Eberhard Zeidler

Quantum Field Theory III: Gauge Theory

A Bridge between Mathematicians and Physicists

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TO KRZYSZTOF MAURIN
IN GRATITUDE

Preface

> Sein Geist drang in die tiefsten Geheimnisse der Zahl, des Raumes und der Natur; er maß den Lauf der Gestirne, die Gestalt und die Kräfte der Erde; die Entwicklung der mathematischen Wissenschaft eines kommenden Jahrhunderts trug er in sich.[1]
>
> Lines under the portrait of Carl Friedrich Gauss (1777–1855)
> in the German Museum in Munich

> *Force equals curvature.*
> The basic principle of modern physics

> A theory is the more impressive, the simpler are its premises, the more distinct are the things it connects, and the broader is the range of applicability.
>
> Albert Einstein (1879–1955)

> Textbooks should be attractive by showing the beauty of the subject.
> Johann Wolfgang von Goethe (1749–1832)

The present book is the third volume of a comprehensive introduction to the mathematical and physical aspects of modern quantum field theory which comprises the following six volumes:

Volume I: Basics in Mathematics and Physics
Volume II: Quantum Electrodynamics
Volume III: Gauge Theory
Volume IV: Quantum Mathematics
Volume V: The Physics of the Standard Model
Volume VI: Quantum Gravitation and String Theory.

It is our goal to build a bridge between mathematicians and physicists based on challenging questions concerning the fundamental forces in

- the macrocosmos (the universe) and
- the microcosmos (the world of elementary particles).

[1] His mind pierced the deepest secrets of numbers, space, and nature; he measured the orbits of the planets, the form and the forces of the earth; in his mind he carried the mathematical science of a coming century.

The six volumes address a broad audience of readers, including both undergraduate and graduate students, as well as experienced scientists who want to become familiar with quantum field theory, which is a fascinating topic in modern mathematics and physics, full of many crucial open questions.

For students of mathematics, detailed knowledge of the physical background helps to enliven mathematical subjects and to discover interesting interrelationships between quite different mathematical topics. For students of physics, fairly advanced mathematical subjects are presented that go beyond the usual curriculum in physics. The strategies and the structure of the six volumes are thoroughly discussed in the Prologue to Volume I. In particular, we will try to help the reader to understand the basic ideas behind the technicalities. In this connection, the famous ancient story of Ariadne's thread is discussed in the Preface to Volume I:

> *In terms of this story, we want to put the beginning of Ariadne's thread in quantum field theory into the hands of the reader.*

There are four fundamental forces in the universe, namely,

- gravitation,
- electromagnetic interaction (e.g., light),
- strong interaction (e.g., the binding force of the proton),
- weak interaction (e.g., radioactive decay).

In modern physics, these four fundamental forces are described by

- Einstein's theory of general relativity (gravitation), and
- the Standard Model in elementary particle physics (electromagnetic, strong, and weak interaction).

The basic mathematical framework is provided by gauge theory:

> *The main idea is to describe the four fundamental forces by the curvature of appropriate fiber bundles.*

In this way, the universal principle *force equals curvature* is implemented. There are many open questions:

- A mathematically rigorous quantum field theory for the quantized version of the Standard Model in elementary particles has yet to be found.
- We do not know how to combine gravitation with the Standard Model in elementary particle physics (the challenge of quantum gravitation).
- Astrophysical observations show that 96 percent of the universe consists of both dark matter and dark energy. However, both the physical structure and the mathematical description of dark matter and dark energy are unknown.

> *One of the greatest challenges of the human intellect is the discovery of a unified theory for the four fundamental forces in nature based on first principles in physics and rigorous mathematics.*

In the present volume, we concentrate on the *classical aspects* of gauge theory related to curvature. These have to be supplemented by the crucial, but elusive quantization procedure. The quantization of the Maxwell–Dirac system leads to quantum electrodynamics (see Vol. II). The quantization of both the full Standard Model in elementary particle physics and the quantization of gravitation will be studied in the volumes to come.

> *One cannot grasp modern physics without understanding gauge theory, which tells us that the fundamental interactions in nature are based on parallel transport, and in which forces are described by curvature, which measures the path-dependence of the parallel transport.*

Gauge theory is the result of a fascinating long-term development in both mathematics and physics. Gauge transformations correspond to a change of potentials, and physical quantities measured in experiments are invariants under gauge transformations. Let us briefly discuss this.

