Omega\},\\
\mathscr{H}_T\left(\Omega;\Lambda^k\right) &= \{f \in \mathscr{H}\left(\Omega;\Lambda^k\right) : \nu \wedge f = 0 \quad on\ \partial\Omega\},\\
\mathscr{H}_N\left(\Omega;\Lambda^k\right) &= \{f \in \mathscr{H}\left(\Omega;\Lambda^k\right) : \nu \lrcorner f = 0 \quad on\ \partial\Omega\}.
\end{aligned}
$$

We now list (cf. Section 6.1) some properties of the harmonic fields.

**Theorem 1.13.** *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then*

$$\mathscr{H}\left(\Omega;\Lambda^k\right) \subset C^\infty\left(\Omega;\Lambda^k\right).$$

*Moreover if $\Omega$ is bounded and smooth, then the next statements are valid.*

*(i) The following inclusion holds:*

$$\mathscr{H}_T\left(\Omega;\Lambda^k\right) \cup \mathscr{H}_N\left(\Omega;\Lambda^k\right) \subset C^\infty\left(\overline{\Omega};\Lambda^k\right).$$

*Furthermore, if $r \geq 0$ is an integer and $0 \leq \alpha \leq 1$, then there exists $C = C(r,\Omega)$ such that for every $\omega \in \mathscr{H}_T\left(\Omega;\Lambda^k\right) \cup \mathscr{H}_N\left(\Omega;\Lambda^k\right)$,*

$$\|\omega\|_{W^{r,2}} \leq C\|\omega\|_{L^2} \quad \text{and} \quad \|\omega\|_{C^{r,\alpha}} \leq C\|\omega\|_{C^0}.$$

*(ii) The spaces $\mathscr{H}_T\left(\Omega;\Lambda^k\right)$ and $\mathscr{H}_N\left(\Omega;\Lambda^k\right)$ are finite dimensional and closed in $L^2\left(\Omega;\Lambda^k\right)$.*

*(iii) Furthermore, if $\Omega$ is contractible (cf. Definition 6.1), then*

$$
\begin{aligned}
\mathscr{H}_T\left(\Omega;\Lambda^k\right) &= \{0\} \quad \text{if } 0 \leq k \leq n-1,\\
\mathscr{H}_N\left(\Omega;\Lambda^k\right) &= \{0\} \quad \text{if } 1 \leq k \leq n.
\end{aligned}
$$

*(iv) If $k = 0$ or $k = n$ and $h \in \mathscr{H}\left(\Omega;\Lambda^k\right)$, then $h$ is constant on each connected component of $\Omega$. In particular, $\mathscr{H}_T\left(\Omega;\Lambda^0\right) = \{0\}$ and $\mathscr{H}_N\left(\Omega;\Lambda^n\right) = \{0\}$.*

*Remark 1.14.* If $k = 1$ and assuming that $\Omega$ is smooth, then the sets $\mathscr{H}_T$ and $\mathscr{H}_N$ can be rewritten, as usual by abuse of notations, as

$$
\begin{aligned}
\mathscr{H}_T\left(\Omega;\Lambda^1\right) &= \left\{ f \in C^\infty\left(\overline{\Omega};\mathbb{R}^n\right) : \begin{bmatrix} \operatorname{curl} f = 0 \text{ and } \operatorname{div} f = 0 \\ f_i\nu_j - f_j\nu_i = 0,\ \forall\, 1 \leq i < j \leq n \end{bmatrix} \right\},\\
\mathscr{H}_N\left(\Omega;\Lambda^1\right) &= \left\{ f \in C^\infty\left(\overline{\Omega};\mathbb{R}^n\right) : \begin{bmatrix} \operatorname{curl} f = 0 \text{ and } \operatorname{div} f = 0 \\ \sum_{i=1}^n f_i\nu_i = 0 \end{bmatrix} \right\}.
\end{aligned}
$$

Moreover, if $\Omega$ is simply connected, then

$$\mathscr{H}_T\left(\Omega;\Lambda^1\right)=\mathscr{H}_N\left(\Omega;\Lambda^1\right)=\{0\}.$$

## 1.3 Hodge–Morrey Decomposition and Poincaré Lemma

### *1.3.1 A General Identity and Gaffney Inequality*

In the proof of Morrey of the Hodge decomposition, one of the key points to get compactness is the following inequality (cf. Theorem 5.16).

