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Rafail Khasminskii

Stochastic Stability of Differential Equations

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Stochastic Stability of Differential Equations

With contributions by G.N. Milstein and M.B. Nevelson

Completely Revised and Enlarged 2nd Edition

Springer

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Preface to the Second Edition

After the publication of the first edition of this book, stochastic stability of differential equations has become a very popular theme of recent research in mathematics and its applications. It is enough to mention the Lecture Notes in Mathematics, Nos 294, 1186 and 1486, devoted to the stability of stochastic dynamical systems and Lyapunov Exponents, the books of L. Arnold [3], A. Borovkov [35], S. Meyn and R. Tweedie [196], among many others.

Nevertheless I think that this book is still useful for those researchers who would like to learn this subject, to start their research in this area or to study properties of concrete mechanical systems subjected to random perturbations. In particular, the method of Lyapunov functions for the analysis of qualitative behavior of stochastic differential equations (SDEs), the exact formulas for the Lyapunov exponent for linear SDEs, which are presented in this book, provide some very powerful instruments in the study of stability properties for concrete stochastic dynamical systems, conditions of existence the stationary solutions of SDEs and related problems.

The study of exponential stability of the moments (see Sects. 5.7, 6.3, 6.4 here) makes natural the consideration of certain properties of the moment Lyapunov exponents. This very important concept was first proposed by S. Molchanov [204], and was later studied in detail by L. Arnold, E. Oeljeklaus, E. Pardoux [8], P. Baxendale [19] and many other researchers (see, e.g., [136]).

Another important characteristic for stability (or instability) of the stochastic systems is the stability index, studied by Arnold, Baxendale and the author. For the reader's convenience I decided to include the main results on the moment Lyapunov exponents and the stability index in the Appendix B to this edition. The Appendix B was mainly written by G. Milstein, who is an accomplished researcher in this area. I thank him whole-heartily for his generous help and support.

I have many thanks to the Institute for the Problems Information Transmission, Russian Academy of Sciences, and to the Wayne State University, Detroit, for their support during my work on this edition. I also have many thanks to B.A. Amosov for his essential help in the preparation of this edition.

In conclusion I will enumerate some other changes in this edition.

1. Derivation of the often used in the book Feynman–Kac formula is added to Sect. 3.6.
2. A much improved version of Theorem 4.6 is proven in Chap. 4.
3. The Arcsine Law and its generalization are added in 4.12.
4. Sect. A.4 in the Appendix B to the first edition is shortened.
5. New books and papers, related to the content of this book, added to the bibliography.
6. Some footnotes are added and misprints are corrected.

Moscow
March 2011

Rafail Khasminskii

Preface to the First English Edition

I am very pleased to witness the printing of an English edition of this book by Noordhoff International Publishing. Since the date of the first Russian edition in 1969 there have appeared no less than two specialist texts devoted at least partly to the problems dealt with in the present book [38, 211]. There have also appeared a large number of research papers on our subject. Also worth mentioning is the monograph of Sagirov [243] containing applications of some of the results of this book to cosmology.

In the hope of bringing the book somewhat more up to date we have written, jointly with M.B. Nevelson, an Appendix A containing an exposition of recent results. Also, we have in some places improved the original text of the book and have made some corrections. Among these changes, the following two are especially worth mentioning: A new version of Sect. 8.4, generalizing and simplifying the previous exposition, and a new presentation of Theorem 7.8.

Finally, there have been added about thirty new titles to the list of references. In connection with this we would like to mention the following. In the first Russian edition we tried to give as complete as possible a list of references to works concerning the subject. This list was up to date in 1967. Since then the annual output of publications on stability of stochastic systems has increased so considerably that the task of supplying this book with a totally up to date and complete bibliography became very difficult indeed. Therefore we have chosen to limit ourselves to listing only those titles which pertain directly to the contents of this book. We have also mentioned some more recent papers which were published in Russian, assuming that those will be less known to the western reader.

I would like to conclude this preface by expressing my gratitude to M.B. Nevelson for his help in the preparation of this new edition of the book.

Moscow
September 1979

Rafail Khasminskii

Preface to the Russian Edition

This monograph is devoted to the study of the qualitative theory of differential equations with random right-hand side. More specifically, we shall consider here problems concerning the behavior of solutions of systems of ordinary differential equations whose right-hand sides involve stochastic processes. Among these the following questions will receive most of our attention.

1. When is each solution of the system defined with probability 1 for all $t > 0$ (i.e., the solution does not "escape to infinity" in a finite time)?
2. If the function $X(t) \equiv 0$ is a solution of the system, under which conditions is this solution stable in some stochastic sense?
3. Which systems admit only bounded for all $t > 0$ (again in some stochastic sense) solutions?
4. If the right-hand side of the system is a stationary (or periodic) stochastic process, under which additional assumptions does the system have a stationary (periodic) solution?
5. If the system has a stationary (or periodic) solution, under which circumstances will every other solution converge to it?

The above problems are also meaningful (and motivated by practical interest) for deterministic systems of differential equations. In that case, they received detailed attention in [154, 155, 178, 188, 191, 228], and others.

Problems 3–5 have been thoroughly investigated for linear systems of the type $\dot{x} = Ax + \xi(t)$, where A is a constant or time dependent matrix and $\xi(t)$ a stochastic process. For that case one can obtain not only qualitative but also quantitative results (i.e., the moment, correlation and spectral characteristics of the output process $x(t)$) in terms of the corresponding characteristics of the input process $\xi(t)$. Methods leading to this end are presented e.g., in [177, 233], etc. In view of this, we shall concentrate our attention in the present volume primarily on non-linear systems, and on linear systems whose parameters (the elements of the matrix A) are subjected to random perturbations.

In his celebrated memoir Lyapunov [188] applied his method of auxiliary functions (Lyapunov functions) to the study of stability. His method proved later to be applicable also to many other problems in the qualitative theory of differential equations. Also in this book we shall utilize an appropriate modification of the method of Lyapunov functions when discussing the solutions to the above mentioned problems.

In Chaps. 1 and 2 we shall study problems 1–5 without making any specific assumptions on the form of the stochastic process on the right side of the special equation. We shall be predominantly concerned with systems of the type $\dot{x} = F(x,t) + \sigma(x,t)\xi(t)$ in Euclidean l-space. We shall discuss their solutions, using the Lyapunov functions of the truncated system $\dot{x} = F(x,t)$. In this we shall try to impose as few restrictions as possible on the stochastic process $\xi(t)$; e.g., we may require only that the expectation of $|\xi(t)|$ be bounded. It seems convenient to take this approach, first, because sophisticated methods are available for constructing Lyapunov functions for deterministic systems, and second, because the results so obtained will be applicable also when the properties of the process $\xi(t)$ are not completely known, as is often the case.

