Systems & Control: Foundations & Applications

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# The Robust Maximum Principle

## Theory and Applications

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*To Erica and Tatyana with love.*

# Preface

*Optimal Control* is a rapidly expanding field developed during the last half-century to analyze the optimal behavior of a constrained process that evolves in time according to prescribed laws. Its applications now embrace a variety of new disciplines such as economics and production planning. The main feature of *Classical Optimal Control Theory* (OCT) is that the mathematical technique, especially designed for the analysis and synthesis of an optimal control of dynamic models, is based on the assumption that a designer (or an analyst) possesses *complete information* on a considered model as well as on an environment where this controlled model has to evolve.

There exist two principal approaches to solving *optimal control problems* in the presence of complete information on the dynamic models considered:

– the first one is the *Maximum Principle* (MP) of L. Pontryagin (Boltyanski et al. 1956)
– and the second one is the *Dynamic Programming Method* (DPM) of R. Bellman (Bellman 1957)

The **Maximum Principle** is a basic instrument to derive a set of *necessary conditions* which should be satisfied by any optimal solution (see also Boltyanski 1975, 1978; Dubovitski and Milyutin 1971; Sussman 1987a, 1987b, 1987c). Thus, to solve a static optimization problem in a finite-dimensional space, one should obtain the so-called *zero-derivative condition* (in the case of unconstrained optimization) and the *Kuhn–Tucker conditions* (in the case of constrained optimization). These conditions become sufficient under certain convexity assumptions related to the objective as well as to constraint functions. Optimal control problems, on the other hand, may be regarded as optimization problems in the corresponding infinite-dimensional (Hilbert or, in general, Banach) spaces. The Maximum Principle is really a milestone of modern optimal control theory. It states that any dynamic system, closed by an optimal control strategy or, simply, by an optimal control, is a Hamiltonian system (with a doubled dimension) described by a system of forward-backward ordinary differential equations; in addition, an optimal control maximizes a function called the Hamiltonian. Its mathematical importance is derived from the following fact: the maximization of the Hamiltonian with respect to a control variable given in a finite-dimensional space looks and really is much easier than the original optimization problem formulated in an infinite-dimensional space. The key idea of the original version of the Maximum Principle comes from classical variational calculus. To derive the main MP formulation, first one needs to perturb slightly an optimal control using the so-called needle-shape (spike) variations and, second, to consider the first-order term in a Taylor expansion with respect to this perturbation. Letting the perturbations go to zero, some variational inequalities may be obtained. Then the final result follows directly from duality. The same formulation can be arrived at based on more general concepts related to some geometric representation and separability theorems in Banach space. This approach is called the *Tent Method*. It is a key mathematical apparatus used in this book.

The **Dynamic Programming Method** (DPM) is another powerful approach to solve optimal control problems. It provides *sufficient conditions* for testing whether a control is optimal or not. The basic idea of this approach consists of considering a family of optimal control problems with different initial conditions (times and states) and obtaining some relationships among them via the so-called *Hamilton–Jacobi–Bellman equation* (HJB), which is a nonlinear first-order partial differential equation. If this HJB equation is solvable (analytically or even numerically), then the optimal control can be obtained by maximization (or minimization) of the corresponding generalized Hamiltonian. Such optimal controllers turn out to be given by a nonlinear feedback depending on the optimized plant nonlinearities as well as on the solution of the corresponding HJB equation. Such an approach actually provides solutions to the entire family of optimization problems, and, in particular, the original problem. Such a technique is called "*Invariant Embedding*." The major drawback of the classical HJB method is that it requires that this partial differential equation admits a smooth enough solution. Unfortunately this is not the case even for some very simple situations. To overcome this problem the so-called *viscosity solutions* have been introduced (Crandall and Lions 1983). These solutions are some sort of nonsmooth solutions with a key function to replace the conventional derivatives by a set-valued super/subdifferential maintaining the uniqueness of the solutions under very mild conditions. This approach not only saves the DPM as a mathematical method, but also makes it a powerful tool in optimal control tackling. In this book we will briefly touch on this approach and also discuss the gap between necessary (MP) and sufficient conditions (DPM), while applying this consideration to some particular problems.

