Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge
A Series of Modern Surveys in Mathematics 56

Franc Forstnerič

Stein Manifolds and Holomorphic Mappings

The Homotopy Principle in Complex Analysis

Springer

Ergebnisse der Mathematik und ihrer Grenzgebiete Volume 56

3. Folge

A Series of Modern Surveys in Mathematics

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ISSN 0071-1136 Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics
ISBN 978-3-642-22249-8 e-ISBN 978-3-642-22250-4
DOI 10.1007/978-3-642-22250-4
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2011936225

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Preface

This book is an attempt to present a coherent account of Oka theory, from the classical Oka-Grauert theory originating in the works of Kiyoshi Oka and Hans Grauert to the contemporary developments initiated by Mikhael Gromov.

At the core of Oka theory lies the heuristic *Oka principle*, a term coined by Jean-Pierre Serre in 1951: *Analytic problems on Stein manifolds admit analytic solutions if there are no topological obstructions.* The Cartan-Serre Theorems A and B are primary examples. The main exponent of the classical Oka-Grauert theory is the equivalence between topological and holomorphic classification of principal fiber bundles over Stein spaces. On the interface with affine algebraic geometry the Oka principle holds only rarely, while in projective geometry we have Serre's GAGA principle, the equivalence of analytic and algebraic coherent sheaves on compact projective algebraic varieties. In smooth geometry there is the analogous *homotopy principle* originating in the Smale-Hirsch homotopy classification of smooth immersions.

Modern Oka theory focuses on those properties of a complex manifold Y which insure that any continuous map $X \to Y$ from a Stein source space X can be deformed to a holomorphic map; the same property is considered for sections of a holomorphic submersion $Y \to X$. By including the Runge approximation and the Cartan extension condition one obtains several ostensibly different Oka properties. Gromov's main result is that a geometric condition called ellipticity – the existence of a dominating holomorphic spray on Y – implies all forms of the Oka principle for maps or sections $X \to Y$. Subsequent research culminated in the result that all Oka properties of a complex manifold Y are equivalent to the following Runge approximation property:

A complex manifold Y is said to be an Oka manifold *if every holomorphic map $f\colon K \to Y$ from a neighborhood of a compact convex set $K \subset \mathbb{C}^n$ to Y can be approximated uniformly on K by entire maps $\mathbb{C}^n \to Y$.*

The related concept of an *Oka map* pertains to the Oka principle for lifting holomorphic maps from Stein sources. The class of Oka manifolds is dual to the class of Stein manifolds in a sense that can be made precise by means of abstract homotopy theory. Finnur Lárusson constructed a model category containing all complex manifolds in which Stein manifolds are cofibrant, Oka manifolds are fibrant, and Oka maps are fibrations. This means that

Stein manifolds are the natural sources of holomorphic maps, while Oka manifolds are the natural targets.

Oka manifolds seem to be few and special; in particular, no compact complex manifold of Kodaira general type is Oka. However, special and highly symmetric objects are often more interesting than average generic ones.

A few words about the content. Chapter 1 contains some preparatory material, and Chapter 2 is a brief survey of Stein space theory. In Chapter 3 we construct open Stein neighborhoods of certain types of sets in complex spaces that are used in Oka theory. Chapter 4 contains an exposition of the theory of holomorphic automorphisms of Euclidean spaces and of the density property, a subject closely intertwined with our main theme. In Chapter 5 we develop Oka theory for stratified fiber bundles with Oka fibers (this includes the classical Oka-Grauert theory), and in Chapter 6 we treat Oka-Gromov theory for stratified subelliptic submersions over Stein spaces. Chapters 7 and 8 contain applications ranging from classical to the recent ones. In Chapter 8 we present results on regular holomorphic maps of Stein manifolds; highlights include the optimal embedding theorems for Stein manifolds and Stein spaces, proper holomorphic embeddings of some bordered Riemann surfaces into $\mathbb{C}^2$, and the construction of noncritical holomorphic functions, submersions and foliations on Stein manifolds. In Chapter 9 we explore implications of Seiberg-Witten theory to the geometry of Stein surfaces, and we present the Eliashberg-Gompf construction of Stein structures on manifolds with suitable handlebody decomposition. A part of this story is the *Soft Oka principle.*

This book would not have existed without my collaboration with Jasna Prezelj who explained parts of Gromov's work on the Oka principle in her dissertation (University of Ljubljana, 2000). Josip Globevnik suggested that we look into this subject, while many years earlier Edgar Lee Stout proposed that I study the Oka-Grauert principle. My very special thanks go to the colleagues who read parts of the text and offered suggestions for improvements: Barbara Drinovec-Drnovšek, Frank Kutzschebauch, Finnur Lárusson, Takeo Ohsawa, Marko Slapar, and Erlend Fornæss Wold. I am grateful to Reinhold Remmert for his invitation to write a volume for the Ergebnisse series, and to the staff of Springer-Verlag for their professional help.

Finally, I thank Angela Gheorghiu for all those incomparably beautiful arias, and my family for their patience.

Ljubljana, *Franc Forstnerič*
May 1, 2011

Contents

'Say at last – who art thou?'
'That Power I serve which wills forever evil yet does forever good.'

J. W. Goethe, *Faust*

'Forgive me, but I don't believe you,' said Woland. 'That cannot be. Manuscripts don't burn.'

M. A. Bulgakov, *The Master and Margarita*

1

Preliminaries

This preliminary chapter is a brief review of the basic notions and constructions that are indispensable for reading the book. A comprehensive account is available in a number of excellent sources; for smooth manifolds see [5] and [503]; for complex and algebraic manifolds see [103, 233, 241, 370, 229, 508], among others; and for the theory of Stein manifolds and Stein spaces see the monographs [228, 241], and [267].

1.1 Complex Manifolds and Holomorphic Mappings

We denote by $\mathbb{R}$ the field of real numbers and by $\mathbb{C}$ the field of complex numbers. Let $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ be a positive integer. The model n-dimensional complex manifold is the complex Euclidean space $\mathbb{C}^n$, the Cartesian product of n copies of $\mathbb{C}$. Let $z = (z_1, \ldots, z_n)$ denote the complex coordinates on $\mathbb{C}^n$. Write $z_j = x_j + \mathrm{i}\, y_j$, where $x_j, y_j \in \mathbb{R}$ and $\mathrm{i} = \sqrt{-1}$. Given a differentiable complex valued function $f \colon D \to \mathbb{C}$ on a domain $D \subset \mathbb{C}^n$, the differential $\mathrm{d}f$ splits as the sum of the $\mathbb{C}$-linear part ∂f and the $\mathbb{C}$-antilinear part $\overline{\partial} f$:

$$\mathrm{d}f = \partial f + \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, \mathrm{d}z_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar z_j}\, \mathrm{d}\bar z_j. \tag{1.1}$$

Here $\mathrm{d}z_j = \mathrm{d}x_j + \mathrm{i}\, \mathrm{d}y_j$, $\mathrm{d}\bar z_j = \mathrm{d}x_j - \mathrm{i}\mathrm{d}y_j$, and

$$\frac{\partial f}{\partial z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} - \mathrm{i}\frac{\partial f}{\partial y_j}\right), \qquad \frac{\partial f}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + \mathrm{i}\frac{\partial f}{\partial y_j}\right). \tag{1.2}$$