Evidently, to obtain more detailed results, we shall have to restrict the class of stochastic processes $\xi(t)$ that may appear on the right side of the equation. Thus in Chaps. 3 through 7 we shall study the solutions of the equation $\dot{x} = F(x,t) + \sigma(x,t)\xi(t)$ where $\xi(t)$ is a white noise, i.e. a Gaussian process such that $\mathbf{E}\xi(t) = 0$, $\mathbf{E}[\xi(s)\xi(t)] = \delta(t-s)$. We have chosen this process, because:

1. In many real situations physical noise can be well approximated by white noise.
2. Even under conditions different from white noise, but when the noise acting upon the system has a finite memory interval τ (i.e., the values of the noise at times t_1 and t_2 such that $|t_2 - t_1| > \tau$ are virtually independent), it is often possible after changing the time scale to find an approximating system, perturbed by the white noise.
3. When solutions of an equation are sought in the form of a process, continuous in time and without after-effects, the assumption that the noise in the system is "white" is essential. The investigation is facilitated by the existence of a well developed theory of processes without after-effects (Markov processes).

Shortly after the publication of Kolmogorov's paper [144], which laid the foundations for the modern analytical theory of Markov processes, Andronov, Pontryagin and Vitt [229] pointed out that actual noise in dynamic systems can be replaced by white noise, thus showing that the theory of Markov processes is a convenient tool for the study of such systems.

Certain difficulties in the investigation of the equation $\dot{x} = F(x,t) + \sigma(x,t)\xi(t)$, where $\xi(t)$ is white noise are caused by the fact that, strictly speaking, "white" noise processes do not exist; other difficulties arise because of the many ways of interpreting the equation itself. These difficulties have been largely overcome by the efforts of Bernshtein, Gikhman and Itô. In Chap. 3 we shall state without proof a theorem on the existence and uniqueness of the Markov process determined by an equation with the white noise. We shall assume a certain interpretation of this equation. For a detailed proof we refer the reader to [56, 64, 92].

However, we shall consider in Chap. 3 various other issues in great detail, such as sufficient conditions for a sample path of the process not to "escape to infinity" in a finite time, or to reach a given bounded region with probability 1. It turns out that such conditions are often conveniently formulated in terms of certain auxiliary functions analogous to Lyapunov functions. Instead of the Lyapunov operator (the derivative along the path) one uses the infinitesimal generator of the corresponding Markov process.

In Chap. 4 we examine conditions under which a solution of a differential equation where $\xi(t)$ is white noise, converges to a stationary process. We show how this is related to the ergodic theory of dynamic systems and to the problem of stabilization of the solution of a Cauchy problem for partial differential equations of parabolic type.

Chapters 5–8 I contain the elements of stability theory of stochastic systems without after-effects. This theory has been created in the last few years for the purpose of studying the stabilization of controlled motion in systems perturbed by random noise. Its origins date from the 1960 paper by Kac and Krasovskii [111] which has stimulated considerable further research. More specifically, in Chap. 5 we generalize the theorems of Lyapunov's second method; Chapter 6 is devoted to a detailed investigation of linear systems, and in Chap. 7 we prove theorems on stability and instability in the first approximation. We do this, keeping in view applications to stochastic approximation and certain other problems.

Chapter 8 is devoted to application of the results of Chaps. 5 to 7 to optimal stabilization of controlled systems. It was written by the author in collaboration with M.B. Nevelson. In preparing this chapter we have been influenced by Krasovskii's excellent Appendix IV in [191].

As far as we know, there exists only one other monograph on stochastic stability. It was published in the U.S.A. in 1967 by Kushner [168], and its translation into Russian is now ready for print. Kushner's book contains many interesting theorems and examples. They overlap partly with the results of Sect. 3.7 and Sects. 5.1–5.5 of this book.

Though our presentation of the material is abstract, the reader who is primarily interested in applications should bear in mind that many of the results admit a directly "technical" interpretation. For example, problem 4, stated above, concerning the question of the existence of a stationary solution, is equivalent to the problem of determining when stationary operating conditions can prevail within a given, generally non-linear, automatic control system, whose parameters experience random perturbations and whose input process is also stochastic. Similarly, the convergence of each solution to a stationary solution (see Chap. 4) means that each output process of the system will ultimately "settle down" to stationary conditions.

In order not to deviate from the main purpose of the book, we shall present without proof many facts from analysis and from the general theory of stochastic process. However, in all such cases we shall mention either in the text or in a footnote where the proof can be found. For the reader's convenience, such references will usually be not to the original papers but rather to more accessible textbooks and monographs. On the other hand, in the rather narrow range of the actual subject matter we have tried to give precise references to the original research. Most of the references appear in footnotes.

Part of the book is devoted to the theory of stability of solutions of stochastic equations (Sects. 1.5–1.8, Chaps. 5–8). This appears to be an important subject which has recently been receiving growing attention. The volume of the relevant literature is increasing steadily. Unfortunately, in this area various authors have published results overlapping significantly with those of others. This is apparently due to the fact that the field is being studied by mathematicians, physicists, and engineers, and each of these groups publishes in journals not read by the others. Therefore the bibliography given at the end of this book lists, besides the books and papers cited in the text, various other publications on the stability of stochastic systems known to the author, which appeared prior to 1967. For the reason given above, this list is far from complete, and the author wishes to apologize to authors whose research he might have overlooked.

The book is intended for mathematicians and physicists. It may be of particular interest to those who specialize in mechanics, in particular in the applications of the theory of stochastic processes to problems in oscillation theory, automatic control and related fields. Certain sections may appeal to specialists in the theory of stochastic processes and differential equations. The author hopes that the book will also be of use to specialized engineers interested in the theoretical aspects of the effect of random noise on the operation of mechanical and radio-engineering systems and in problems relating to the control of systems perturbed by random noise.

To study the first two chapters it is sufficient to have an acquaintance with the elements of the theory of differential equations and probability theory, to the extent generally given in higher technical schools (the requisite material from the theory of stochastic processes is given in the text without proofs).

The heaviest mathematical demands on the reader are made in Chaps. 3 and 4. To read them, he will need an acquaintance with the elements of the theory of Markov processes to the extent given, e.g., in Chap. VIII of [92].

The reader interested only in the stability of stochastic systems might proceed directly from Chap. 2 to Chaps. 5–7, familiarizing himself with the results of Chaps. 3 and 4 as the need arises.

The origin of this monograph dates back to some fruitful conversations which the author had with N.N. Krasovskii. In the subsequent research, here described, the author has used the remarks and advice offered by his teachers A.N. Kolmogorov and E.B. Dynkin, to whom he is deeply indebted.

This book also owes much to the efforts of its editor, M.B. Nevelson, who not only took part in writing Chap. 8 and indicated several possible improvements, but also placed some of his yet unpublished examples at the author's disposal. I am grateful to him for this assistance. I also would like to thank V.N. Tutubalin, V.B. Kolmanovskii and A.S. Holevo for many critical remarks, and to R.N. Stepanova for her work on the preparation of the manuscript.