When we do not have complete information on a dynamic model to be controlled, the main problem entails designing an acceptable control which remains "close to the optimal one" (having a low sensitivity with respect to an unknown (unpredictable) parameter or input belonging to a given possible set). In other words, the desired control should be *robust* with respect to the unknown factors. In the presence of any sort of uncertainties (parametric type, unmodeled dynamics, and external perturbations), the main approach to obtaining a solution suitable for a class of given models is to formulate a corresponding *Min-Max control* problem, where maximization is taken over a set of uncertainties and minimization is taken over control actions within a given set. The Min-Max controller design for different classes of nonlinear systems has been a hot topic of research over the last two decades.

One of the important components of *Min-Max Control Theory* is the game-theoretic approach (Basar and Bernhard 1991). In terms of game theory, control and model uncertainty are strategies employed by opposing players in a game: control is chosen to minimize a cost function and uncertainty is chosen to maximize it. To the best of our knowledge, the earliest publications in this direction were the papers of Dorato and Drenick (1966) and Krasovskii (1969, in Russian). Subsequently, in the book by Kurjanskii (1977), the *Lagrange Multiplier Approach* was applied to problems of control and observations under incomplete information. They were formulated as corresponding Min-Max problems.

Starting from the pioneering work of Zames (1981), which dealt with frequency domain methods to minimize the norm of the transfer function between the disturbance inputs and the performance output, the minimax controller design is formulated as an $H^{\infty}$*-optimization* problem. As was shown in Basar and Bernhard (1991), this specific problem can be successfully solved in the time domain, leading to rapprochement with dynamic game theory and the establishment of a relationship with risk-sensitivity quadratic stochastic control (Doyle et al. 1989; Glover and Doyle 1988; Limebeer et al. 1989; Khargonekar 1991). The paper by Limebeer et al. (1989) presented a control design method for continuous-time plants whose uncertain parameters in the output matrix are only known to lie within an ellipsoidal set. An algorithm for Min-Max control, which at every iteration approximately minimizes the defined Hamiltonian, is presented in Pytlak (1990). In the publication by Didinsky and Basar (1994), using "the cost-to-come" method, the authors showed that the original problem with incomplete information can be converted into an equivalent full information Min-Max control problem of a higher dimension, which can be solved using the Dynamic Programming Approach. Min-Max control of a class of dynamic systems with mixed uncertainties was investigated in Basar (1994). A continuous deterministic uncertainty which affects system dynamics and discrete *stochastic uncertainties* leading to jumps in the system structure at random times were also studied. The solution involves a finite-dimensional compensator using two finite sets of partial differential equations. The robust controller for linear time-varying systems given by a stochastic differential equation was studied in Poznyak and Taksar (1996). The solution was based on stochastic Lyapunov-like analysis with a martingale technique implementation.

Another class of problems dealing with discrete-time models of a deterministic and/or stochastic nature and their corresponding solutions was discussed in Didinsky and Basar (1991), Blom and Everdij (1993), and Bernhard (1994). A comprehensive survey of various parameter space methods for robust control design can be found in Siljak (1989).

In this book we present a new version of the Maximum Principle recently developed, particularly, for the construction of optimal control strategies for the class of uncertain systems given by *a system of ordinary differential equations with unknown parameters* belonging to a given set (finite or compact) which corresponds to different scenarios of the possible dynamics. Such problems, dealing with finite uncertainty sets, are very common, for example, in Reliability Theory, where some of the sensors or actuators may fail, leading to a complete change in the structure of the system to be controlled (each of the possible structures can be associated with one of the fixed parameter values). The problem under consideration belongs to the class of optimization problems of the Min-Max type. The proof is based on the Tent Method (Boltyanski 1975, 1987), which is discussed in the following text. We show that in the general case the original problem can be converted into the analysis of nonsolid convex cones, which leads to the inapplicability of the Dubovitski–Milyutin method (Dubovitski and Milyutin 1965) for deriving the corresponding necessary conditions of optimality whenever the Tent Method still remains operative.