**Theorem 1.15 (Gaffney inequality).** *Let $\Omega\subset\mathbb{R}^n$ be a bounded open smooth set. Then there exists a constant $C=C(\Omega)>0$ such that*

$$\|\omega\|_{W^{1,2}}^2\le C\left(\|d\omega\|_{L^2}^2+\|\delta\omega\|_{L^2}^2+\|\omega\|_{L^2}^2\right)$$

*for every $\omega\in W_T^{1,2}(\Omega;\Lambda^k)\cup W_N^{1,2}(\Omega;\Lambda^k)$.*

*Remark 1.16.* When $k=1$, the inequality says, identifying 1-forms with vector fields,

$$\|\omega\|_{W^{1,2}}^2\le C\left(\|\operatorname{curl}\omega\|_{L^2}^2+\|\operatorname{div}\omega\|_{L^2}^2+\|\omega\|_{L^2}^2\right)$$

for every $\omega\in W^{1,2}(\Omega;\mathbb{R}^n)$ satisfying either one of the following two conditions:

$$\nu\wedge\omega=0\ \Leftrightarrow\ \omega_i\nu_j-\omega_j\nu_i=0,\ \forall\,1\le i<j\le n,$$

$$\nu\lrcorner\,\omega=\langle\nu;\omega\rangle=\sum_{i=1}^n\omega_i\nu_i=0.$$

The inequality, as stated above, has been proved by Morrey [76, 77], generalizing results of Gaffney [44, 45]. We will prove in Section 5.3 the inequality appealing to a very general identity (see Theorem 5.7) proved by Csató and Dacorogna [24].

**Theorem 1.17 (A general identity).** *Let $0\le k\le n$ and let $\Omega\subset\mathbb{R}^n$ be a bounded open smooth set and with exterior unit normal $\nu$. Then every $\alpha,\beta\in C^1(\overline{\Omega};\Lambda^k)$ satisfy the equation*

$$\begin{aligned}&\int_\Omega\left(\langle d\alpha;d\beta\rangle+\langle\delta\alpha;\delta\beta\rangle-\langle\nabla\alpha;\nabla\beta\rangle\right)\\&\quad=-\int_{\partial\Omega}\left(\langle\nu\wedge d(\nu\lrcorner\,\alpha);\nu\wedge\beta\rangle+\langle\nu\lrcorner\,\delta(\nu\wedge\alpha);\nu\lrcorner\,\beta\rangle\right)\\&\qquad+\int_{\partial\Omega}\left(\langle L^\nu(\nu\wedge\alpha);\nu\wedge\beta\rangle+\langle K^\nu(\nu\lrcorner\,\alpha);\nu\lrcorner\,\beta\rangle\right).\end{aligned}$$

The operators $L^\nu$ and $K^\nu$ (cf. Definition 5.1) can be seen as matrices acting on $(k+1)$-forms and $(k-1)$-forms respectively (identifying, as usual, a $k$-form with a $\binom{n}{k}$ vector). They depend only on the geometry of $\Omega$ and on the degree $k$ of the form. They can easily be calculated explicitly for general $k$-forms and, when $\Omega$ is a ball of radius $R$ (cf. Corollary 5.9), it turns out that

$$L^\nu(\nu \wedge \omega) = \frac{k}{R}\,\nu \wedge \omega \quad \text{and} \quad K^\nu(\nu \lrcorner \omega) = \frac{n-k}{R}\,\nu \lrcorner \omega$$

and, thus,

$$\langle L^\nu(\nu \wedge \omega); \nu \wedge \omega\rangle = \frac{k}{R}\,|\nu \wedge \omega|^2 \quad \text{and} \quad \langle K^\nu(\nu \lrcorner \omega); \nu \lrcorner \omega\rangle = \frac{n-k}{R}\,|\nu \lrcorner \omega|^2.$$

In the case of a 1-form and for general open sets $\Omega$ (cf. Proposition 5.11), it can be shown that $K^\nu$ is a scalar and it is a multiple of $\kappa$, the mean curvature of the hypersurface $\partial\Omega$, namely

$$K^\nu = (n-1)\,\kappa.$$

Summarizing the results for a 1-form $\omega$ in $\mathbb{R}^n$ (cf. Corollary 5.12) with vanishing tangential component (i.e., $\nu \wedge \omega = 0$ on $\partial\Omega$), we have

$$\int_\Omega \left(|\operatorname{curl}\omega|^2 + |\operatorname{div}\omega|^2 - |\nabla\omega|^2\right) = (n-1)\int_{\partial\Omega} \kappa\,[\langle \nu;\omega\rangle]^2,$$

where $\kappa$ is the mean curvature of the hypersurface $\partial\Omega$ and $\langle .;.\rangle$ denotes the scalar product in $\mathbb{R}^n$.