Moscow
September, 1967

Rafail Khasminskii

Contents

Basic Notation

I_T	$=\{t : 0 \le t < T\}$, set of points t such that $0 \le t < T$, p. 1
I	$= I_\infty$, p. 1
U_R	$= \{x : \|x\| < R\}$, p. 4
$\mathbb{R}^l$	Euclidean l-space, p. 2
E	$= \mathbb{R}^l \times I$, p. 4
$\mathbf{L}$	class of functions $f(t)$ absolutely integrable on every finite interval, p. 4
$\mathbf{C}_2$	class of functions $V(t, x)$ twice continuously differentiable with respect to x and once continuously differentiable with respect to t, p. 72
$\mathbf{C}_2^0(U)$	class of functions $V(t, x)$ twice continuously differentiable with respect to $x \in U$ and once continuously differentiable with respect to $t \in I$ everywhere except possibly at the point $x = 0$, p. 146
$\mathbf{C}$	class of functions $V(t, x)$ absolutely continuous in t and satisfying a local Lipschitz condition, p. 6
$\mathbf{C}_0$	class of functions $V(t, x) \in \mathbf{C}$ satisfying a global Lipschitz condition, p. 6
$\mathfrak{A}$	σ-algebra of Borel sets in the initial probability space, p. 1
$\mathfrak{B}$	σ-algebra of Borel sets in Euclidean space, p. 47
V_R	$= \inf_{t \ge t_0,\, x \ge R} V(t, x)$, p. 7
$V^{(\delta)}$	$= \sup_{t \ge t_0,\, \|x\| < \delta} V(t, x)$, p. 28
A^c	complement to the set A, p. 1
$\frac{d^0 V}{dt}$	Lyapunov operator for ODE, p. 6
$U_\delta(\Gamma)$	δ-neighborhood of the set Γ, p. 149
J	identity matrix, p. 97
$\mathcal{N}_s$	family of σ-algebras defined on the p. 60
$\widetilde{\mathcal{N}}_t$	family of σ-algebras defined on the p. 68
$\mathbb{1}_A(\cdot)$	indicator function of the set A, p. 62

Chapter 1
Boundedness in Probability and Stability of Stochastic Processes Defined by Differential Equations

1.1 Brief Review of Prerequisites from Probability Theory

Let $\Omega = \{\omega\}$ be a space with a family of subsets $\mathfrak{A}$ such that, for any finite or countable sequence of sets $A_i \in \mathfrak{A}$, the intersection $\bigcap_i A_i$, union $\bigcup_i A_i$ and complement A_i^c (with respect to Ω) are also in $\mathfrak{A}$. Suppose moreover that $\Omega \in \mathfrak{A}$. A family of subsets possessing these properties is known as a σ-algebra. If a probability measure $\mathbf{P}$ is defined on the σ-algebra $\mathfrak{A}$ (i.e. $\mathbf{P}$ is a non-negative countably additive set function on $\mathfrak{A}$ such that $\mathbf{P}(\Omega) = 1$), then the triple $(\Omega, \mathfrak{A}, \mathbf{P})$ is called a probability space and the sets in $\mathfrak{A}$ are called random events. (For more details, see [56, 64, 185].)

The following standard properties of measures will be used without any further reference:

1. If $A \in \mathfrak{A}$, $B \in \mathfrak{A}$, $A \subset B$, then $\mathbf{P}(A) \le \mathbf{P}(B)$.
2. For any finite or countable sequence A_n in $\mathfrak{A}$,

$$\mathbf{P}\Big(\bigcup_n A_n\Big) \le \sum_n \mathbf{P}(A_n).$$

3. If $A_n \in \mathfrak{A}$ and $A_1 \subset A_2 \subset \cdots \subset A_n \subset A_{n+1} \subset \cdots$, then

$$\mathbf{P}\Big(\bigcup_n A_n\Big) = \lim_{n\to\infty} \mathbf{P}(A_n).$$

4. If $A_n \in \mathfrak{A}$ and $A_1 \supset A_2 \supset A_3 \supset \cdots \supset A_n \supset \cdots$, then

$$\mathbf{P}\Big(\bigcap_n A_n\Big) = \lim_{n\to\infty} \mathbf{P}(A_n).$$

Proofs of these properties may be found in any textbook on probability theory, such as [95, §8]; or [92, Sect. 1.1].

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A random variable is a function $\xi(\omega)$ on Ω which is $\mathfrak{A}$ measurable and almost everywhere finite.[1] In this book we shall consider only random variables which take on values in Euclidean l-space $\mathbb{R}^l$ i.e., such that $\xi(\omega) = (\xi_1(\omega), \dots, \xi_l(\omega))$ is a vector in $\mathbb{R}^l$ ($l = 1, 2, \dots$). A vector-valued random variable $\xi(\omega)$ may be defined by its joint distribution function $F(x_1, \dots, x_l)$, that is, by specifying the probability of the event $\{\xi_1(\omega) < x_1; \dots; \xi_l(\omega) < x_l\}$. Given any vector $x \in \mathbb{R}^l$ or a $k \times l$ matrix $\sigma = ((\sigma_{ij}))$ ($i = 1, \dots, k$; $j = 1, \dots, l$) we shall denote, as usual,

$$|x| = (x_1^2 + \cdots + x_l^2)^{1/2}, \qquad \|\sigma\| = \left(\sum_{i=1}^{k} \sum_{j=1}^{l} \sigma_{ij}^2 \right)^{1/2}.$$

Then we have the well-known inequalities $|\sigma x| \le \|\sigma\| |x|$, $\|\sigma_1 \sigma_2\| \le \|\sigma_1\| \|\sigma_2\|$.

The expectation of a random variable $\xi(\omega)$ is defined to be the integral

$$\mathbf{E}\xi = \int_{\Omega} \xi(\omega)\, P(d\omega),$$

provided the function $|\xi(\omega)|$ is integrable.