The function f is *holomorphic* if $\mathrm{d}f = \partial f$ on D; that is, if the differential $\mathrm{d}f_z$ is $\mathbb{C}$-linear at every point $z \in D$. Equivalently, f is holomorphic if and only if $\overline{\partial} f = 0$, and this is equivalent to the n equations

$$\frac{\partial f}{\partial \bar z_j} = 0, \quad j = 1, \ldots, n.$$

F. Forstnerič, *Stein Manifolds and Holomorphic Mappings*,
Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics 56, DOI 10.1007/978-3-642-22250-4_1,

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Writing $f = u + \mathrm{i}v$ with u and v real, the equation $\partial f/\partial \bar{z}_j = 0$ is equivalent to the system of Cauchy-Riemann equations

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial y_j}, \qquad \frac{\partial u}{\partial y_j} = -\frac{\partial v}{\partial x_j}. \tag{1.3}$$

A mapping $f = (f_1, f_2, \ldots, f_m) \colon D \to \mathbb{C}^m$ is holomorphic if each component function f_j is such. When $m = n$, f is *biholomorphic* onto its image $D' = f(D) \subset \mathbb{C}^n$ if it is bijective and its inverse $f^{-1} \colon D' \to D$ is also holomorphic. An injective holomorphic map of a domain $D \subset \mathbb{C}^n$ to $\mathbb{C}^n$ is always biholomorphic onto its image [233, p. 19].

A *topological manifold of dimension* n is a second countable Hausdorff topological space which is locally Euclidean, in the sense that each point has an open neighborhood homeomorphic to an open set in $\mathbb{R}^n$. Such a space is metrizable, countably compact, and paracompact.

Assume now that X is a topological manifold of even dimension $2n$. A *complex atlas* on X is a collection $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$, where $\{U_\alpha\}_{\alpha \in A}$ is an open cover of X and ϕ_α is a homeomorphism of U_α onto an open subset U'_α in $\mathbb{R}^{2n} = \mathbb{C}^n$ such that for every pair of indexes $\alpha, \beta \in A$ the *transition map*

$$\phi_{\alpha,\beta} = \phi_\alpha \circ \phi_\beta^{-1} \colon \phi_\beta(U_{\alpha,\beta}) \to \phi_\alpha(U_{\alpha,\beta}) \tag{1.4}$$

is biholomorphic. Here $U_{\alpha,\beta} = U_\alpha \cap U_\beta$. An element (U_α, ϕ_α) of a complex atlas is called a *complex chart*, or a *local holomorphic coordinate system* on X. We also say that charts in a complex atlas are *holomorphically compatible*. For any three indexes $\alpha, \beta, \gamma \in A$ we have

$$\phi_{\alpha,\alpha} = \mathrm{Id}, \quad \phi_{\alpha,\beta} = \phi_{\beta,\alpha}^{-1}, \qquad \phi_{\alpha,\beta} \circ \phi_{\beta,\gamma} = \phi_{\alpha,\gamma} \tag{1.5}$$

on the respective domains of these maps. Two complex atlases $\mathcal{U}$, $\mathcal{V}$ on a topological manifold X are said to be *holomorphically compatible* if their union $\mathcal{U} \cup \mathcal{V}$ is also a complex atlas. This is an equivalence relation on the set of all complex atlases on X. Each equivalence class contains a unique *maximal complex atlas* – the union of all complex atlases in the given class.

A *complex manifold of complex dimension* n is a topological manifold X of real dimension $2n$ equipped with a complex atlas. Two complex atlases determine the same complex structure on X if and only if they are holomorphically compatible. We write $n = \dim_{\mathbb{C}} X$. A complex manifold of dimension one is called a *Riemann surface*, or a *complex curve* when it is seen as a complex submanifold in another complex manifold. A *complex surface* is a complex manifold of dimension $n = 2$.

A function $f \colon X \to \mathbb{C}$ on a complex manifold is said to be holomorphic if for any chart (U, ϕ) from the maximal atlas on X the function $f \circ \phi^{-1} \colon \phi(U) \to \mathbb{C}$ is holomorphic on the open set $\phi(U) \subset \mathbb{C}^n$. We denote by $\mathcal{O}(X)$ the Fréchet algebra of all holomorphic functions on X with the compact-open topology.

If D is a relatively compact domain with $\mathcal{C}^r$ smooth boundary in a complex manifold X for some $r \in \mathbb{N}$ then $\mathcal{A}^r(D)$ denotes the Banach algebra of all functions $\bar{D} \to \mathbb{C}$ of class $\mathcal{C}^r$ that are holomorphic on D.

Let X and Y be complex manifolds of dimensions n and m, respectively. A continuous map $f\colon X \to Y$ is said to be *holomorphic* if for any point $p \in X$ there are complex charts (U, ϕ) on X and (V, ψ) on Y such that $p \in U$, $f(U) \subset V$, and the map $\widetilde{f} = \psi \circ f \circ \phi^{-1}\colon \phi(U) \longrightarrow \psi(V) \subset \mathbb{C}^m$ is holomorphic on the open set $\phi(U) \subset \mathbb{C}^n$. Since the charts in a complex atlas are holomorphically compatible, the choice of charts is not important.

We adopt the convention that a map $X \to Y$ is *holomorphic on a compact set* K in X if it is holomorphic in an open neighborhood of K in X; two such maps are identified if they agree in some neighborhood of K. A family of maps is holomorphic on K if every map in the family is holomorphic in an open neighborhood of K that is independent of the map.

A map $f\colon X \to Y$ is *biholomorphic* if it is bijective and if both f and its inverse $f^{-1}\colon Y \to X$ are holomorphic. (This requires that $\dim X = \dim Y$.) As before, the latter condition is superfluous — a bijective holomorphic map between complex manifolds is actually biholomorphic. Note that every local chart $\phi\colon U \to \mathbb{C}^n$ on X is a biholomorphic map of U onto $\phi(U) \subset \mathbb{C}^n$.

A biholomorphic self-map $f\colon X \to X$ is called a *holomorphic automorphism* of X; the collection of all automorphisms is the *holomorphic automorphism group* $\mathrm{Aut}\, X = \mathrm{Aut}_{\mathrm{hol}}\, X$. We denote by $\mathrm{Aut}_{\mathrm{alg}}\, X$ the group of all algebraic automorphisms of an algebraic manifold. In many cases $\mathrm{Aut}\, X$ has the structure of a real or complex *Lie group* (see Example 1.2.4 below). For instance, $\mathrm{Aut}\, \mathbb{C}$ consists of all affine linear maps $z \mapsto \alpha z + \beta$ ($\alpha \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\beta \in \mathbb{C}$) and is a complex two dimensional Lie group. The automorphism group of any bounded domain $D \subset \mathbb{C}^n$ is a finite dimensional real Lie group; the maximal dimension is obtained when D is the ball $\mathbb{B}^n = \{z \in \mathbb{C}^n\colon |z|^2 = \sum_{j=1}^n |z_j|^2 < 1\}$. The group $\mathrm{Aut}\, \mathbb{B}^n$ acts transitively on $\mathbb{B}^n$, and the isotropy group of the origin $0 \in \mathbb{B}^n$ is the unitary group $U(n)$ (see [423]). Most bounded domains $D \subset \mathbb{C}^n$ have no automorphisms other than the identity. On the other hand, for $n > 1$ the group $\mathrm{Aut}\, \mathbb{C}^n$ is infinite dimensional (see Chapter 4).