This book is for experts, scientists, and researchers in the field of Control Theory. However, it may also be of interest to scholars who want to use the results of Control Theory in complex cases, in engineering, and management science. It will also be useful for students who pursue Ph.D.-level or advanced graduate-level courses. It may also serve for training and research purposes.

The present book is both a refinement and an extension of the authors' earlier publications and consists of four complementary parts.

Part I: **Topics of Classical Optimal Control**.
Part II: **The Tent Method**.
Part III: **Robust Maximum Principle for Deterministic Systems**.
Part IV: **Robust Maximum Principle for Stochastic Systems**.

**Part I** presents a review of *Classical Optimal Control Theory* and includes two main topics: the Maximum Principle and Dynamic Programming. Two important subproblems such as Linear Quadratic Optimal Control and Time Optimization are considered in more detail. This part of the book can be considered as independent and may be recommended (adding more examples) for a postgraduate course in Optimal Control Theory as well as for self-study by wide groups of electrical and mechanical engineers.

**Part II** introduces the reader to the *Tent Method*, which, in fact, is a basic mathematical tool for the rigorous proof and justification of one of the main results of Optimal Control Theory. The Tent Method is shown to be a general tool for solving extremal problems profoundly justifying the so-called Separation Principle. First, it was developed in finite-dimensional spaces, using topology theory to justify some results in variational calculus. A short historical remark on the Tent Method is made, and the idea of the proof of the Maximum Principle is explained, paying special attention to the necessary topological tools. The finite-dimensional version of the Tent Method allows one to establish the Maximum Principle and a generalization of the Kuhn–Tucker Theorem in Euclidean spaces. In this part, we also present a version of the Tent Method in Banach spaces and demonstrate its application to a generalization of the Kuhn–Tucker Theorem and the Lagrange Principle for infinite-dimensional spaces.

This part is much more advanced than the others and is accessible only to readers with a strong background in mathematics, particularly in topology. Those who find it difficult to follow topological (homology) arguments can omit the proofs of the basic theorems, trying to understand only their principal statements.

**Part III** is the central part of this book. It presents a *robust version of the Maximum Principle* dealing with the construction of Min-Max control strategies for the class of uncertain systems described by an ordinary differential equation with unknown parameters from a given compact set. A finite collection of parameters corresponds to different scenarios of possible dynamics. The proof is based on the Tent Method described in the previous part of the book. The Min-Max Linear Quadratic (LQ) Control Problem is considered in detail. It is shown that the design of the Min-Max optimal controller in this case may be reduced to a finite-dimensional optimization problem given at the corresponding simplex set containing the weight parameters to be found. The robust LQ optimal control may be interpreted as a mixture (with optimal weights) of the controls which are optimal for each fixed parameter value. Robust time optimality is also considered (as a particular case of the Lagrange problem). Usually, the Robust Maximum Principle appears only as a necessary condition for robust optimality. But the specific character of the linear time-optimization problem permits us to obtain more profound results. In particular, in this case the Robust Maximum Principle appears as both a *necessary and a sufficient condition*. Moreover, for linear robust time optimality, it is possible to establish some additional results: the *existence* and *uniqueness* of robust controls, *piecewise constant* robust controls for the polyhedral resource set, and a Feldbaum-type estimate for *the number of intervals of constancy* (or “switching”). All these aspects are studied in detail in this part of the book. *Dynamic Programming for Min-Max problems* is also derived. A comparison of optimal controllers, designed by the Maximum Principle and Dynamic Programming for LQ problems, is carried out. Applications of results obtained to Multimodel Sliding Mode Control and Multimodel Differential Games are also presented.