Let $\mathcal{B}$ be a σ-algebra of Borel subsets of a closed interval $[s_0, s_1]$, $\mathcal{B} \times \mathfrak{A}$ the minimal σ-algebra of subsets of $I \times \Omega$ containing all subsets of the type $\{t \in \Delta,\ \omega \in A\}$, where $\Delta \in \mathcal{B}$, $A \in \mathfrak{A}$. A function $\xi(t, \omega) \in \mathbb{R}^l$ is called a measurable stochastic process (random function) defined on $[s_0, s_1]$ with values in $\mathbb{R}^l$ if it is $\mathcal{B} \times \mathfrak{A}$-measurable and $\xi(t, \omega)$ is a random variable for each $t \in [s_0, s_1]$. For fixed ω, we shall call the function $\xi(t, \omega)$ a trajectory or sample function of the stochastic process. In the sequel we shall consider only separable stochastic processes, i.e., processes whose behavior for all $t \in [s_0, s_1]$ is determined up to an event of probability zero by its behavior on some dense subset $\Lambda \in [s_0, s_1]$. To be precise, a process $\xi(t, \omega)$ is said to be *separable* if, for some countable dense subset $\Lambda \in [s_0, s_1]$, there exists an event A of probability 0 such that for each closed subset $C \subset \mathbb{R}^l$ and each open subset $\Delta \subset [s_0, s_1]$ the event

$$\{\xi(t_j, \omega) \in C;\ t_j \in \Lambda \cap \Delta\}$$

implies the event

$$A \cup \{\xi(t, \omega) \in C;\ t \in \Delta\}.$$

A process $\xi(t, \omega)$ is stochastically continuous at a point $s \in [s_0, s_1]$ if for each $\varepsilon > 0$

$$\lim_{t \to s} \mathbf{P}\{|\xi(t, \omega) - \xi(s, \omega)| > \varepsilon\} = 0.$$

The definitions of right and left stochastic continuity are analogous.

It can be proved (see [56, Chap. II, Theorem 2.6]) that for each process $\xi(t, \omega)$ which is stochastically continuous throughout $[s_0, s_1]$, except for a countable subset

[1] Sometimes (see Chap. 3), but only when this is explicitly mentioned, we shall find it convenient to consider random variables which can take on the values $\pm\infty$ with positive probability.

of $[s_0, s_1]$, there exists a separable measurable process $\tilde{\xi}(t, \omega)$ such that for every $t \in [s_0, s_1]$

$$\mathbf{P}\{\xi(t, \omega) = \tilde{\xi}(t, \omega)\} = 1 \qquad (\xi(t, \omega) = \tilde{\xi}(t, \omega) \text{ almost surely}).$$

If $\xi(t, \omega)$ is a measurable stochastic process, then for fixed ω the function $\xi(t, \omega)$, as a function of t, is almost surely Lebesgue-measurable. If, moreover, $\mathbf{E}\xi(t, \omega) = m(t)$ exists, then $m(t)$ is Lebesgue-measurable, and the inequality

$$\int_A \mathbf{E}|\xi(t, \omega)|\, dt < \infty$$

implies that the process $\xi(t, \omega)$ is almost surely integrable over A [56, Chap. II, Theorem 2.7].

On the σ-algebra $\mathcal{B} \times \mathfrak{A}$ there is defined the direct product $\mu \times \mathbf{P}$ of the Lebesgue measure μ and the probability measure $\mathbf{P}$. If some relation holds for $(t, \omega) \in A$ and $\mu \times \mathbf{P}(A^c) = 0$, the relation will be said to hold for almost all t, ω. Let $A_1, \ldots, A_n$ be Borel sets in $\mathbb{R}^l$, and $t_1, \ldots, t_n \in [s_0, s_1]$; the probabilities

$$\mathbf{P}(t_1, \ldots, t_n, A_1, \ldots, A_n) = \mathbf{P}\{\xi(t_1, \omega) \in A_1, \ldots, \xi(t_n, \omega) \in A_n\}$$

are the values of the n-dimensional distributions of the process $\xi(t, \omega)$. Kolmogorov has shown that any compatible family of distributions $\mathbf{P}(t_1, \ldots, t_n, A_1, \ldots, A_n)$ is the family of the finite-dimensional distributions of some stochastic process.

The following theorem of Kolmogorov will play an important role in the sequel.

Theorem 1.1 *If α, β, k are positive numbers such that whenever $t_1, t_2 \in [s_0, s_1]$,*

$$\mathbf{E}|\xi(t_2, \omega) - \xi(t_1, \omega)|^\alpha < k|t_1 - t_2|^{1+\beta}$$

and $\xi(t, \omega)$ is separable, then the process $\xi(t, \omega)$ has continuous sample functions almost surely (a.s.).

Let $\xi(t, \omega)$ be a stochastic process defined for $t \geq t_0$. The process is said to satisfy the law of large numbers if for each $\varepsilon > 0$, $\delta > 0$ there exists a $T > 0$ such that for all $t > T$

$$\mathbf{P}\left\{\left|\frac{1}{t}\int_{t_0}^{t_0+t} \xi(s, \omega)\, ds - \frac{1}{t}\int_{t_0}^{t_0+t} \mathbf{E}\xi(s, \omega)\, ds\right| > \delta\right\} < \varepsilon. \tag{1.1}$$

A stochastic process $\xi(t, \omega)$ satisfies the *strong* law of large numbers if

$$\mathbf{P}\left\{\frac{1}{t}\int_{t_0}^{t_0+t} \xi(s, \omega)\, ds - \frac{1}{t}\int_{t_0}^{t_0+t} \mathbf{E}\xi(s, \omega)\, ds \underset{t\to\infty}{\longrightarrow} 0\right\} = 1. \tag{1.2}$$

The most important characteristics of a stochastic process are its expectation $m(t) = \mathbf{E}\xi(t, \omega)$ and covariance matrix

$$K(s, t) = \operatorname{cov}(\xi(s), \xi(t)) = ((\mathbf{E}[(\xi_i(s) - m_i(s))(\xi_j(t) - m_j(t))])).$$

In particular, all the finite-dimensional distributions of a Gaussian process can be reconstructed from the function $m(t)$ and $K(s,t)$. A Gaussian process is stationary if

$$m(t) = \text{const}, \qquad K(s,t) = K(t-s). \tag{1.3}$$

A stochastic process $\xi(t,\omega)$ satisfying condition (1.3) is said to be stationary in the wide sense. The Fourier transform of the matrix $K(\tau)$ is called the spectral density of the process $\xi(t,\omega)$. It is clear that the spectral density $f(\lambda)$ exists and is bounded if the function $\|K(\tau)\|$ is absolutely integrable.

1.2 Dissipative Systems of Differential Equations

In this section we prove some theorems from the theory of differential equations that we shall need later. We begin with a few definitions.

Let I_T denote the set $0 < t < T$, $I = I_\infty$, $E = \mathbb{R}^l \times I$; U_R the ball $|x| < R$ and U_R^c its complement in $\mathbb{R}^l$. If $f(t)$ is a function defined on I, we write $f \in \mathbf{L}$ if $f(t)$ is absolutely integrable over every finite interval. The same notation $f \in \mathbf{L}$ will be retained for a stochastic function $f(t,\omega)$ which is almost surely absolutely integrable over every finite interval.

Let $F(x,t) = (F_1(x,t), \ldots, F_l(x,t))$ be a Borel-measurable function defined for $(x,t) \in E$. Let us assume that for each $R > 0$ there exist functions $M_R(t) \in \mathbf{L}$ and $B_R(t) \in \mathbf{L}$ such that

$$|F(x,t)| \le M_R(t), \tag{1.4}$$

$$|F(x_2,t) - F(x_1,t)| \le B_R(t)|x_2 - x_1| \tag{1.5}$$

for $x, x_i \in U_R$.