Given a holomorphic map $f = (f_1, \ldots, f_m)\colon D \to \mathbb{C}^m$ on a domain $D \subset \mathbb{C}^n$, we denote by $\mathrm{rank}_p f$ the complex rank of f at a point $p \in D$; that is, the rank of the complex $m \times n$ Jacobian matrix

$$f'(p) = \left(\frac{\partial f_j}{\partial z_k}(p) \right). \tag{1.6}$$

This matrix represents the differential $\mathrm{d}f_p = \partial f_p\colon \mathrm{T}_p\mathbb{C}^n \to \mathrm{T}_{f(p)}\mathbb{C}^m$ in standard bases on the tangent spaces $\mathrm{T}_p\mathbb{C}^n$, $\mathrm{T}_{f(p)}\mathbb{C}^m$, respectively. (See §1.6.) Clearly $\mathrm{rank}_p f \le \min\{m, n\}$. The map f is an *immersion* at p if $\mathrm{rank}_p f = n$, and is a *submersion* at p if $\mathrm{rank}_p f = m$. These notions coincide when $n = m$, and in this case f is said to be *locally biholomorphic* at p. These notions, being local, extend to holomorphic maps between complex manifolds.

Let X be a complex manifold of dimension n. A subset M of X is a *complex submanifold* of dimension $m \in \{0, 1, \ldots, n\}$ (and codimension $d = n - m$) if every point $p \in M$ admits an open neighborhood $U \subset X$ and a holomorphic chart $\phi: U \to U' \subset \mathbb{C}^n$ such that $\phi(U \cap M) = U' \cap (\mathbb{C}^m \times \{0\}^{n-m})$. Any such chart (U, ϕ) on X is said to be *adapted* to M. Let $\pi: \mathbb{C}^n \to \mathbb{C}^m$ denote the coordinate projection $\pi(z_1, \ldots, z_m, \ldots, z_n) = (z_1, \ldots, z_m)$. For each adapted complex chart (U, ϕ) on X we get a complex chart $(U \cap M, \pi \circ \phi|_{U \cap M})$ on M with values in $\mathbb{C}^m$. The collection of all such charts is a complex atlas on M, and the corresponding complex structure on M is the *complex submanifold structure* induced by the inclusion map $\iota: M \hookrightarrow X$. Considering M with this submanifold structure as a complex manifold in its own right, the inclusion ι is a *holomorphic embedding* of M in X, that is, an injective holomorphic immersion of M onto the complex submanifold $\iota(M)$ of X.

The image of an injective holomorphic immersion $f: M \to X$ need not be a submanifold of X, not even a topological one, due to possible accumulation of the image on itself. The following important property prevents this behavior.

Definition 1.1.1. *A continuous map $f: X \to Y$ of topological spaces is said to be* proper *if the preimage $f^{-1}(K)$ of any compact set $K \subset Y$ is compact.*

A map $f: X \to Y$ between manifolds is proper if and only if it maps any discrete sequence in X to a discrete sequence in Y. If X and Y are complex manifolds and $f: X \to Y$ is a proper injective holomorphic immersion, then $f(X)$ is a closed complex submanifold of Y; such f is called a *proper holomorphic embedding*. More generally, if X and Y are complex spaces (see §1.3 below) and $f: X \to Y$ is a proper holomorphic map then $f(X)$ is a closed complex subvariety of Y according to a theorem of Remmert [412].

1.2 Examples of Complex Manifolds

Example 1.2.1. (*Riemann surfaces.*) These are one dimensional complex manifolds. By the Riemann-Koebe uniformization theorem [303, 250, 251] the only connected and simply connected Riemann surfaces up to a biholomorphism are the complex plane $\mathbb{C}$, the *Riemann sphere* $\mathbb{C} \cup \{\infty\} = \mathbb{P}^1$, and the disc $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$. If R is a connected Riemann surface then its universal covering space X is one of the surfaces $\mathbb{P}^1, \mathbb{C}, \mathbb{D}$, and R is biholomorphic to the quotient X/Γ for some group $\Gamma \subset \mathrm{Aut}\, X$ acting without fixed points and properly discontinuously on X. The automorphism group $\mathrm{Aut}\, \mathbb{P}^1 = \{z \mapsto \frac{az+b}{cz+d}: \ ad - bc = 1\}$ does not contain any nontrivial subgroups with these properties; hence $\mathbb{P}^1$ has no nontrivial holomorphic quotients. The only subgroups $\Gamma \subset \mathrm{Aut}\, \mathbb{C}$ with the required properties are *lattices*, i.e., discrete $\mathbb{Z}$-submodules of $\mathbb{C}$ acting on $\mathbb{C}$ by translations. Such Γ has either one or two generators: $\Gamma = \mathbb{Z}a$ ($a \neq 0$), or $\Gamma = \mathbb{Z}a + \mathbb{Z}b$ where $a, b \in \mathbb{C}$ are nonzero numbers with $ab^{-1} \notin \mathbb{R}$. The quotient $\mathbb{C}/\Gamma$ is $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$ in the case of a single generator, and is a complex one dimensional torus in the case of two generators. All other Riemann surfaces are quotients of the disc $\mathbb{D}$. □

Example 1.2.2. (*Affine algebraic manifolds.*) An *affine algebraic variety* in $\mathbb{C}^n$ is the common zero set of finitely many holomorphic polynomials in n complex variables. An affine algebraic variety without singular points is called an *affine algebraic manifold.* □

Example 1.2.3. (*Stein manifolds.*) The class of Stein manifolds was introduced by Karl Stein in 1951 [460] (under the name of *holomorphically complete manifolds*) by a system of three axioms postulating the existence of many global holomorphic functions, in analogy to the properties of domains of holomorphy (see Def. 2.2.1 on p. 47). The simplest characterization of this class is given by the Remmert embedding theorem [411]: A complex manifold is Stein if and only if it is biholomorphic to a closed complex submanifold of a Euclidean space $\mathbb{C}^N$. (For a more precise result see Theorem 2.2.8.) Hence Stein manifolds are holomorphic analogues of affine algebraic manifolds, a fact that is made precise by the algebraic approximations theorems (see p. 50). Analytic properties of Stein manifolds are in many aspects close to those of smooth manifolds, and are very different from those of compact complex manifolds. The main topic of this book is the theory of holomorphic mappings from Stein manifolds and Stein spaces to other complex manifolds. □

Example 1.2.4. (*Lie groups and homogeneous manifolds.*) A complex manifold G that is also a group with holomorphic group operations is called a *complex Lie group*. The main examples include the *general linear group* $GL_n(\mathbb{C})$ (the group of invertible complex $n \times n$ matrices) and its subgroups such as $SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) \colon \det A = 1\}$; the symplectic subgroup $Sp_n(\mathbb{C}) \subset GL_{2n}(\mathbb{C})$; certain quotients such as the *projective linear group* $PGL_n(\mathbb{C}) = GL_{n+1}(\mathbb{C})/\mathbb{C}^* = \mathrm{Aut}\,\mathbb{P}^n$ (the holomorphic automorphism group of $\mathbb{P}^n$); universal coverings of Lie groups, etc. A complex manifold X is said to be *G-homogeneous* if there exists a transitive holomorphic action $G \times X \to X$ of G on X by holomorphic automorphisms. Fixing a point $p \in X$, we see that X is biholomorphic to the quotient G/H where $H = \{g \in G \colon g(p) = p\}$ is the isotropy subgroup of the point p. For results on this subject see e.g. [7, 56]. □