## 1.3.2 The Hodge–Morrey Decomposition

We now turn to the celebrated Hodge–Morrey decomposition (cf. Theorem 6.9).

**Theorem 1.18 (Hodge–Morrey decomposition).** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $0 \le k \le n$ and $f \in L^2(\Omega;\Lambda^k)$. Then there exist*

$$\alpha \in W_T^{1,2}(\Omega;\Lambda^{k-1}), \quad \beta \in W_T^{1,2}(\Omega;\Lambda^{k+1}),$$
$$h \in \mathcal{H}_T(\Omega;\Lambda^k) \quad \text{and} \quad \omega \in W_T^{2,2}(\Omega;\Lambda^k)$$

*such that, in $\Omega$,*

$$f = d\alpha + \delta\beta + h, \quad \alpha = \delta\omega \quad \text{and} \quad \beta = d\omega.$$

*Remark 1.19.* (i) We have quoted only one of the three decompositions (cf. Theorem 6.9 for details). Another one, completely similar, is by replacing $T$ by $N$ and the other one mixing both $T$ and $N$.

(ii) If $k \le n-1$ and if $\Omega$ is contractible, then $h = 0$.

(iii) If $k = 0$, then the theorem reads as

$$f = \delta\beta = \delta d\omega = \Delta\omega \text{ in } \Omega \quad \text{with} \quad \omega = 0 \text{ on } \partial\Omega.$$

(iv) When $k=1$ and $n=3$, the decomposition reads as follows. Let $\nu$ be the exterior unit normal. For any $f \in L^2(\Omega;\mathbb{R}^3)$, there exist

$$\omega \in W^{2,2}\left(\Omega;\mathbb{R}^3\right) \quad \text{with } \omega_i\nu_j - \omega_j\nu_i = 0 \text{ on } \partial\Omega,\ \forall\, 1 \le i < j \le 3$$
$$\alpha \in W_0^{1,2}(\Omega) \quad \text{and} \quad \alpha = \operatorname{div}\omega,$$
$$\beta \in W^{1,2}\left(\Omega;\mathbb{R}^3\right) \quad \text{with } \beta = -\operatorname{curl}\omega \quad \text{and} \quad \langle \nu;\beta\rangle = 0 \text{ on } \partial\Omega$$
$$h \in \left\{ h \in C^\infty\left(\overline{\Omega};\mathbb{R}^3\right) : \left[ \begin{array}{c} \operatorname{curl} h = 0 \text{ and } \operatorname{div} h = 0 \\ h_i\nu_j - h_j\nu_i = 0,\ \forall\, 1 \le i < j \le 3 \end{array} \right. \right\}$$

such that

$$f = \operatorname{grad}\alpha + \operatorname{curl}\beta + h \text{ in } \Omega.$$

Furthermore, if $\Omega$ is simply connected, then $h = 0$.

(v) If $f$ is more regular than in $L^2$, then $\alpha, \beta$ and $\omega$ are in the corresponding class of regularity (cf. Theorem 6.12). More precisely if, for example, $r \ge 0$ is an integer, $0 < q < 1$ and $f \in C^{r,q}(\overline{\Omega};\Lambda^k)$, then

$$\alpha \in C^{r+1,q}\left(\overline{\Omega};\Lambda^{k-1}\right), \quad \beta \in C^{r+1,q}\left(\overline{\Omega};\Lambda^{k+1}\right) \quad \text{and} \quad \omega \in C^{r+2,q}\left(\overline{\Omega};\Lambda^{k}\right).$$

(vi) The proof of Morrey (cf. Theorem 6.7) uses the direct methods of the calculus of variations. One minimizes

$$D_f(\omega) = \int_\Omega \left( \frac{1}{2}|d\omega|^2 + \frac{1}{2}|\delta\omega|^2 + \langle f;\omega\rangle \right)$$

in an appropriate space, Gaffney inequality giving the coercivity of the integral.