We shall say that a function $x(t)$ is a solution of the equation

$$\frac{dx}{dt} = F(x,t), \tag{1.6}$$

satisfying the initial condition

$$x(t_0) = x_0 \quad (t_0 \ge 0) \tag{1.7}$$

on the interval $[t_0, t_1]$, if for all $t \in [t_0, t_1]$

$$x(t) = x_0 + \int_{t_0}^{t} F(x(s), s)\, ds. \tag{1.8}$$

In cases where solutions are being considered under varying initial conditions, we shall denote this solution by $x(t, x_0, t_0)$.

The function $x(t)$ is evidently absolutely continuous, and at all points of continuity of $F(x,t)$ it also satisfies (1.6).

Theorem 1.2 *If conditions* (1.4) *and* (1.5) *are satisfied, then the solution* $x(t)$ *of problem* (1.6), (1.7) *exists and is unique in some neighborhood of* t_0. *Suppose moreover that for every solution* $x(t)$ (*if a solution exists*) *and some function* τ_R *which tends to infinity as* $R \to \infty$, *we have the following "a priori estimate"*:

$$\inf\{t : t \geq t_0;\ |x(t)| > R\} \geq \tau_R. \tag{1.9}$$

Then the solution of the problem (1.6), (1.7) *exists and is unique for all* $t \geq t_0$ (*i.e., the solution can be unlimitedly continued for* $t \geq t_0$).

Proof We may assume without loss of generality that the function $M_R(t)$ in (1.4) satisfies the inequality

$$|M_R(t)| > 1. \tag{1.10}$$

Therefore we can find numbers R and $t_1 > t_0$ such that $|x_0| \leq R/2$ and

$$\Phi(t_0, t_1) = \int_{t_0}^{t_1} M_R(s)\,ds \,\exp\left\{\int_{t_0}^{t_1} B_R(s)\,ds\right\} = \frac{R}{2}. \tag{1.11}$$

Applying the method of successive approximations to (1.8) on the interval $[t_0, t_1]$,

$$x^{(n+1)}(t) = x_0 + \int_{t_0}^{t} F(x^{(n)}(s), s)\,ds, \quad x^0(t) \equiv x_0,$$

and using (1.4), (1.5) and (1.11), we get the estimates

$$|x^{(1)}(t) - x_0| \leq \int_{t_0}^{t} M_R(s)\,ds \leq \frac{R}{2},$$

$$|x^{(n+1)}(t) - x^{(n)}(t)| \leq \int_{t_0}^{t} B_R(s)|x^{(n)}(s) - x^{(n-1)}(s)|\,ds.$$

Together with (1.11), these imply the inequality

$$|x^{(n+1)}(t) - x^{(n)}(t)| \leq \int_{t_0}^{t} M_R(s)\,ds \frac{[\int_{t_0}^{t} B_R(s)\,ds]^n}{n!}. \tag{1.12}$$

It follows from (1.12) that $\lim_{n\to\infty} x^{(n)}(t)$ exists and that it satisfies (1.8). The proof of uniqueness is similar.

Now consider an arbitrary $T > t_0$ and choose R so that, besides the relations $|x_0| < R/2$ and (1.11), we also have $\tau_{R/2} > T$. Then by (1.9), it follows that $x(t_1) \leq R/2$ and thus the solutions can be continued to a point t_2 such that $\Phi(t_1, t_2) = R/2$. Repeating this procedure, we get $t_n \geq T$ for some n, since the functions $M_R(t)$ and $L_R(t)$ are integrable over every finite interval. This completes the proof. □

If the function $M_R(t)$ is independent of t and its rate of increase in R is at most linear, i.e.,

$$|F(x, t)| \leq c_1|x| + c_2, \tag{1.13}$$

we get the following estimate for the solution of problem (1.6), (1.7), valid for $t \geq t_0$ and some $c_3 > 0$:

$$|x(t)| \leq |x_0| c_3 e^{c_1(t-t_0)}.$$

We omit the proof now, since we shall later prove a more general theorem. But if condition (1.13) fails to hold, the solution will generally "escape to infinity" in a finite time. (As for example, the solution $x = (1-t)^{-1}$ of the problem $dx/dt = x^2$, $x(0) = 1$.) Since condition (1.13) fails to cover many cases of practical importance, we shall need a more general condition implying that the solution can be unlimitedly continued. We present first some definitions.

The Lyapunov operator associated with (1.6) is the operator d^0/dt defined by

$$\frac{d^0 V(x,t)}{dt} = \overline{\lim_{h \to +0}} \frac{1}{h}[V(x(t+h,x,t),t+h) - V(x,t)]. \tag{1.14}$$

It is obvious that if $V(x,t)$ is continuously differentiable with respect to x and t, then for almost all t the action of the Lyapunov operator

$$\frac{d^0 V}{dt} = \frac{\partial V}{\partial t} + \sum_{i=1}^{l} \frac{\partial V}{\partial x_i} F_i(x,t) = \frac{\partial V}{\partial t} + \left(\frac{\partial}{\partial x} V, F\right) \tag{1.15}$$

is simply a differentiation of the function V along the trajectory of the system (1.6).

In his classical work [188], Lyapunov discussed the stability of systems of differential equations by considering non-negative functions for which $d^0 V/dt$ satisfies certain inequalities.

These functions will be called *Lyapunov functions* here.

In Sects. 1.5, 1.6, 1.8, and also in Chaps. 5 to 7 we shall apply Lyapunov's ideas to stability problems for random perturbations.

In this and the next sections we shall use method of Lyapunov functions to find conditions under which the solution can be continued for all $t > 0$ and to conditions of boundedness solution. All Lyapunov functions figuring in the discussion will be henceforth assumed to be absolutely continuous in t, uniformly in x in the neighborhood of every point. Moreover we shall assume a Lipschitz condition with respect to x:

$$|V(x_2,t) - V(x_1,t)| < B|x_2 - x_1| \tag{1.16}$$

in the domain $U_R \times I_T$, with a Lipschitz constant which generally depends on R and T. We shall write $V \in \mathbf{C}$ in this case. If the function V satisfies condition (1.16) with a constant B not depending on R and T, we shall write $V \in \mathbf{C}_0$.

If $V \in \mathbf{C}$ and the function $y(t)$ is absolutely continuous, then it is easily verified that the function $V(y(t),t)$ is also absolutely continuous. Hence, for almost all t,

$$\frac{d^0 V(x,t)}{dt} = \frac{d}{dt} V(x(t),t)\bigg|_{x(t)=x},$$

where $x(t)$ is the solution of (1.6). We shall use this fact frequently without further reference.