Example 1.2.5. (*Complex projective spaces.*) The complex projective spaces $\mathbb{P}^n = \mathbb{CP}^n = \mathbb{P}(\mathbb{C}^{n+1})$ play the analogous role in algebraic geometry as the Euclidean spaces play in affine and Stein geometry. As a set, $\mathbb{P}^n$ consists of all complex lines through the origin in $\mathbb{C}^{n+1}$. A complex line $\lambda \subset \mathbb{C}^{n+1}$ is determined by any point $0 \neq z = (z_0, \ldots, z_n) \in \lambda$; we denote this line by $[z] = [z_0 \colon z_1 \colon \cdots \colon z_n]$ and call these the *homogeneous coordinates* on $\mathbb{P}^n$. Clearly $[z] = [w]$ if and only if $w = tz$ for some $t \in \mathbb{C}^*$. There is a unique complex manifold structure on $\mathbb{P}^n$ in which the projection $\pi \colon \mathbb{C}^{n+1}_* = \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{P}^n$,

$\pi(z) = [z] \in \mathbb{P}^n$ is holomorphic. A complex atlas is given by the collection (U_j, ϕ_j) $(j = 0, 1, \ldots, n)$ where $U_j = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n : z_j \neq 0\}$ and

$$\phi_j\left([z_0 : z_1 : \cdots : z_n]\right) = \left(\frac{z_0}{z_j}, \ldots, \frac{z_{j-1}}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_n}{z_j}\right) \in \mathbb{C}^n.$$

It is immediate that ϕ_j maps U_j bijectively onto $\mathbb{C}^n$ and that the transition maps $\phi_i \circ \phi_j^{-1}$ are linear fractional. □

Example 1.2.6. (*Projective manifolds and varieties.*) A nonzero holomorphic polynomial $P(z_0, \ldots, z_n)$ is *homogeneous* of degree $d \in \mathbb{N}$ if $P(tz_0, \ldots, tz_n) = t^d P(z_0, \ldots, z_n)$ for all $t \in \mathbb{C}$. Such P determines a complex hypersurface

$$V = V(P) = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{P}^n : P(z_0, \ldots, z_n) = 0\}.$$

More generally, homogeneous polynomials $P_1, \ldots, P_m$ on $\mathbb{C}^{n+1}$ determine a complex subvariety $V(P_1, \ldots, P_m) = V(P_1) \cap \cdots \cap V(P_m) \subset \mathbb{P}^n$. Subvarieties of this type in $\mathbb{P}^n$ are called *projective varieties*, or *projective manifolds* when they are nonsingular. A *quasi-projective variety* is a variety of the form $V = X \backslash Y$, where X and Y are closed complex subvarieties of $\mathbb{P}^n$. By Chow's theorem [85, 233, 241] every closed complex subvariety of $\mathbb{P}^n$ equals $V(P_1, \ldots, P_m)$ for some homogeneous polynomials in $n+1$ variables. A compact complex manifold (resp. a complex space) is said to be *projective algebraic* if it is biholomorphic to a projective manifold (resp. to projective subvariety) in some $\mathbb{P}^n$. A considerable extension of Chow's theorem is the *GAGA principle* of J.-P. Serre [441] concerning the equivalence between analytic and algebraic coherent sheaves over projective algebraic varieties. □

Example 1.2.7. (*Stiefel manifolds.*) Pick integers $1 \leq k \leq n$. The complex *Stiefel manifold* $V_{k,n}$ consists of all complex $k \times n$ matrices $A \in M_{k,n}(\mathbb{C}) \cong \mathbb{C}^{kn}$ with $\mathrm{rank} A = k$. Clearly $V_{k,n}$ is an open subset of $M_{k,n} = M_{k,n}(\mathbb{C})$. The group $GL_n(\mathbb{C})$ acts transitively on $V_{k,n}$ by right multiplication, so $V_{k,n}$ is a complex homogeneous manifold. We have $V_{k,n} = M_{k,n} \backslash \Sigma_{k,n}$ where $\Sigma_{k,n}$ consists of all complex $k \times n$ matrices of less than maximal rank. Note that $\Sigma_{k,n}$ is an algebraic subvariety of $M_{k,n}(\mathbb{C}) \cong \mathbb{C}^{kn}$ defined by the vanishing of all maximal $k \times k$ minors; these are homogeneous polynomial equations of order k (so $\Sigma_{k,n}$ is a complex cone in $\mathbb{C}^{kn}$), and at every point of $\Sigma_{k,n}$ at least $n-k+1$ of these equations are independent. In fact we have a stratification $\Sigma_{k,n} = \Sigma^1_{k,n} \supset \Sigma^2_{k,n} \supset \cdots$ where for every $i = 1, \ldots, k$ the set

$$\Sigma^i_{k,n} = \{A \in M_{k,n} : \mathrm{rank} A = k - i\}$$

is an algebraic subvariety of complex codimension $\mathrm{codim}\, \Sigma^i_{k,n} = i(n-k+i)$. (See e.g. [214, Proposition 5.3, p. 60].) In particular, $\mathrm{codim}\, \Sigma_{k,n} = n - k + 1$. It follows from the transversality theorem that the homotopy groups of $V_{k,n}$ vanish in the range up to $2(n-k)$:

$$\pi_q(V_{k,n}) = 0, \quad q = 1, 2, \ldots, 2n - 2k. \tag{1.7}$$

Example 1.2.8. (*Grassmann manifolds.*) The complex *Grassmann manifold* $G_{k,n} = G_k(\mathbb{C}^n)$ is the set of all k-dimensional complex linear subspaces of $\mathbb{C}^n$. (Thus $G_{1,n} = \mathbb{P}^{n-1}$.) Let $V_{k,n}$ be the Stiefel manifold (Example 1.2.7 above). We have a surjective map $\pi\colon V_{k,n} \to G_{k,n}$ which sends $A \in V_{k,n}$ to the $\mathbb{C}$-linear span of the row vectors in A. There is a unique complex structure on $G_{k,n}$ which makes this projection holomorphic. The group $GL_k(\mathbb{C})$ acts on $V_{k,n}$ by left multiplication, and we have $\pi(A) = \pi(B)$ for $A, B \in V_{k,n}$ if and only if $A = GB$ for some $G \in GL_k(\mathbb{C})$, so the Grassman manifold $G_{k,n}$ is the leaf space of this action. Grassmann manifolds are projective algebraic; the *Plücker embedding* $G_{k,n}(\mathbb{C}) \to \mathbb{P}(\wedge^k \mathbb{C}^n)$ is induced by the map $V_{k,n} \to \wedge^k \mathbb{C}^n$ sending a matrix $A \in V_{k,n}$ with rows $a_1, \ldots, a_k$ to $a_1 \wedge \cdots \wedge a_k \in \wedge^k \mathbb{C}^n$ [508, p. 11]. An important property for us is that every point in $G_{k,n}$ contains a Zariski open neighborhood isomorphic to $\mathbb{C}^{k(n-k)}$. $\square$