### *1.3.3 First-Order Systems of Cauchy–Riemann Type*

It turns out that the Hodge–Morrey decomposition is in fact equivalent (cf. Proposition 7.9) to solving the first-order system

$$\begin{cases} d\omega = f \quad \text{and} \quad \delta\omega = g & \text{in } \Omega, \\ \nu \wedge \omega = \nu \wedge \omega_0 & \text{on } \partial\Omega \end{cases}$$

or the similar one,

$$\begin{cases} d\omega = f \quad \text{and} \quad \delta\omega = g & \text{in } \Omega, \\ \nu \lrcorner \omega = \nu \lrcorner \omega_0 & \text{on } \partial\Omega. \end{cases}$$

Both systems are discussed in Theorems 7.2 and 7.4. We here state a simplified version of the first one.

**Theorem 1.20.** *Let $r \geq 0$ and $1 \leq k \leq n-2$ be integers, $0 < q < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded contractible open smooth set and with exterior unit normal $\nu$. Let $g \in C^{r,q}\left(\overline{\Omega};\Lambda^{k-1}\right)$ and $f \in C^{r,q}\left(\overline{\Omega};\Lambda^{k+1}\right)$ be such that*

$$\delta g = 0 \text{ in } \Omega, \quad df = 0 \text{ in } \Omega \quad \text{and} \quad \nu \wedge f = 0 \text{ on } \partial\Omega.$$

*Then there exists $\omega \in C^{r+1,q}\left(\overline{\Omega};\Lambda^k\right)$, such that*

$$\begin{cases} d\omega = f \quad \text{and} \quad \delta\omega = g & \text{in } \Omega, \\ \qquad \nu \wedge \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

*Remark 1.21.* (i) It turns out that the sufficient conditions are also necessary (cf. Theorems 7.2 and 7.4).

(ii) When $k = n-1$, the result is valid provided

$$\int_\Omega f = 0.$$

Note that in this case the conditions $df = 0$ and $\nu \wedge f = 0$ are automatically fulfilled.

(iii) Completely analogous results are given in Theorems 7.2 and 7.4 for Sobolev spaces.

(iv) If $\Omega$ is not contractible, then additional necessary conditions have to be added.

(v) When $k = 1$ and $n = 3$, the theorem reads as follows. Let $\Omega \subset \mathbb{R}^3$ be a bounded contractible smooth open set, $g \in C^{r,q}\left(\overline{\Omega}\right)$ and $f \in C^{r,q}\left(\overline{\Omega};\mathbb{R}^3\right)$ be such that

$$\operatorname{div} f = 0 \text{ in } \Omega \quad \text{and} \quad \langle f;\nu\rangle = 0 \text{ on } \partial\Omega.$$

Then there exists $\omega \in C^{r+1,q}\left(\overline{\Omega};\mathbb{R}^3\right)$ such that

$$\begin{cases} \operatorname{curl}\omega = f \quad \text{and} \quad \operatorname{div}\omega = g & \text{in } \Omega, \\ \omega_i\nu_j - \omega_j\nu_i = 0 \ \forall\, 1 \leq i < j \leq 3 & \text{on } \partial\Omega. \end{cases}$$

### *1.3.4 Poincaré Lemma*

We start with the classical Poincaré lemma (cf. Theorem 8.1).

**Theorem 1.22 (Poincaré lemma).** *Let $r \geq 1$ and $0 \leq k \leq n-1$ be integers and $\Omega \subset \mathbb{R}^n$ be an open contractible set. Let $g \in C^r\left(\Omega;\Lambda^{k+1}\right)$ with $dg = 0$ in $\Omega$. Then there exists $G \in C^r\left(\Omega;\Lambda^k\right)$ such that*

$$dG = g \quad \text{in } \Omega.$$

With the help of the Hodge–Morrey decomposition, the result can be improved (cf. Theorem 8.3) in two directions. First, one can consider general sets $\Omega$, not only contractible sets. Moreover, one can get sharp regularity in Hölder and in Sobolev spaces. We quote here only the case of Hölder spaces. We also give the theorem with the $d$ operator. Analogous results are also valid for the $\delta$ operator; see Theorem 8.4.

**Theorem 1.23.** *Let $r \geq 0$ and $0 \leq k \leq n-1$ be integers, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $f : \overline{\Omega} \to \Lambda^{k+1}$. The following statements are equivalent:*

*(i) Let $f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^{k+1}\right)$ be such that*

$$df = 0 \text{ in } \Omega \quad \text{and} \quad \int_\Omega \langle f;\psi\rangle = 0 \text{ for every } \psi \in \mathscr{H}_N\left(\Omega;\Lambda^{k+1}\right).$$

*(ii) There exists $\omega \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^{k}\right)$ such that*

$$d\omega = f \quad \text{in } \Omega.$$

*Remark 1.24.* (i) When $k = n-1$, there is no restriction on the solvability of $d\omega = f$.