Theorem 1.3 [2] *Assume that there exists a Lyapunov function* $V \in \mathbf{C}$ *defined on the domain* $\mathbb{R}^l \times \{t > t_0\}$ *such that for some* $c_1 > 0$

$$V_R = \inf_{(x,t)\in U_R^c \times \{t>t_0\}} V(x,t) \to \infty \quad \text{as } R \to \infty, \tag{1.17}$$

$$\frac{d^0 V}{dt} \le c_1 V, \tag{1.18}$$

and let the function F *satisfy conditions* (1.4), (1.5).

Then the solution of problem (1.6), (1.7) *can be extended for all* $t \ge t_0$.

The proof of this theorem employs the following well-known lemma, which will be used repeatedly.

Lemma 1.1 *Let the function* $y(t)$ *be absolutely continuous for* $t \ge t_0$ *and let the derivative* dy/dt *satisfy the inequality*

$$\frac{dy}{dt} < A(t)y + B(t) \tag{1.19}$$

for almost all $t \ge t_0$*, where* $A(t)$ *and* $B(t)$ *are almost everywhere continuous functions integrable over every finite interval. Then for* $t > t_0$

$$y(t) < y(t_0)\exp\left\{\int_{t_0}^t A(s)\,ds\right\} + \int_{t_0}^t \exp\left\{\int_s^t A(u)\,du\right\} B(s)\,ds. \tag{1.20}$$

Proof It follows from (1.19) that for almost all $t \ge t_0$

$$\frac{d}{dt}\left(y(t)\exp\left\{-\int_{t_0}^t A(s)\,ds\right\}\right) < B(t)\exp\left\{-\int_{t_0}^t A(s)\,ds\right\}.$$

Integration of this inequality yields (1.20). □

Proof of Theorem 1.3 It follows from (1.18) that for almost all t we have $dV(x(t),t)/dt \le c_1 V(x(t),t)$. Hence, by Lemma 1.1, it follows that for $t > t_0$

$$V(x(t),t) \le V(x_0,t_0)\exp\{c_1(t-t_0)\}.$$

If τ_R denotes a solution of the equation

$$V(x_0,t_0)\exp\{c_1(\tau_R - t_0)\} = V_R,$$

then condition (1.9) is obviously satisfied. Thus all assumptions of Theorem 1.2 are now satisfied. This completes the proof. □

[2] General conditions for every solution to be unboundedly continuable have been obtained by Okamura and are described in [178]. These results imply Theorem 1.3.

Let us now consider conditions under which the solutions of (1.6) are bounded for $t > 0$. There exist in the literature various definitions of boundedness. We shall adopt here only one which is most suitable for our purposes, referring the reader for more details to [285], [178], and [51, 52].

The system (1.6) is said to be *dissipative* for $t > 0$ if there exists a positive number $R > 0$ such that for each $r > 0$, beginning from some time $T(r, t_0) \geq t_0$, the solution $x(t, x_0, t_0)$ of problem (1.6), (1.7), $x_0 \in U_r$, $t_0 > 0$, lies in the domain U_R. (Yoshizawa [285] calls the solutions of such a system equi-ultimately bounded.)

Theorem 1.4 [3] *A sufficient condition for the system* (1.6) *to be dissipative is that there exist a nonnegative Lyapunov function* $V(x, t) \in \mathbf{C}$ *on* E *with the properties*

$$V_R = \inf_{(x,t)\in U_R^c \times I} V(x, t) \to \infty \quad \text{as } R \to \infty, \tag{1.21}$$

$$\frac{d^0 V}{dt} < -cV \quad (c = \text{const} > 0). \tag{1.22}$$

Proof It follows from Lemma 1.1 and from (1.22) that for $t > t_0$, $x_0 \in U_r$,

$$V(x(t), t) \leq V(x_0, t_0)e^{-c(t-t_0)} \leq e^{-c(t-t_0)} \sup_{|x_0|<r} V(x_0, t_0).$$

Therefore $V(x(t), t) < 1$ for $t > T(t_0, r)$. This inequality and (1.21) imply the statement of the theorem. □

Remark 1.1 The converse theorem is also valid: Yoshizawa [285] proves that for each system which is dissipative in the above sense there exists a nonnegative function V with properties (1.21), (1.22), provided $F(x, t)$ satisfies a Lipschitz condition in every bounded subset of E.

Remark 1.2 It is easy to show that the conclusion of Theorem 1.4 remains valid if it is merely assumed that (1.22) holds in a domain U_R^c for some $R > 0$, and in the domain U_R the functions V and $d^0 V/dt$ are bounded above. To prove this, it is enough to apply Lemma 1.1 to the inequality

$$\frac{d^0 V}{dt} < -cV + c_1,$$

which is valid under the above assumptions for some positive constant c_1 and for $(x, t) \in E$.

In the sequel we shall need a certain frequently used estimate; its proof, analogous to the proof of Lemma 1.1, may be found, e.g., in [23].

[3] See [285].

Lemma 1.2 (Gronwall–Bellman Lemma) *Let $u(t)$ and $v(t)$ be nonnegative functions and let k be a positive constant such that for $t \geq s$*

$$u(t) \leq k + \int_s^t u(t_1)v(t_1)\,dt_1.$$

Then for $t \geq s$

$$u(t) \leq k \exp\left\{\int_s^t v(t_1)\,dt_1\right\}.$$

1.3 Stochastic Processes as Solutions of Differential Equations

Let $\xi(t,\omega)$ $(t \geq 0)$ be a separable measurable stochastic process with values in $\mathbb{R}^k$, and let $G(x,t,z)$ $(x \in \mathbb{R}^l,\ t \geq 0,\ z \in \mathbb{R}^k)$ be a Borel-measurable function of (x,t,z) satisfying the following conditions:

1. There exists a stochastic process $B(t,\omega) \in \mathbf{L}$ such that for all $x_i \in \mathbb{R}^l$

$$|G(x_2,t,\xi(t,\omega)) - G(x_1,t,\xi(t,\omega))| \leq B(t,\omega)|x_1 - x_2|. \tag{1.23}$$

2. The process $G(0,t,\xi(t,\omega))$ is in $\mathbf{L}$, i.e., for every $T > 0$,

$$\mathbf{P}\left\{\int_0^T |G(0,t,\xi(t,\omega))|\,dt < \infty\right\} = 1. \tag{1.24}$$

We shall show presently that under these assumptions the equation

$$\frac{dx}{dt} = G(x,t,\xi(t,\omega)) \tag{1.25}$$

with initial condition

$$x(t_0) = x_0(\omega) \tag{1.26}$$

determines a new stochastic process in $\mathbb{R}^l$ for $t \geq t_0$.

Theorem 1.5 *If conditions* (1.23) *and* (1.24) *are satisfied, then problem* (1.25), (1.26) *has a unique solution $x(t,\omega)$, determining a stochastic process which is almost surely absolutely continuous for all $t \geq t_0$. For each $t \geq t_0$, this solution admits the estimate*

$$|x(t,\omega) - x_0(\omega)| \leq \int_{t_0}^t |G(x_0(\omega),s,\xi(s,\omega))|\,ds\ \exp\left\{\int_{t_0}^t B(s,\omega)\,ds\right\}. \tag{1.27}$$

The proof is analogous to that of Theorem 1.2.