Example 1.2.9. (Complexifications.) For every real analytic manifold M there exists a complex manifold X obtained by complexifying the transition maps defining M [58]. Such X contains M as a maximal totally real submanifold, and it can be chosen Stein according to Grauert [224, §3]. (See §3.5.) $\square$

Example 1.2.10. (Hyperbolic manifolds.) The *Kobayashi-Royden pseudometric* on a complex manifold X is the largest pseudometric which equals the Poincaré metric on the unit disc and such that holomorphic maps are distance decreasing. A complex manifold X is said to be *Kobayashi hyperbolic* if the Kobayashi-Royden pseudometric on X is a metric, and is *complete hyperbolic* if this metric is complete (see [299, 300] for precise definitions). A complex manifold X is *Brody k-hyperbolic* for some $k \in \{1, \ldots, \dim X\}$ if every holomorphic map $\mathbb{C}^k \to X$ has rank $< k$; for $k = 1$ this means that every map $\mathbb{C} \to X$ is constant. For $k = \dim X$ this property is called (Brody) *volume hyperbolicity.* A compact complex manifold is Brody 1-hyperbolic if and only if it is Kobayashi hyperbolic [55]. For the notion of the Kobayashi-Eisenman form and hyperbolicity see [126, 287]. $\square$

1.3 Subvarieties and Complex Spaces

Let X be a complex manifold. We denote by $\mathcal{O}_x = \mathcal{O}_{X,x}$ the ring of germs of holomorphic functions at a point $x \in X$. A germ $[f]_x \in \mathcal{O}_x$ is represented by a holomorphic function in an open neighborhood of x; two such functions determine the same germ at x if and only if they agree in some neighborhood of x. The ring $\mathcal{O}_{X,x}$ is isomorphic to the ring $\mathcal{O}_{\mathbb{C}^n,0}$ via any holomorphic coordinate map sending x to 0. We can identify $\mathcal{O}_{\mathbb{C}^n,0}$ with the ring of convergent power series in n complex variables $(z_1, \ldots, z_n)$. This ring is Noetherian and a unique factorization domain. Its units are precisely the germs that do not vanish at 0. The set of germs vanishing at 0 is the unique maximal ideal $\mathfrak{m}_0 \subset \mathcal{O}_{\mathbb{C}^n,0}$ and $\mathcal{O}_{\mathbb{C}^n,0}/\mathfrak{m}_0 = \mathbb{C}$. For further properties of the local ring see

[103, 229, 233, 241]. The disjoint union $\mathcal{O}_X = \cup_{x\in X}\mathcal{O}_{X,x}$ is equipped with the topology whose basis is given by sets $\{[f]_x : x \in U\}$, where $f: U \to \mathbb{C}$ is a holomorphic function on an open set $U \subset X$. This makes $\mathcal{O}_X$ into a sheaf of commutative rings, called the *sheaf of germs of holomorphic functions* or *the structure sheaf* of X. The identity principle shows that the sheaf $\mathcal{O}_X$ is Hausdorff. We denote by $\mathcal{C}_X$ the sheaf of germs of continuous functions on X.

Since the ring $\mathcal{O}_x$ has no zero divisors, we can form its quotient field $\mathcal{M}_x$, called the field of germs of *meromorphic functions* on X at the point x. Thus a meromorphic function on X is locally at every point $x \in X$ given as the quotient f/g of two holomorphic functions whose germs at x are coprime. Such function is holomorphic off the zero locus of g, also called the *polar set* of f/g, and its *indeterminacy set* is $\{f = 0,\ g = 0\}$.

A subset A of a complex manifold X is a *complex (analytic) subvariety* of X if for every point $p \in A$ there exist a neighborhood $U \subset X$ of p and functions $f_1, \ldots, f_d \in \mathcal{O}(U)$ such that

$$A \cap U = \{x \in U : f_1(x) = 0, \ldots, f_d(x) = 0\}.$$

If such A is topologically closed in X then A is a *closed complex subvariety* of X. Since the local ring $\mathcal{O}_x$ is Noetherian, a subset of X that is locally defined by infinitely many holomorphic equations is still a subvariety and can be locally defined by finitely many equations.

A point p in a subvariety A is a *regular* (or *smooth*) point if A is a complex submanifold near p; the set of all regular points is denoted A_{reg}. A point $p \in A\backslash A_{\mathrm{reg}} = A_{\mathrm{sing}}$ is a *singular point* of A.

Let A be a closed complex subvariety X. For every point $x \in X$ we denote by $\mathcal{J}_{A,x}$ the ideal in $\mathcal{O}_x$ consisting of all holomorphic function germs at x whose restriction to A vanishes. In particular, $\mathcal{J}_{A,x} = \mathcal{O}_x$ for every $x \in X\backslash A$. The corresponding sheaf $\mathcal{J}_A = \cup_{x\in X}\mathcal{J}_{A,x}$ is the *sheaf of ideals* (or the *ideal sheaf*, or simply the *ideal*) of A in X. The restriction of the quotient sheaf $\mathcal{O}_X/\mathcal{J}_A = \cup_{x\in X}\mathcal{O}_{X,x}/\mathcal{J}_{A,x}$ to A is the sheaf of germs of holomorphic functions on A, denoted $\mathcal{O}_A$ and called the structure sheaf of A.

The notion of a *complex space* was first introduced in 1951 by H. Behnke and K. Stein [42] and H. Cartan [75]; their definitions correspond to what is now called a *normal complex space* (see [227]). The definition which is accepted as the standard one, and which is also used in this book, was given by J.-P. Serre in his GAGA paper [441]:

A *reduced complex space* is a pair $(X, \mathcal{O}_X)$, where X is a paracompact Hausdorff topological space and $\mathcal{O}_X$ is a sheaf of rings of continuous functions on X (a subsheaf of the sheaf $\mathcal{C}_X$ of germs of continuous functions) such that for every point $x \in X$ there is a neighborhood $U \subset X$ and a homeomorphism $\phi: U \to A \subset \mathbb{C}^n$ onto a locally closed complex subvariety of $\mathbb{C}^n$ so that the homomorphism $\phi^*: \mathcal{C}_A \to \mathcal{C}_X$, $f \mapsto f \circ \phi$, induces an isomorphism of $\mathcal{O}_A$ onto $\mathcal{O}_U = \mathcal{O}_X|_U$. Intuitively speaking, X is obtained by gluing pieces of subvarieties in Euclidean spaces using biholomorphic transition maps. Similarly one defines an *algebraic spaces* [441]. We get a *nonreduced complex space* by allowing local models $(A, \mathcal{F})$, where A is a closed complex subvariety in an open set $\Omega \subset \mathbb{C}^n$ and $\mathcal{F} = (\mathcal{O}_\Omega/\mathcal{I})|_A$ for some sheaf of ideals $\mathcal{I} \subset \mathcal{J}_A$ supported on A (i.e., $\mathcal{I}_x = \mathcal{O}_x$ for $x \notin A$). The ring $\mathcal{F}_x$ may have nilpotent elements. By the Nullstellensatz the radical $\sqrt{\mathcal{I}_x}$ of any such ideal equals $\mathcal{J}_{A,x}$.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be complex spaces. A continuous map $f\colon X \to Y$ is said to be holomorphic if for every $x \in X$ the composition $\mathcal{C}_{Y,f(x)} \ni g \mapsto g \circ f \in \mathcal{C}_{X,x}$ defines a homomorphism $f_x^*\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. For such map we can define the differential $\mathrm{d}f_x\colon \mathrm{T}_x X \to \mathrm{T}_{f(x)} Y$ as a $\mathbb{C}$-linear map on the *Zariski tangent space* (see (1.29) on p. 21). This is the usual differential at smooth points, while at singular points we locally embed the two spaces as complex subvarieties of Euclidean space of minimal dimension $n_x = \mathrm{embdim}_x X$, $m_y = \mathrm{embdim}_y Y$, respectively, and take the differential $\mathrm{d}F_x$ of the local holomorphic extension F of f (a $\mathbb{C}$-linear map $\mathbb{C}^{n_x} \to \mathbb{C}^{m_y}$).