**Part IV** deals with designing the *Robust Maximum Principle for Stochastic Systems* described by stochastic differential equations (with the Itô integral implementation) and subject to terminal constraints. The main goal of this part is to illustrate the possibilities of the MP approach for a class of Min-Max control problems for uncertain systems given by a system of linear stochastic differential equations with *controlled drift* and *diffusion terms* and unknown parameters within a given finite and, in general, compact uncertainty set, supplemented by a given measure. If in the deterministic case the adjoint equations are backward ordinary differential equations and represent, in some sense, the same forward equation but in reverse time, then in the stochastic case such an interpretation is not valid because any time reversal may destroy the nonanticipative character of the stochastic solutions, that is, any obtained robust control should be independent of the future. The proof of the

Robust Maximum Principle is also based on the use of the Tent Method, but with a special technique specific to stochastic calculus. The Hamiltonian function used for these constructions is equal to the Lebesgue integral over the given uncertainty set of the standard stochastic Hamiltonians corresponding to a fixed value of the uncertain parameter. Two illustrative examples, dealing with production planning and reinsurance-dividend management, conclude this part.

Most of the material given in this book has been tested in class at the Steklov Mathematical Institute (Moscow, 1962–1980), the Institute of Control Sciences (Moscow, 1978–1993), the Mathematical Investigation Center of Mexico (CIMAT, Guanojuato, 1995–2006), and the Center of Investigation and Advanced Education of IPN (CINVESTAV, Mexico, 1993–2009). Some studies, dealing with multimodel sliding-mode control and multimodel differential games, present the main results of Ph.D. theses of our students defended during the last few years.

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Mexico, D.F., Mexico

Vladimir G. Boltyanski
Alexander S. Poznyak

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# Chapter 1
# Introduction

In this book **our main purpose** is to obtain the *Min-Max control* arising whenever the state of a system at time $t \in [0, T]$ as described by a vector

$$x(t) \in \left(x_1(t), \ldots, x_n(t)\right)^{\mathrm{T}} \in \mathbb{R}^n$$

evolves according to a prescribed law, usually given in the form of a first-order vector ordinary differential equation

$$\dot{x}(t) = f^{\alpha}\left(x(t), u(t), t\right) \tag{1.1}$$

under the assignment of a vector valued control function

$$u(t) = \left(u_1(t), \ldots, u_r(t)\right)^{\mathrm{T}} \in \mathbb{R}^r,$$

which is the control that may run over a given control region $U \subset \mathbb{R}^r$, and $\alpha$ is a parameter that may run over a given parametric set $\mathcal{A}$. On the right-hand side, where

$$f^{\alpha}(x, u, t) = \left(f_1^{\alpha}(x, u, t), \ldots, f_n^{\alpha}(x, u, t)\right)^{\mathrm{T}} \in \mathbb{R}^n, \tag{1.2}$$

we impose the usual restrictions: *continuity* with respect to the arguments $x$, $u$, measurability on $t$, and *differentiability* (or the Lipschitz condition) with respect to $x$. Here we will assume that the admissible $u(t)$ may be only piecewise continuous at each time interval from $[0, T]$ ($T$ is permitted to vary). Controls that have the same values except at common points of discontinuity will be considered as identical.

The **Min-Max control**, which we are interested in, consists of finding an admissible control $\{u^*(\cdot)\}_{t \in [0,T]}$ which for a given initial condition $x(0) = x_0$ and a terminal condition $x^{\alpha}(T) \in \mathcal{M}$ ($\alpha \in \mathcal{A}$) ($\mathcal{M}$ is a given compact from $\mathbb{R}^n$) provides

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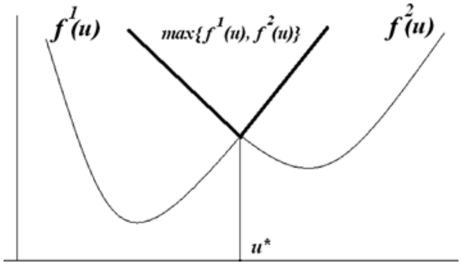


**Fig. 1.1** Min-Max optimized function

us with the following *optimality property*:

$$\{u^*(\cdot)\}_{t\in[0,T]} \in \underset{\text{admissible } \{u(\cdot)\}_{t\in[0,T]}}{\arg\min} \max_{\alpha\in\mathcal{A}} J^\alpha\big(u(\cdot)\big),$$

$$J\big(u(\cdot)\big) := h_0\big(x^\alpha(T)\big) + \int_{t=0}^{T} h\big(x^\alpha(t), u(t), t\big)\,\mathrm{d}t, \tag{1.3}$$

where $h_0 : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}_+ \to \mathbb{R}$ are functions that are smooth enough and characterize the *loss functional* $J^\alpha(u(\cdot))$ for each fixed value of the parameter $\alpha \in \mathcal{A}$.