Example 1.1 Consider the linear system

$$\frac{dx}{dt} = A(t,\omega)x + b(t,\omega), \qquad x(0) = x_0(\omega).$$

If $\|A(t,\omega)\|, |b(t,\omega)| \in L$, then it follows from Theorem 1.5 that this system has a solution which is a continuous stochastic process for all $t > 0$.

The global Lipschitz condition (1.23) fails to hold in many important applications. Most frequently the following local Lipschitz condition holds: For each $R > 0$, there exists a stochastic process $B_R(t,\omega) \in \mathbf{L}$ such that if $x_i \in U_R$, then

$$|G(x_2,t,\xi(t,\omega)) - G(x_1,t,\xi(t,\omega))| \leq B_R(t,\omega)|x_2 - x_1|. \tag{1.28}$$

As we have already noted in Sect. 1.2, condition (1.28) does not prevent the sample function escaping to infinity in a finite time, even in the deterministic case. However, we have the following theorem which is a direct corollary of Theorem 1.2.

Theorem 1.6 *Let $\tau(R,\omega)$ be a family of random variables such that $\tau(R,\omega) \uparrow \infty$ almost surely as $R \to \infty$. Suppose that these random variables satisfy almost surely for each solution $x(t,\omega)$ of problem* (1.25), (1.26) (*if a solution exists*) *the following inequality*:

$$\inf\{t : |x(t,\omega)| \geq R\} \geq \tau(R,\omega). \tag{1.29}$$

Assume moreover that conditions (1.24) *and* (1.28) *are satisfied. Then the solution of problem* (1.25), (1.26) *is almost surely unique and it determines an absolutely continuous stochastic process for all $t \geq t_0$* (*unboundedly continuable for $t \geq t_0$*).

Assume that the function G in (1.25) depends linearly on the third variable, i.e.,

$$\frac{dx}{dt} = F(x,t) + \sigma(x,t)\xi(t,\omega). \tag{1.30}$$

(Here σ is a $k \times l$ matrix, ξ a vector in $\mathbb{R}^k$ and k a positive integer.) Then the solution of (1.30) can be unboundedly continued if there exists a Lyapunov function of the truncated system

$$\frac{dx}{dt} = F(x,t). \tag{1.31}$$

Let us use $d^{(1)}/dt$ to denote the Lyapunov operator of the system (1.30), retaining the notation d^0/dt for the Lyapunov operator of the system (1.31).

Theorem 1.7 *Let $\xi(t,\omega) \in \mathbf{L}$ be a stochastic process, F a vector and σ a matrix satisfying the local Lipschitz condition* (1.16), *where $F(0,t) \in \mathbf{L}$ and*

$$\sup_{\mathbb{R}^l \times \{t > t_0\}} \|\sigma(x,t)\| < c_2. \tag{1.32}$$

Assume that a Lyapunov function $V(x,t) \in \mathbf{C}_0$ of the system (1.31) exists with

$$V_R = \inf_{U_R^c \times \{t > t_0\}} V(x,t) \to \infty \quad \text{as } R \to \infty, \tag{1.33}$$

$$\frac{d^0 V}{dt} < c_1 V. \tag{1.34}$$

Then the solution of problem (1.30), (1.26) exists and determines an absolutely continuous stochastic process for all $t \geq t_0$.

To prove this theorem we need the following lemma.

Lemma 1.3 *If $V(x,t) \in \mathbf{C}_0$, then for almost all t the following relation holds almost surely*:

$$\frac{d^{(1)} V(x,t)}{dt} \leq \frac{d^0 V(x,t)}{dt} + B\|\sigma(x,t)\||\xi(t,\omega)|, \tag{1.35}$$

where B is the constant in the condition (1.16).

Proof It can be easily verified that the difference $x(t+h,\omega,x,t) - x(t+h,x,t)$ between solutions of (1.30) and (1.31) with the initial condition $x(t) = x$, satisfies for almost all t, ω the inequality

$$|x(t+h,\omega,x,t) - x(t+h,x,t)| \leq h\|\sigma(x,t)\||\xi(t,\omega)| + o(h) \quad (h \to 0).$$

This inequality, together with (1.16), implies (1.35). □

Proof of Theorem 1.7 We shall show that the assumptions of Theorem 1.6 are satisfied. Since conditions (1.24) and (1.28) are obviously satisfied, it will suffice to prove (1.29). Let $x(t,\omega)$ be a solution of problem (1.30), (1.26). It follows from the assumptions of the theorem and from Lemma 1.3 that the function $V(x(t,\omega),t)$ is absolutely continuous, and for almost all t, ω

$$\begin{aligned}\frac{dV(x(t,\omega),t)}{dt} &\leq \frac{d^0 V(x(t,\omega),t)}{dt} + B\|\sigma(x(t,\omega),t)\||\xi(t,\omega)| \\ &\leq c_1 V(x(t,\omega),t) + Bc_2|\xi(t,\omega)|.\end{aligned}$$

Combining this with Lemma 1.1 we get that almost surely

$$V(x(t,\omega),t) \leq e^{c_1(t-t_0)}\left[V(x_0(\omega),t_0) + Bc_2\int_{t_0}^{t} |\xi(s,\omega)|\,ds\right]. \tag{1.36}$$

Let $\tau_R(\omega)$ denote a solution of the equation

$$e^{c_1(\tau_R - t_0)}\left[V(s_0(\omega),t_0) + Bc_2\int_{t_0}^{\tau_R} |\xi(s,\omega)|\,ds\right] = V_R. \tag{1.37}$$

It now follows from the relation $\xi(t,\omega) \in \mathbf{L}$ and from (1.33) that $\tau_R \uparrow \infty$ almost surely as $R \to \infty$. (1.29) follows now from (1.36) and (1.37). Thus all assumptions of Theorem 1.6 are satisfied. □

Remark 1.3 If the relation $|\xi(t,\omega)|^{(1+\varepsilon)/\varepsilon} \in \mathbf{L}$ holds for some $\varepsilon > 0$, condition (1.32) can be slightly weakened and replaced by the condition

$$\|\sigma(x,t)\|^{1+\varepsilon} \le c_3 V(x,t). \tag{1.38}$$

To prove this, we need only use Young's inequality

$$|ab| < \frac{|a|^p}{p} + \frac{|b|^q}{q} \quad \left(\frac{1}{p} + \frac{1}{q} = 1,\ p,q > 0\right) \tag{1.39}$$

for estimating $\|\sigma\||\xi|$. In particular, if for each $T > 0$, there is a constant c such that the process $\xi(t,\omega)$ satisfies the condition

$$\mathbf{P}\left\{\sup_{0\le t\le T} |\xi(t,\omega)| < c\right\} = 1,$$

then it is enough to require that inequality (1.38) holds for sufficiently small $\varepsilon > 0$.