Among the most fundamental results in the theory of complex spaces is Remmert's theorem [412] saying that the image $f(X)$ of a proper holomorphic map $f\colon X \to Y$ is a closed analytic subvariety of Y. A more general result of Grauert (see [229]) gives the coherence of the direct image $f_*\mathcal{F}$ of any coherent analytic sheaf $\mathcal{F}$ under a proper holomorphic map. (See also p. 53 below.)

Definition 1.3.1. *Let Z and X be reduced complex spaces. A holomorphic map $\pi\colon Z \to X$ is a* holomorphic submersion *if for every point $z_0 \in Z$ there exist an open neighborhood $V \subset Z$ of z_0, an open neighborhood $U \subset X$ of $x_0 = \pi(z_0)$, an open set W in $\mathbb{C}^p$, and a biholomorphic map $\phi\colon V \to U \times W$ such that $pr_1 \circ \phi = \pi$. (Here $pr_1\colon U \times W \to U$ is the projection on the first factor.) Each such local chart ϕ will be called* adapted to π.

Note that each fiber $Z_x = \pi^{-1}(x)$ $(x \in X)$ of a holomorphic submersion is a closed complex submanifold of Z, and the dimension $\dim Z_x$ is constant on every connected component of Z.

Definition 1.3.2. *Assume that $h\colon Z \to X$ is a holomorphic submersion onto a complex space X, X' is a closed complex subvariety of X, and $\mathcal{S} \subset \mathcal{O}_X$ is a sheaf of ideals with support X', i.e., $\mathcal{S}_x = \mathcal{O}_{X,x}$ when $x \in X \backslash X'$. Local holomorphic sections f_0, f_1 of $h\colon Z \to X$ in a neighborhood of a point $x \in X'$ are* $\mathcal{S}$-tangent at x *if there is a neighborhood $V \subset Z$ of the point $z = f_0(x) = f_1(x) \in Z$ and a holomorphic embedding $\phi\colon V \hookrightarrow \mathbb{C}^N$ such that the germ at x of any component of the map $\phi f_0 - \phi f_1\colon U \to \mathbb{C}^N$ belongs to $\mathcal{S}_x$. If f_0 and f_1 are holomorphic in a neighborhood of X' and $\mathcal{S}$-tangent at each point $x \in X'$, then we say that f_0 and f_1 are $\mathcal{S}$-tangent and write $\delta(f_0, f_1) \in \mathcal{S}$. If this holds for the r-th power of the ideal sheaf $\mathcal{J}_{X'}$ of the subvariety X' then f_0 and f_1 are said to be* tangent to order r along X'.

Definition 1.3.3. *A* stratification *of a finite dimensional complex space X is a finite descending sequence $X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$ of closed complex* *subvarieties such that each connected component S (stratum) of a difference $X_k \backslash X_{k+1}$ is a complex manifold and $\overline{S} \backslash S \subset X_{k+1}$.*

Every finite dimensional complex space admits a stratification [514, p. 227]: Take X_1 to be the union of the singular locus of $X = X_0$ and of all irreducible components of X_0 of less than maximal dimension; define X_2 in the same way with respect to X_1, etc. By considering substratifications we can ask for many additional properties. For example, a finite dimensional Stein space admits a stratification whose strata are Stein manifolds. Whitney's condition (a) is used in transversality theorems proved in §7.8.

1.4 Holomorphic Fiber Bundles

Fiber bundles represent one of the most important constructions of new manifolds from the existing ones.

Definition 1.4.1. *A* holomorphic fiber bundle *is a triple (Z, π, X), where X and Z are complex spaces and $\pi: Z \to X$ is a holomorphic map of Z onto X such that there exist a complex manifold Y, an open cover $\mathcal{U} = \{U_\alpha\}$ of X, and for every α a biholomorphic map*

$$\theta_\alpha: Z|_{U_\alpha} = \pi^{-1}(U_\alpha) \to U_\alpha \times Y, \quad \theta_\alpha(z) = (\pi(z), \vartheta_\alpha(z)). \tag{1.8}$$

The manifold Z is the total space*, X is the* base space*, and Y is the* fiber*.*

A holomorphic submersion $\pi: Z \to X$ is a stratified holomorphic fiber bundle *if there is a stratification $X = X_0 \supset X_1 \supset \cdots \supset X_m = \emptyset$ of X such that the restriction of Z to every stratum $S \subset X_k \backslash X_{k+1}$ is a holomorphic fiber bundle over S. (Fibers over different strata may be different.)*

The simplest example is a *product bundle* $\pi: Z = X \times Y \to X$, $(x, y) \mapsto x$. By definition, every fiber bundle is isomorphic to the product bundle over small open sets in X. The fiber $Z_x = \pi^{-1}(x)$ over any point $x \in X$ is biholomorphic to Y. A map θ_α (1.8) is a *fiber bundle chart* on Z, and the collection $\{(U_\alpha, \theta_\alpha)\}$ is a *holomorphic fiber bundle atlas* on Z. The *transition maps* $\theta_{\alpha,\beta} = \theta_\alpha \circ \theta_\beta^{-1}: U_{\alpha,\beta} \times Y \to U_{\alpha,\beta} \times Y$ are of the form

$$\theta_{\alpha,\beta}(x, y) = \big(x, \vartheta_{\alpha,\beta}(x, y)\big), \quad x \in U_{\alpha,\beta},\ y \in Y, \tag{1.9}$$

and they satisfy the *cocycle condition*

$$\theta_{\alpha,\alpha} = \mathrm{Id}, \quad \theta_{\alpha,\beta} \circ \theta_{\beta,\gamma} \circ \theta_{\gamma,\alpha} = \mathrm{Id} \quad \text{on } U_{\alpha,\beta,\gamma} \times Y. \tag{1.10}$$

For every fixed $x \in U_{\alpha,\beta}$ we have $\vartheta_{\alpha,\beta}(x, \cdot\,) \in \mathrm{Aut}\, Y$. Conversely, given an open cover $\mathcal{U} = \{U_\alpha\}$ of X and a collection of biholomorphic self-maps (1.9) satisfying the cocycle condition (1.10), we get a holomorphic fiber bundle $Z \to X$ with these transition maps by taking Z to be the disjoint union of all $U_\alpha \times Y$, modulo the identifications provided by the transition maps.