In fact, the Min-Max problem (1.3) is an *optimization problem in a Banach* (infinite-dimensional) space. So it would be interesting to consider first a Min-Max problem in a finite-dimensional Euclidean space and to understand which specific features of a Min-Max solution arise and what we may expect from their expansion to infinite-dimensional Min-Max problems; also to verify whether these properties remain valid or not.

**The parametric set** $A$ **is finite** Consider the following simple *static single-dimensional optimization problem*:

$$\min_{u\in\mathbb{R}} \max_{\alpha\in\mathcal{A}} h^\alpha(u), \tag{1.4}$$

where $h^\alpha : \mathbb{R} \to \mathbb{R}$ is a differentiable *strictly convex* function, and

$$\mathcal{A} = \{\alpha_1 \equiv 1, \alpha_2 \equiv 2, \ldots, \alpha_N \equiv N\}$$

is a simple finite set containing only $N$ possible parameter values, that is,

$$\min_{u\in\mathbb{R}} \max\big\{h^1(u), h^2(u), \ldots, h^N(u)\big\}. \tag{1.5}$$

To find specific features of this problem let us reformulate it in a manner that is a little bit different. Namely, it is not difficult to see that the problem (1.4) is equivalent to the following one, which, in fact, is a conditional minimization problem that does not contain any maximization operation, that is,

$$\boxed{\begin{array}{l} \displaystyle\min_{u\in\mathbb{R}, v\geq 0} \; v \\ \text{subject to} \quad h^\alpha(u) \leq v \quad \text{for all } \alpha \in \mathcal{A}. \end{array}} \tag{1.6}$$

Figure 1.1 gives a clear illustration of this problem for the case $\mathcal{A} = \{1, 2\}$. To solve the optimization problem (1.6) let us apply the *Lagrange Multiplier Method* (see, for example, Sect. 21.3.3 in Poznyak 2008) and let us consider the following unconditional optimization problem:

$$\begin{aligned} L(u, v, \lambda) &:= v + \sum_{i=1}^{N} \lambda_i \big(h^i(u) - v\big) \\ &= v\left(1 - \sum_{i=1}^{N} \lambda_i\right) + \sum_{i=1}^{N} \lambda_i h^i(u) \to \min_{u, v \in \mathbb{R}} \max_{\lambda_i \geq 0 \ (i=1,\dots,N)}. \end{aligned} \tag{1.7}$$

Notice that if $\sum_{i=1}^{N} \lambda_i \neq 1$, for example,

$$1 - \sum_{i=1}^{N} \lambda_i > 0,$$

one can take $v \to -\infty$, which means that the minimum of $L(u, v, \lambda)$ does not exist. This contradicts our assumption that a minimum of the initial problem (1.6) does exist (since the functions $h^\alpha$ are strictly convex). The same is valid if

$$1 - \sum_{i=1}^{N} \lambda_i > 0$$

and we take $v \to \infty$. So, the unique option leading to the existence of the solution is

$$\lambda \in S_N := \left\{ \lambda \in \mathbb{R}^N : \ \lambda_i \geq 0 \ (i = 1, \dots, N), \ \sum_{i=1}^{N} \lambda_i = 1 \right\}, \tag{1.8}$$

which implies that the initial optimization problem (1.6) is reduced to the following one:

$$L(u, v, \lambda) = \sum_{i=1}^{N} \lambda_i h^i(u) \to \min_{u \in \mathbb{R}} \max_{\lambda \in S_N}; \tag{1.9}$$

that is, the Lagrange function $L(u, v, \lambda)$ to be minimized, according to (1.9), is equal to the weighted sum (with weights $\lambda_i$) of the individual loss functions $h^i(u)$ $(i = 1, \dots, N)$. Defining the joint Hamiltonian function $H(u, \lambda)$ and the individual Hamiltonians $H_i(u, \lambda_i)$ by

$$\begin{aligned} H(u, \lambda) = -L(u, v, \lambda) = -\sum_{i=1}^{N} \lambda_i h^i(u) = \sum_{i=1}^{N} H_i(u, \lambda_i), \\ H_i(u, \lambda_i) := -\lambda_i h^i(u), \end{aligned} \tag{1.10}$$

we can represent problem (1.9) in the Hamiltonian form

$$\boxed{H(u,\lambda) \to \max_{u\in\mathbb{R}} \min_{\lambda\in S_N}.} \tag{1.11}$$