Gauss discovered that the curvature of a two-dimensional surface is an intrinsic property of the surface. This means that the Gaussian curvature of the surface can be determined by using measurements on the surface (e.g., on the earth) without using the surrounding three-dimensional space. The precise formulation is provided by Gauss' theorema egregium (the egregious theorem). Bernhard Riemann (1826–1866) and Élie Cartan (1859–1951) formulated far-reaching generalizations of the theorema egregium which lie at the heart of

- modern differential geometry (the curvature of general fiber bundles), and
- modern physics (gauge theories).

Interestingly enough, in this way,

- Einstein's theory of general relativity (the curvature of the four-dimensional space-time manifold), and
- the Standard Model in elementary particle physics (the curvature of a specific fiber bundle with the symmetry group $U(1) \times SU(2) \times SU(3)$)

can be traced back to Gauss' theorema egregium.

In classical mechanics, a large class of forces can be described by the differentiation of potentials. This simplifies the solution of Newton's equation of motion and leads to the concept of potential energy together with energy conservation (for the sum of kinetic and potential energy). In the 1860s, Maxwell determined that the computation of electromagnetic fields can be substantially simplified by introducing potentials for both the electric and the magnetic field (the electromagnetic four-potential).

> *Gauge theory generalizes this by describing forces (interactions) by the differentiation of generalized potentials (also called connections).*

The point is that gauge transformations change the generalized potentials, but not the essential physical effects.

> *Physical quantities, which can be measured in experiments, have to be invariant under gauge transformations.*

Parallel to this physical situation, in mathematics the *Riemann curvature tensor* can be described by the differentiation of the Christoffel symbols (also called connection coefficients or geometric potentials). The notion of the Riemann curvature tensor was introduced by Riemann in order to generalize Gauss' theorema egregium to higher dimensions. In 1915, Einstein discovered that the Riemann curvature tensor of a four-dimensional space-time manifold can be used to describe gravitation in the framework of the theory of general relativity.

> *The basic idea of gauge theory is the transport of physical information along curves (also called parallel transport).*

This generalizes the parallel transport of vectors in the three-dimensional Euclidean space of our intuition.

> *In 1917, it was discovered by Levi-Civita that the study of curved manifolds in differential geometry can be based on the notion of parallel transport of tangent vectors (velocity vectors).*

In particular, curvature can be measured intrinsically by transporting a tangent vector along a closed path. This idea was further developed by Élie Cartan in the 1920s (the method of moving frames) and by Ehresmann in the 1950s (the connection of both principal fiber bundles and their associated vector bundles). The very close relation between

- gauge theory in modern physics (the transport of local $SU(2)$-phase factors investigated by Yang and Mills in 1954), and
- the formulation of differential geometry in terms of fiber bundles in modern mathematics

was only noticed by physicists in 1975 (see T. Wu and C. Yang, Concept of nonintegrable phase factors and global formulation of gauge fields, Phys. Rev. **D12** (1975), 3845–3857).

The present Volume III on gauge theory and the following Volume IV on quantum mathematics form a unified whole. The two volumes cover the following topics:

Volume III: Gauge Theory

Part I: The Euclidean Manifold as a Paradigm

Chapter 1: The Euclidean Space E_3 (Hilbert Space and Lie Algebra Structure)
Chapter 2: Algebras and Duality (Tensor Algebra, Grassmann Algebra, Clifford Algebra, Lie Algebra)
Chapter 3: Representations of Symmetries in Mathematics and Physics
Chapter 4: The Euclidean Manifold $\mathbb{E}^3$
Chapter 5: The Lie Group $U(1)$ as a Paradigm in Harmonic Analysis and Geometry
Chapter 6: Infinitesimal Rotations and Constraints in Physics
Chapter 7: Rotations, Quaternions, the Universal Covering Group, and the Electron Spin
Chapter 8: Changing Observers – A Glance at Invariant Theory Based on the Principle of the Correct Index Picture
Chapter 9: Applications of Invariant Theory to the Rotation Group
Chapter 10: Temperature Fields on the Euclidean Manifold $\mathbb{E}^3$
Chapter 11: Velocity Vector Fields on the Euclidean Manifold $\mathbb{E}^3$
Chapter 12: Covector Fields on the Euclidean Manifold $\mathbb{E}^3$ and Cartan's Exterior Differential – the Beauty of Differential Forms