Remark 1.4 The conditions of Theorem 1.7 guarantee that the solutions of (1.30) are unboundedly continuable, uniformly in the following sense: For all initial conditions $x_0(\omega)$ which satisfy the relation

$$\mathbf{P}\{|x_0(\omega)| < K\} = 1 \tag{1.40}$$

for some K, one can find a family of random variables $\tau(R,\omega)$ satisfying condition (1.29). Since

$$\mathbf{P}\left\{\max_{0\ge t\ge T} |x(t,\omega,x_0(\omega))| > R\right\} \le \mathbf{P}\{\tau_R < T\},$$

this implies in particular that for every $\varepsilon > 0$, $T > 0$ and $K > 0$ there exists an $R > 0$ such that

$$\mathbf{P}\left\{\max_{0\ge t\ge T} |x(t,\omega,x_0(\omega))| > R\right\} > \varepsilon$$

for all $x_0(\omega)$ satisfying condition (1.40).

Example 1.2 In the one-dimensional case with the Lyapunov function $V(x,t) = |x| + 1$ we get the following result. If $F \in \mathbf{C}$, $\sigma \in \mathbf{C}$, σ satisfies the condition (1.32), while $\xi(t,\omega)$, $F(0,t) \in \mathbf{L}$, then a sufficient condition for the solutions of problem (1.30), (1.26) to be unboundedly extendable is that $F(x,t)\operatorname{sign} x < c(|x|+1)$ for some $c > 0$.

Example 1.3 Consider the equation

$$x'' + f(x)x' + g(x) = \sigma(x,x')\xi(t,\omega). \tag{1.41}$$

This equation describes the process "at the output" of many mechanical systems driven by a stochastic process. In particular, for $f(x) = x^2 - 1$, $g(x) = x$ and $\sigma(x, x') = 1$, the output process is that of a system described by a Van der Pol equation. Let the function $f(x)$ be bounded from below and assume that

$$|\sigma(x, x')| < c_1, \qquad \left|\frac{g(x)}{x}\right| < c_2.$$

Then

$$V(x, y) = (x^2 + y^2)^{1/2} \in \mathbf{C}_0$$

is obviously a Lyapunov function for the system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -f(x)y - g(x).$$

Moreover V satisfies conditions (1.33) and (1.34). Applying Theorem 1.7, we see that the process in (1.41) exists for all $t \geq t_0$ provided that $\xi(t, \omega) \in \mathbf{L}$.

1.4 Boundedness in Probability of Stochastic Processes Defined by Systems of Differential Equations

A stochastic process $\xi(t, \omega)$ $(t \geq 0)$ is said to be *bounded in probability* if the random variables $|\xi(t, \omega)|$ are bounded in probability uniformly in t, i.e.,

$$\sup_{t>0} \mathbf{P}\{|\xi(t, \omega)| > R\} \to 0 \quad \text{as } R \to \infty.$$

We shall say that a random variable $x_0(\omega)$ is in the class A_{R_0} if

$$\mathbf{P}\{|x_0(\omega)| < R_0\} = 1. \tag{1.42}$$

The system (1.25) will be called *dissipative* if the random variables $|x(t, \omega, x_0, t_0)|$ are bounded in probability, uniformly in $t \geq t_0$ whenever $x_0(\omega) \in A_R$ for some $R > 0$.

It is readily seen that this definition agrees with that of a deterministic dissipative system (see Sect. 1.2).

Theorem 1.8 *Let $V(x, t) \in \mathbf{C}_0$ be a non-negative Lyapunov function, defined on the domain E which satisfies condition* (1.33) *and the condition*

$$\frac{d^0 V}{dt} \leq -c_1 V \quad (c_1 = \text{const} > 0). \tag{1.43}$$

Let F and σ satisfy a local Lipschitz condition (1.16), *and let σ also satisfy condition* (1.32).

Then the system (1.30) *is dissipative for every stochastic process* $\xi(t,\omega)$ *such that*

$$\sup_{t>0} \mathbf{E}|\xi(t,\omega)| < \infty. \tag{1.44}$$

Before proving this theorem, we shall prove a lemma which yields a convenient for further applications form of Chebyshev's inequality.

Lemma 1.4 *Let* $V(x,t)$ *be a nonnegative function and* $\eta(t,\omega)$ *a stochastic process such that* $\mathbf{E}V(\eta(t,\omega),t)$ *exists.*

Then

$$\mathbf{P}\{|\eta(t,\omega)| > R\} \le \frac{\mathbf{E}V(\eta(t,\omega),t)}{\inf_{U_R^c \times \{s>t_0\}} V(x,s)}. \tag{1.45}$$

The proof follows from the following chain of inequalities:

$$\begin{aligned}\mathbf{E}V(\eta(t,\omega),t) &\ge \int_{|\eta(t,\omega)|>R} V(\eta(t,\omega),t)\,\mathbf{P}(d\omega)\\ &\ge \inf_{U_R^c \times \{s>t_0\}} V(x,s)\mathbf{P}\{|\eta(t,\omega)| > R\}.\end{aligned}$$ □

Proof of Theorem 1.8 Let $x(t,\omega)$ be a solution of problem (1.30), (1.26). Then the function $V(x(t,\omega),t)$ is differentiable for almost all t, ω. By Lemma 1.3 and by (1.43),

$$\begin{aligned}\frac{dV(x(t,\omega),t)}{dt} &\le \frac{d^0V(x(t,\omega),t)}{dt} + Bc_2|\xi(t,\omega)|\\ &\le -c_1V(x(t,\omega),t) + Bc_2|\xi(t,\omega)|.\end{aligned}$$

Hence, by Lemma 1.1,

$$V(x(t,\omega),t) \le V(x_0(\omega),t_0)e^{c_1(t-t_0)} + Bc_2\int_{t_0}^{t} e^{c_1(s-t)}|\xi(s,\omega)|\,ds.$$

Calculating the expectation of both sides of this inequality and using (1.44), we see that the function $\mathbf{E}V(x(t,\omega),t)$ is bounded uniformly for $t \ge t_0$ and for all $x_0(\omega)$ satisfying condition (1.42). Together with (1.45), this implies the theorem. □

Remark 1.5 It is clear from Remark 1.1 that the existence of a function V satisfying conditions (1.33), (1.43) is not only sufficient but also necessary for the system (1.30) to be dissipative for each stochastic process $\xi(t,\omega)$ satisfying (1.44).

Remark 1.6 If for some $\varepsilon > 0$

$$\sup_{t>0} \mathbf{E}|\xi(t,\omega)|^{(1+\varepsilon)/\varepsilon} < \infty,$$