As can be seen from Fig. 1.1 the optimal solution $u^*$ in the case $N=2$ satisfies the condition

$$h^1(u^*) = h^2(u^*). \tag{1.12}$$

This property is true also in the general case. Indeed, the complementary slackness conditions (see Theorem 21.12 in Poznyak 2008) for this problem are

$$\lambda_i^*\big(h^i(u^*) - v\big) = 0 \quad \text{for any } i = 1,\dots,N, \tag{1.13}$$

which means that for any active indices $i$, $j$, corresponding to $\lambda_i^*, \lambda_j^* > 0$, we have

$$h^i(u^*) = h^j(u^*) = v \tag{1.14}$$

or, in other words, for the optimal solution $u^*$ we find all loss functions $h^i(u^*)$ for which $\lambda_i^* > 0$ to be equal. So, one can see that the following *two basic properties* (formulated here as a proposition) of the Min-Max solution $u^*$ exist.

**Proposition 1.1**

- *The joint Hamiltonian $H(u,\lambda)$ (1.10) of the initial optimization problem is equal to the sum of the individual Hamiltonians $H_i(u,\lambda_i)$ $(i=1,\dots,N)$.*
- *In the optimal point $u^*$ all loss functions $h^i(u^*)$, corresponding to the active indices for which $\lambda_i^* > 0$, are equal.*

**The parametric set $A$ is a compact** In this case, when we deal with the original Min-Max problem (1.4), written in the form (1.6), the corresponding Lagrange function has the form

$$\begin{aligned} L(u,v,\lambda) &:= v + \int_{\alpha\in\mathcal{A}} \lambda_\alpha\big(h^\alpha(u) - v\big)\,\mathrm{d}\alpha \\ &= v\left(1 - \int_{\alpha\in\mathcal{A}} \lambda_\alpha\,\mathrm{d}\alpha\right) + \int_{\alpha\in\mathcal{A}} \lambda_\alpha h^\alpha(u)\,\mathrm{d}\alpha \to \min_{u,v\in\mathbb{R}} \max_{\lambda_\alpha\ge 0,\alpha\in\mathcal{A}}. \end{aligned} \tag{1.15}$$

By the same argument as for a finite parametric set, the only possibility here to have a finite solution for the problem considered is to take

$$\int_{\alpha\in\mathcal{A}} \lambda_\alpha\,\mathrm{d}\alpha = 1, \tag{1.16}$$

which, together with the nonnegativity of the multipliers $\lambda_\alpha$, permits us to refer to them as a "*distribution*" of the index $\alpha$ on the set $\mathcal{A}$. Define the set of all possible distributions on $\mathcal{A}$ as

$$\mathcal{D} = \left\{ \lambda_\alpha,\ \alpha \in A:\ \lambda_\alpha \geq 0,\ \int_{\alpha \in \mathcal{A}} \lambda_\alpha \, d\alpha = 1 \right\}. \tag{1.17}$$

Then problem (1.15) becomes

$$L(u, v, \lambda) = \int_{\alpha \in \mathcal{A}} \lambda_\alpha h^\alpha(u) \, d\alpha \to \min_{u, v \in \mathbb{R}} \max_{\lambda_\alpha \geq 0, \alpha \in \mathcal{A}} \tag{1.18}$$

or, in the corresponding Hamiltonian form,

$$H(u, \lambda) \to \max_{u, v \in \mathbb{R}} \min_{\lambda_\alpha \geq 0, \alpha \in \mathcal{A}}, \tag{1.19}$$

where

$$\begin{aligned} H(u, \lambda) &= -L(u, v, \lambda) \\ &= \int_{\alpha \in \mathcal{A}} \lambda_\alpha h^\alpha(u) \, d\alpha = \int_{\alpha \in \mathcal{A}} H_\alpha(u, \lambda_\alpha) \, d\alpha, \\ H_\alpha(u, \lambda_\alpha) &:= -\lambda_\alpha h^\alpha(u). \end{aligned} \tag{1.20}$$