Part II: Ariadne's Thread in Gauge Theory

Chapter 13: The Commutative Weyl $U(1)$-Gauge Theory and the Electromagnetic Field
Chapter 14: Symmetry Breaking
Chapter 15: The Noncommutative Yang–Mills $SU(N)$-Gauge Theory
Chapter 16: Cocycles and Observers
Chapter 17: The Axiomatic Geometric Approach to Vector Bundles and Principal Bundles

Part III: Einstein's Theory of Special Relativity

Chapter 18: Inertial Systems and Einstein's Principle of Special Relativity
Chapter 19: The Relativistic Invariance of the Maxwell Equations
Chapter 20: The Relativistic Invariance of the Dirac Equations and the Electron Spin

Part IV: Ariadne's Thread in Cohomology

Chapter 21: Exact Sequences
Chapter 22: Electrical Circuits as a Paradigm in Homology and Cohomology
Chapter 23: The Electromagnetic Field and the de Rham Cohomology.

Volume IV: Quantum Mathematics

Part I: The Hydrogen Atom as a Paradigm

Chapter 1: The Non-Relativistic Hydrogen Atom via Lie Algebra, Gauss's Hypergeometric Functions, von Neuman's Functional Analytic Approach, the Weyl–Kodaira Theory, Gelfand's Generalized Eigenfunctions, and Supersymmetry
Chapter 2: The Dirac Equation and the Relativistic Hydrogen Atom via the Clifford Algebra of the Minkowski Space

Part II: The Four Fundamental Forces in the Universe

Chapter 3: Relativistic Invariance and the Energy–Momentum Tensor in Classical Field Theories
Chapter 4: The Standard Model for Electroweak and Strong Interaction in Particle Physics
Chapter 5: Gravitation, Einstein's Theory of General Relativity, and the Standard Model in Cosmology

Part III: Lowest-Order Radiative Corrections in Quantum Electrodynamics (QED)

Chapter 6: Dimensional Regularization for the Feynman Propagators in QED (Quantum Electrodynamics)
Chapter 7: The Electron in an External Electromagnetic Field (Renormalization of Electron Mass and Electron Charge)
Chapter 8: The Lamb Shift

Part IV: Conformal Symmetry

Chapter 9: Conformal Transformations According to Gauss, Riemann, and Lichtenstein
Chapter 10: Compact Riemann Surfaces
Chapter 11: Minimal Surfaces
Chapter 12: Strings and the Graviton
Chapter 13: Complex Function Theory and Conformal Quantum Field Theory

Part V: Models in Quantum Field Theory

Part VI: Distributions and the Epstein–Glaser Approach to Perturbative Quantum Field Theory

Part VII: Nets of Operator Algebras and the Haag–Kastler Approach to Quantum Field Theory

Part VIII: Symmetry and Quantization – the BRST Approach to Quantum Field Theory

Part IX: Topology, Quantization, and the Global Structure of Physical Fields

Part X: Quantum Information.

Readers who want to understand modern differential geometry and modern physics as quickly as possible should glance at the Prologue of the present volume and at Chaps. 13 through 17 on Ariadne's thread in gauge theory.

Cohomology plays a fundamental role in modern mathematics and physics.

> *It turns out that cohomology and homology have their roots in the rules for electrical circuits formulated by Kirchhoff in 1847.*

This helps to explain why the Maxwell equations in electrodynamics are closely related to cohomology, namely, de Rham cohomology based on Cartan's calculus for differential forms and the corresponding Hodge duality on the Minkowski space. Since the Standard Model in particle physics is obtained from the Maxwell equations by replacing the commutative gauge group $U(1)$ with the noncommutative gauge group $U(1) \times SU(2) \times SU(3)$, it should come as no great surprise that de Rham cohomology also plays a key role in the Standard Model in particle physics via the theory of characteristic classes (e.g., Chern classes which were invented by Shing-Shen Chern in 1945 in order to generalize the Gauss–Bonnet theorem for two-dimensional manifolds to higher dimensions).

It is our goal to show that the gauge-theoretical formulation of modern physics is closely related to important long-term developments in mathematics pioneered by Gauss, Riemann, Poincaré and Hilbert, as well as Grassmann, Lie, Klein, Cayley, Élie Cartan and Weyl. The prototype of a gauge theory in physics is Maxwell's theory of electromagnetism. The Standard Model in particle physics is based on the principle of local symmetry. In contrast to Maxwell's theory of electromagnetism, the gauge group of the Standard Model in particle physics is a noncommutative Lie group. This generates additional interaction forces which are mathematically described by Lie brackets.