(ii) Recall that if $\Omega$ is contractible and $0 \leq k \leq n-1$, then

$$\mathscr{H}_N\left(\Omega;\Lambda^{k+1}\right) = \{0\}.$$

We finally consider the boundary value problems

$$\begin{cases} d\omega = f & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \delta\omega = g & \text{in } \Omega, \\ \omega = \omega_0 & \text{on } \partial\Omega. \end{cases}$$

We give a result for the first one and for $\omega_0 = 0$ (cf. Theorem 8.16 for general $\omega_0$), but a similar one (cf. Theorem 8.18) exists for the second problem. We only discuss the case of Hölder spaces, but the result is also valid in Sobolev spaces (see Theorems 8.16 and 8.18 for details).

**Theorem 1.25.** *Let $r \geq 0$ and $0 \leq k \leq n-1$ be integers, $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set and with exterior unit normal $\nu$. Then the following statements are equivalent:*

*(i) Let $f \in C^{r,\alpha}\left(\overline{\Omega};\Lambda^{k+1}\right)$ satisfy*

$$df = 0 \text{ in } \Omega, \quad \nu \wedge f = 0 \text{ on } \partial\Omega,$$

*and, for every $\chi \in \mathscr{H}_T\left(\Omega;\Lambda^{k+1}\right)$,*

$$\int_\Omega \langle f;\chi\rangle = 0.$$

*(ii) There exists* $\omega \in C^{r+1,\alpha}\left(\overline{\Omega};\Lambda^k\right)$ *such that*

$$\begin{cases} d\omega = f & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega. \end{cases}$$

## 1.4 The Case of Volume Forms

### *1.4.1 Statement of the Problem*

In Part III, we will discuss the following problem. Given $\Omega$ a bounded open set in $\mathbb{R}^n$ and $f, g : \mathbb{R}^n \to \mathbb{R}$, we want to find $\varphi : \overline{\Omega} \to \mathbb{R}^n$ verifying

$$\begin{cases} g(\varphi(x)) \det \nabla\varphi(x) = f(x) & x \in \Omega, \\ \varphi(x) = x & x \in \partial\Omega. \end{cases} \tag{1.2}$$

Writing the functions $f$ and $g$ as volume forms through the straightforward identification

$$g = g(x)dx^1 \wedge \cdots \wedge dx^n \quad \text{and} \quad f = f(x)dx^1 \wedge \cdots \wedge dx^n,$$

problem (1.2) can be written as

$$\begin{cases} \varphi^*(g) = f & \text{in } \Omega, \\ \varphi = \text{id} & \text{on } \partial\Omega, \end{cases}$$

where $\varphi^*(g)$ is the pullback of $g$ by $\varphi$.

The following preliminary remarks are in order.

(i) The case $n = 1$ is completely elementary and is discussed in Section 1.4.2.

(ii) When $n \geq 2$, the equation in (1.2) is a nonlinear first-order *partial differential equation* homogeneous of degree $n$ in the derivatives. It is *underdetermined*, in the sense that we have $n$ unknowns (the components of $\varphi$) and only one equation. Related to this observation, we have that if there exists a solution to our problem, then there are infinitely many ones. Indeed, for example, if $n = 2$, $\Omega$ is the unit ball and $f = g = 1$, the maps $\varphi_m$ (written in polar and in Cartesian coordinates) defined by

$$\begin{aligned} \varphi_m(x) = \varphi_m(x_1, x_2) &= \begin{pmatrix} r\cos\left(\theta + 2m\pi r^2\right) \\ r\sin\left(\theta + 2m\pi r^2\right) \end{pmatrix} \\ &= \begin{pmatrix} x_1 \cos\left(2m\pi\left(x_1^2 + x_2^2\right)\right) - x_2 \sin\left(2m\pi\left(x_1^2 + x_2^2\right)\right) \\ x_2 \cos\left(2m\pi\left(x_1^2 + x_2^2\right)\right) + x_1 \sin\left(2m\pi\left(x_1^2 + x_2^2\right)\right) \end{pmatrix} \end{aligned}$$

satisfy (1.2) for every $m \in \mathbb{Z}$.