Again, the complementary slackness conditions (see Theorem 21.12 in Poznyak 2008) for this problem are similar to (1.13)

$$\lambda_\alpha^* \big( h^\alpha(u^*) - v \big) = 0 \quad \text{for any } \alpha \in \mathcal{A}, \tag{1.21}$$

which means that for any active indices $\alpha, \tilde{\alpha} \in \mathcal{A}$, corresponding to $\lambda_\alpha^* > 0$, it follows that

$$h^\alpha(u^*) = h^{\tilde{\alpha}}(u^*) = v \tag{1.22}$$

or, in other words, *for the optimal solution* $u^*$ *all loss functions* $h^\alpha(u^*)$, *for which* $\lambda_\alpha^* > 0$, *are equal*. So, again one can state *two basic properties* (formulated as a proposition) characterizing the Min-Max solution $u^*$ on a compact parametric set.

**Proposition 1.2**

- *The joint Hamiltonian* $H(u, \lambda)$ (1.10) *of the initial optimization problem is equal to the integral of the individual Hamiltonians* $H_i(u, \lambda_i)$ $(i = 1, \ldots, N)$ *calculated over the given compact set* $\mathcal{A}$.
- *In the optimal point* $u^*$ *we see that all loss functions* $h^\alpha(u^*)$, *corresponding to active indices for which* $\lambda_\alpha^* > 0$, *are equal. If in the intersection point one function* (*for example*, $f_1$) *is beyond* (*over*) *the other* $f_2$, *then for this case we have the dominating function* $\lambda_1^* = 1$ *and* $\lambda_2^* = 0$.

**The main question** that arises here is: "*Do these two principal properties*, formulated in the propositions above for finite-dimensional Min-Max problems, *remain* *valid for the infinite-dimensional case*, formulated in a Banach space for a Min-Max optimal control problem?”

The answer is: **YES** they do!

The detailed justification of this positive answer forms **the main contribution** of this book.

# Part I
# Topics of Classical Optimal Control

# Chapter 2
# The Maximum Principle

This chapter represents the basic concepts of Classical Optimal Control related to the *Maximum Principle*. The formulation of the general optimal control problem in the Bolza (as well as in the Mayer and the Lagrange) form is presented. The Maximum Principle, which gives the necessary conditions of optimality, for various problems with a fixed and variable horizon is formulated and proven. All necessary mathematical claims are given in the Appendix, which makes this material self-contained.

This chapter is organized as follows. The classical optimal control problems in the Bolza, Lagrange, and Mayer form, are formulated in the next section. Then in Sect. 2.2 the variational inequality is derived based on the needle-shaped variations and Gronwall's inequality. Subsequently, a basic result is presented concerning the necessary conditions of the optimality for the problem considered in the Mayer form with terminal conditions using the duality relations.

## 2.1 Optimal Control Problem

### *2.1.1 Controlled Plant, Cost Functionals, and Terminal Set*

**Definition 2.1** Consider the controlled plant given by the following system of *Ordinary Differential Equations* (ODE):

$$\boxed{\begin{cases} \dot{x}(t) = f\big(x(t), u(t), t\big), & \text{a.e. } t \in [0, T], \\ x(0) = x_0, \end{cases}} \tag{2.1}$$

where $x = (x^1, \dots, x^n)^T \in \mathbb{R}^n$ is its state vector, and $u = (u^1, \dots, u^r)^T \in \mathbb{R}^r$ is the control that may run over a given control region $U \subset \mathbb{R}^r$ with *the cost functional*

$$\boxed{J\big(u(\cdot)\big) := h_0\big(x(T)\big) + \int_{t=0}^{T} h\big(x(t), u(t), t\big)\, \mathrm{d}t} \tag{2.2}$$

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