We also emphasize the methods of invariant theory. In terms of physics, different observers measure different values in their experiments. However, physics does not depend on the choice of observers. Therefore, one needs both an invariant approach and the passage to coordinate systems which correspond to the observers, as emphasized by Einstein in the theory of general relativity and by Dirac in quantum mechanics. The appropriate mathematical tool is provided by invariant theory.

Acknowledgments. In 2003, Jürgen Tolksdorf initiated a series of four International Workshops on the state of the art in quantum field theory and the search for a unified theory concerning the four fundamental interactions in nature. I am very grateful to Felix Finster, Olaf Müller, Marc Nardmann, and Jürgen Tolksdorf for organizing the workshop *Quantum Field Theory and Gravity*, Regensburg, 2010. The following three volumes contain survey articles written by leading experts:

> F. Finster, O. Müller, M. Nardmann, J. Tolksdorf, and E. Zeidler (Eds.), Quantum Field Theory and Gravity: Conceptual and Mathematical Advances in the Search for a Unified Framework, Birkhäuser, Basel (to appear).
>
> B. Fauser, J. Tolksdorf, and E. Zeidler (Eds.), Quantum Field Theory – Competitive Methods, Birkhäuser, Basel, 2008.
>
> B. Fauser, J. Tolksdorf, and E. Zeidler (Eds.), Quantum Gravitation: Mathematical Models and Experimental Bounds, Birkhäuser, Basel, 2006.

These three volumes are recommended as supplements to the material contained in the present monograph. For stimulating discussions and guidance, I would like to thank Sergio Albeverio, Christian Bär, Helga Baum, Christian Brouder, Romeo Brunetti, Detlef Buchholz, Christopher Deninger, Michael Dütsch, Claudia Eberlein, Kurusch Ebrahimi-Fard, William Farris, Bertfried Fauser, Joel Feldman, Chris Fewster, Felix Finster, Christian Fleischhack, Hans Föllmer, Alessandra Frabetti, Klaus Fredenhagen, Harald Fritzsch, Jörg Fröhlich, Peter Gilkey, José Gracia-Bondía, Ivor Grattan-Guiness, Harald Grosse, Stefan Hollands, Arthur Jaffe, Jürgen Jost, Hans Kastrup, Jerzy Kijowski, Klaus Kirsten, Dirk Kreimer, Elliott Lieb, Jan Louis, Dieter Lüst, Kishore Marathe, Matilde Marcolli, Michael Oberguggenberger, Robert Oeckl, Norbert Ortner, Frédéric Patras, Frank Pfäffle, Klaus Rehren, Vincent Rivasseau, Hartmann Römer, Gerd Rudolph, Manfred Salmhofer, Erhard Scholz, Klaus Sibold, Harold Steinacker, Jürgen Tolksdorf, Armin Uhlmann, Rainer Verch, Peter Wagner, Raimar Wulkenhaar, Hans Wußing, and Jacob Yngvason.

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This volume is gratefully dedicated to Professor Krzysztof Maurin in Warsaw. As a young man, I learned from him that mathematics, physics, and philosophy form a unity; they represent marvellous tools for the human intellect in order to approximate step by step the better understanding of the real world, and they have to serve the well-being of human society.

My hometown, Leipzig, is full of the music composed by Johann Sebastian Bach, who worked in Leipzig's Saint Thomas church from 1723 until his death in 1750. In the Preface of his book *Electroweak and Strong Interaction: An Introduction to Theoretical Particle Physics,* Springer, Berlin, 1996, my colleague Florian Scheck from Mainz University adapted Bach's dedication to his "Well-Tempered Clavier" from 1722:

> Written and composed for the benefit and use of young physicists and for the particular diversion of those already advanced in this study.

I would like to use the same quotation, replacing 'physicists' with 'mathematicians and physicists.'

I hope that readers will get a feel for the unity of mathematics and the unity of science. In 1915, John Dewey wrote in his book *The School and Society*, The University of Chicago Press, Chicago, Illinois: "We do not have a series of stratified earths, one of which is mathematical, another physical, another historical, and so on. We should not be able to live very long in any one taken by itself. We live in a world where all sides are bound together; all studies grow out of relations in the one great common world."

Leipzig, Spring 2011 Eberhard Zeidler

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