(iii) An integration by parts, or, what amounts to the same thing, an elementary topological degree argument (see (19.3)), immediately gives the *necessary condition* (independently of the fact that $\varphi$ is a diffeomorphism or not and of the fact that $\varphi(\Omega)$ contains strictly or not $\Omega$)

$$\int_{\Omega} f = \int_{\Omega} g. \tag{1.3}$$

In most of our analysis, it will turn out that this condition is also sufficient.

(iv) We will always assume that $g > 0$. If $g$ is not strictly positive, then hypotheses other than (1.3) are necessary; for example, $f$ cannot be strictly positive. Indeed if, for example, $f \equiv 1$ and $g$ is allowed to vanish even at a single point, then no $C^1$ solution of our problem exists (cf. Proposition 11.6). However, in a very special case (cf. Lemma 11.21), we will deal with functions $f$ and $g$ that *both* change sign.

(v) We will, however, allow $f$ to change sign, but the analysis is very different if $f > 0$ or if $f$ vanishes, even at a single point, let alone if it becomes negative. The first problem will be discussed in Chapter 10, whereas the second one will be dealt with in Chapter 11. One of the main differences is that in the first case, any solution of (1.2) is necessarily a diffeomorphism (cf. Theorem 19.12), whereas this is never true in the second case.

(vi) It is easy to see (cf. Corollary 19.4) that any solution of (1.2) satisfies

$$\varphi(\Omega) \supset \Omega \quad \text{and} \quad \varphi(\overline{\Omega}) \supset \overline{\Omega}. \tag{1.4}$$

If $f > 0$, we have, since $\varphi$ is a diffeomorphism, that (cf. Theorem 19.12)

$$\varphi(\Omega) = \Omega \quad \text{and} \quad \varphi(\overline{\Omega}) = \overline{\Omega}.$$

If this is not the case, then, in general, the inclusions can be strict. We will discuss in Chapter 11 this matter in details.

(vii) Problem (1.2) admits a *weak formulation*. Indeed, if $\varphi$ is a diffeomorphism, we can write (cf. Theorem 19.7) the equation $g(\varphi)\det\nabla\varphi = f$ as

$$\int_{\varphi(E)} g = \int_E f \quad \text{for every open set } E \subset \Omega$$

or, equivalently,

$$\int_{\Omega} g\,\zeta\left(\varphi^{-1}\right) = \int_{\Omega} f\,\zeta \quad \text{for every } \zeta \in C_0^{\infty}(\Omega).$$

We observe that both new writings make sense if $\varphi$ is only a homeomorphism.

(viii) The problem can be seen as a question of *mass transportation*. Indeed, we want to transport the mass distribution $g$ to the mass distribution $f$ without moving the points of the boundary of $\Omega$. In this context, the equation is usually written as

$$\int_E g = \int_{\varphi^{-1}(E)} f \quad \text{for every open set } E \subset \Omega.$$

The problem of *optimal* mass transportation has received considerable attention. We should point out that our analysis is not in this framework. The two main strong points of our analysis are that we are able to find smooth solutions, sometimes with the optimal regularity and to deal with fixed boundary data.

### *1.4.2 The One-Dimensional Case*

As already stated, the case $n = 1$ is completely elementary (cf. Proposition 11.4), but it exhibits some striking differences with the case $n \geq 2$. However, it may shed some light on some issues that we will discuss in the higher-dimensional case. Let $\Omega = (a,b)$,

$$F(x) = \int_a^x f(t)\,dt \quad \text{and} \quad G(x) = \int_a^x g(t)\,dt.$$

Then problem (1.2) becomes

$$\begin{cases} G(\varphi(x)) = F(x) & \text{if } x \in (a,b), \\ \varphi(a) = a \quad \text{and} \quad \varphi(b) = b. \end{cases}$$

If $G$ is invertible and this happens if, for example, $g > 0$ and if

$$F([a,b]) \subset G(\mathbb{R}), \tag{1.5}$$

and this happens if, for example, $g \geq g_0 > 0$, then the problem has the solution

$$\varphi(x) = G^{-1}(F(x)).$$

The necessary condition (1.3)

$$\int_a^b f = \int_a^b g$$

ensures that

$$\varphi(a) = a \quad \text{and} \quad \varphi(b) = b.$$

This very elementary analysis leads to the following conclusions:

1) Contrary to the case $n \geq 2$, the necessary condition (1.3) is not sufficient. We need the extra condition (1.5); see Proposition 11.4 for details.

2) The problem has a *unique* solution, contrary to the case $n \geq 2$.

3) If $f$ and $g$ are in the space $C^r$, then the solution $\varphi$ is in $C^{r+1}$.