A *section* of $\pi\colon Z \to X$ is a map $f\colon X \to Z$ such that $\pi \circ f$ is the identity on X; that is, $f(x) \in Z_x$ for every $x \in X$. Any section of the product bundle $X \times Y \to X$ is of the form $f(x) = (x, g(x))$, where $g\colon X \to Y$ is a map to the fiber. If $\{(U_\alpha, \theta_\alpha)\}$ is a holomorphic fiber bundle atlas on $Z \to X$ with the transition maps $\theta_{\alpha,\beta}$ (1.9), then a holomorphic section $f\colon X \to Z$ is given by a collection of holomorphic maps $f_\alpha\colon U_\alpha \to Y$ satisfying the compatibility conditions

$$f_\alpha(x) = \vartheta_{\alpha,\beta}\big(x, f_\beta(x)\big), \quad x \in U_{\alpha,\beta}. \tag{1.11}$$

Definition 1.4.2. *A* holomorphic isomorphism *of holomorphic fiber bundles* $\pi\colon Z \to X$, $\pi'\colon Z' \to X$ *is a biholomorphic map* $\Phi\colon Z \to Z'$ *such that* $\pi' \circ \Phi = \pi$*; if such* Φ *exists then the bundles are* holomorphically isomorphic. *A fiber bundle is* trivial *if it is isomorphic to the product bundle.*

Isomorphisms of a fiber bundle onto itself are *fiber bundle automorphisms*. Holomorphic automorphisms of a product bundle $X \times Y \to X$ are biholomorphic self-maps of $X \times Y$ of the form $(x, y) \mapsto (x, \varphi(x, y))$, with $\varphi(x, \cdot\,) \in \operatorname{Aut} Y$ for every $x \in X$. In general we choose an open cover $\mathcal{U} = \{U_\alpha\}$ of X and fiber bundle atlases $\{(U_\alpha, \theta_\alpha)\}$ for (Z, π, X), and $\{(U_\alpha, \theta'_\alpha)\}$ for (Z', π', X). A fiber bundle isomorphism Φ of (Z, π, X) to (Z', π', X) is then given by a collection of fiber preserving biholomorphic self-maps $\phi_\alpha\colon U_\alpha \times Y \to U_\alpha \times Y$ of the form $\phi_\alpha(x, y) = (x, \varphi_\alpha(x, y))$ so that the following diagrams commute:

$$\begin{array}{ccc} U_{\alpha,\beta} \times Y & \xleftarrow{\;\theta_{\alpha,\beta}\;} & U_{\alpha,\beta} \times Y \\ \downarrow{\scriptstyle \phi_\alpha} & & \downarrow{\scriptstyle \phi_\beta} \\ U_{\alpha,\beta} \times Y & \xleftarrow{\;\theta'_{\alpha,\beta}\;} & U_{\alpha,\beta} \times Y \end{array}$$

If there exists a fiber bundle atlas on $Z \to X$ (in the given isomorphism class) such that all transition maps belong to a certain subgroup G of $\operatorname{Aut} Y$, then we say G is the *structure group* of the bundle, or that the structure group of the bundle has been reduced to G.

If $\pi\colon Z \to X$ is a holomorphic fiber bundle and X' is a complex subvariety of X then the restriction $\pi\colon Z' = Z|_{X'} \to X'$ is a holomorphic fiber bundle over X', called the *restricted bundle*.

Given a holomorphic fiber bundle $\pi\colon Z \to X$ with fiber Y and a holomorphic map $f\colon W \to X$, the *pull-back bundle* $\pi'\colon f^*Z \to W$ and the map $F\colon f^*Z \to Z$ are defined as follows:

$$f^*Z = \{(w, z) \in W \times Z\colon f(w) = \pi(z)\}, \quad \pi'(w, z) = w, \quad F(w, z) = z. \tag{1.12}$$

Let $\{(U_\alpha,\theta_\alpha)\}$ be a fiber bundle atlas for $\pi\colon Z\to X$ with

$$\theta_\alpha(z)=(\pi(z),\vartheta_\alpha(z))\in U_\alpha\times Y.$$

Set $V_\alpha=f^{-1}(U_\alpha)\subset W$ and define a map $\theta'_\alpha\colon f^*Z|_{V_\alpha}\to V_\alpha\times Y$ by $\theta'_\alpha(w,z)=(w,\vartheta_\alpha(z))$. (The map f appears implicitly in the above definition by the condition $\pi(z)=f(w)$.) This is a holomorphic fiber bundle atlas on $\pi'\colon f^*Z\to W$ with transition maps $\vartheta'_{\alpha,\beta}(w,y)=\vartheta_{\alpha,\beta}(f(w),y)$; hence $f^*Z\to W$ is indeed a holomorphic fiber bundle with fiber Y.

Example 1.4.3. (Vector bundles.) These are fiber bundles with fiber $\mathbb{C}^n$ and structure group $GL_n(\mathbb{C})$. They are considered in the following section. □

Example 1.4.4. (*Affine bundles.*) These are fiber bundles with fiber $\mathbb{C}^n$ and structure group consisting of affine linear maps:

$$v\mapsto a+Bv,\quad a\in\mathbb{C}^n,\ B\in GL_n(\mathbb{C}).$$

By [280] every projective analytic variety X carries an affine bundle $E\to X$ whose total space E is a Stein space. (This will be of interest in §5.16.) We recall the construction in the basic case $X=\mathbb{P}^n$. Consider the Segre embedding $\rho\colon\mathbb{P}^n\times\mathbb{P}^n\hookrightarrow\mathbb{P}^{n^2+2n}$,

$$\big([z_0\colon\cdots\colon z_n],[w_0\colon\cdots\colon w_n]\big)\longmapsto[z_0w_0\colon\cdots\colon z_iw_j\colon\cdots\colon z_nw_n].$$

Let Σ denote the quadratic hypersurface in $\mathbb{P}^n\times\mathbb{P}^n$ defined by the equation $z_0w_0+\cdots+z_nw_n=0$ and set $E=\mathbb{P}^n\times\mathbb{P}^n\backslash\Sigma$. Choose homogeneous coordinates $[\zeta_0\colon\zeta_1\colon\cdots\colon\zeta_{n^2+2n}]$ on $\mathbb{P}^{n^2+2n}$ such that $\zeta_j=z_jw_j$ for $j=0,\ldots,n$ under the embedding ρ. Let H denote the hyperplane in $\mathbb{P}^{n^2+2n}$ given by the equation $\sum_{j=0}^n\zeta_j=0$. Then ρ embeds E properly into $\mathbb{P}^{n^2+2n}\backslash H=\mathbb{C}^{n^2+2n}$, so E is Stein. The projection $\pi\colon E\to\mathbb{P}^n$ onto the first component is an affine bundle with fiber $\mathbb{C}^n$. The restriction $E|_X$ to any closed complex subvariety $X\subset\mathbb{P}^n$ is still an affine bundle whose total space (being a closed analytic subvariety of E) is Stein. For quasi-projective varieties there is some work to see that one can get an affine bundle with Stein total space. □

Example 1.4.5. (*Principal bundles.*) Let G be a finite dimensional complex Lie group. For every $g\in G$ let $\sigma_g\in\operatorname{Aut}G$ be the left multiplication on G by g: $\sigma_g(g')=gg'$ $(g'\in G)$. Set $\Gamma=\{\sigma_g\colon g\in G\}\subset\operatorname{Aut}G$. A *holomorphic principal G-bundle* over X is a holomorphic fiber bundle $\pi\colon Z\to X$ with fiber G and structure group Γ. Such bundle is determined by a 1-cocycle $g_{\alpha,\beta}\colon U_{\alpha,\beta}\to G$ over an open cover $\mathcal{U}=\{U_\alpha\}$ of X; the corresponding transition maps are $\phi_{\alpha,\beta}(x,g)=\big(x,g_{\alpha,\beta}(x)g\big)$. The group G acts holomorphically on the total space Z by the right multiplication on the fibers Z_x, and these fibers are precisely the orbits of the action. See §7.1 and §7.2 for further results. □

Example 1.4.6. (*Fiber bundles associated to principal bundles.*) Assume that a complex Lie group G acts holomorphically on a complex manifold Y. Every holomorphic 1-cocycle $g_{\alpha,\beta}\colon U_{\alpha,\beta} \to G$ on an open cover $\mathcal{U} = \{U_\alpha\}$ of X determines a holomorphic fiber bundle with fiber Y, structure group G and transition maps $\theta_{\alpha,\beta}(x,v) = \big(x, g_{\alpha,\beta}(x)v\big)$. In this way we associate to a principal G-bundle $Z \to X$ a fiber bundle $E \to X$ with fiber Y, structure group G and the same structure cocycle that determines the bundle Z. In particular, to a principal $GL_n(\mathbb{C})$-bundle we associate a holomorphic vector bundle of rank n. Conversely, to a holomorphic vector bundle $E \to X$ of rank n we associate the principal $GL_n(\mathbb{C})$-bundle $Z = F(E) \to X$, called *the frame bundle* of the vector bundle $E \to X$. The elements of $Z_x = F(E_x)$ are *frames* (complex bases) on the vector space $E_x \cong \mathbb{C}^n$. □

Example 1.4.7. (*Bundles with Euclidean fibers.*) These are fiber bundles with fiber $\mathbb{C}^n$ and structure group $\mathrm{Aut}\,\mathbb{C}^n$, the holomorphic automorphism group of $\mathbb{C}^n$. For $n = 1$ these are just affine bundles, but for $n > 1$ the group $\mathrm{Aut}\,\mathbb{C}^n$ is very large and we get many non-affine fiber bundles. In §4.21 we mention examples, due to Skoda, Demailly and Rosay, of such bundles over the disc or $\mathbb{C}$ whose total space is non-Stein. □

Example 1.4.8. (*Flat bundles.*) A holomorphic fiber bundle $Z \to X$ with fiber Y is said to be *flat* if it admits a holomorphic fiber bundle atlas whose transition maps $\vartheta_{\alpha,\beta}\colon U_{\alpha,\beta} \to \mathrm{Aut}\,Y$ are locally constant, and hence constant on every connected component of $U_{\alpha,\beta}$. The trivial horizontal foliations of $U_\alpha \times Y$ with leaves $U_\alpha \times \{y\}$ ($y \in Y$) patch together to a horizontal holomorphic foliation of the total space Z. Flat bundles arise naturally when considering the Chern connection with vanishing curvature tensor on a Hermitian holomorphic vector bundle. A flat holomorphic fiber bundle is trivial over every simply connected open set in the base X, and every holomorphic isomorphism class of flat bundles is determined by a representation of the fundamental group $\pi_1(X,p)$ in the automorphism group $\mathrm{Aut}\,Y$. □

1.5 Holomorphic Vector Bundles

Vector bundles are a principal tool used to linearize problems in analysis and geometry. They are also a subject of intrinsic investigation with a profound impact on modern mathematics. We focus on holomorphic vector bundles, recalling those constructions that will be important to us. Similar constructions apply to other classes of vector bundles (topological, smooth, and with $\mathbb{C}$ replaced by another field such as $\mathbb{R}$).

Definition 1.5.1. *A* holomorphic vector bundle of rank n *over a complex space X is a holomorphic fiber bundle $E \to X$ (Def. 1.4.1) with fiber $Y = \mathbb{C}^n$ and structure group $GL_n(\mathbb{C})$. A vector bundle of rank $n = 1$ is also called a (holomorphic)* line bundle.

This means that we have an open cover $\mathcal{U} = \{U_\alpha\}$ of X and vector bundle charts $\theta_\alpha \colon E|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times \mathbb{C}^n$ with transition maps of the form

$$\theta_\alpha \circ \theta_\beta^{-1}(x,v) = \theta_{\alpha,\beta}(x,v) = \big(x, g_{\alpha,\beta}(x)\, v\big), \quad x \in U_{\alpha,\beta},\ v \in \mathbb{C}^n, \tag{1.13}$$

where $g_{\alpha,\beta} \colon U_{\alpha,\beta} \to GL_n(\mathbb{C})$ is a holomorphic multiplicative 1-cocycle:

$$g_{\alpha,\alpha} = 1, \quad g_{\alpha,\beta}\, g_{\beta,\alpha} = 1, \quad g_{\alpha,\beta}\, g_{\beta,\gamma}\, g_{\gamma,\alpha} = 1. \tag{1.14}$$

Every fiber E_x is a complex vector space such that the fiber bundle charts $\theta_\alpha \colon Z|_{U_\alpha} \to U_\alpha \times \mathbb{C}^n$ are $\mathbb{C}$-vector space isomorphisms on each fiber.

If X is a real manifold of class $\mathcal{C}^r$ and $E \to X$ is a complex vector bundle whose transition maps $g_{\alpha,\beta} \colon U_{\alpha,\beta} \to GL_n(\mathbb{C})$ are of class $\mathcal{C}^r$, then we have a *complex vector bundle of class* $\mathcal{C}^r$ over X. For $r = 0$ we have a *topological complex vector bundle* over X. Replacing $\mathbb{C}$ by any field F we get topological F-vector bundles over topological spaces, or smooth F-vector bundles over smooth manifolds. For $F = \mathbb{R}$ we speak of *real vector bundles.*

Every vector bundle has the *zero section* sending each point $x \in X$ to the origin $0_x \in E_x$. Given a holomorphic vector bundle atlas $\{(U_\alpha, \theta_\alpha)\}$ on E with transition maps $g_{\alpha,\beta}$ (1.13), a section $f \colon X \to E$ is determined by a collection of maps $f_\alpha \colon U_\alpha \to \mathbb{C}^n$ satisfying the compatibility conditions

$$f_\alpha = g_{\alpha,\beta} f_\beta \quad \text{on } U_{\alpha,\beta}. \tag{1.15}$$

Example 1.5.2. (*The tangent bundle.*) Let X be a $\mathcal{C}^r$ manifold of dimension n. Given an atlas $\{(U_\alpha, \phi_\alpha)\}$ on X with transition maps $\phi_{\alpha,\beta}$, the tangent bundle $\mathrm{T}X \to X$ is a real vector bundle of rank n and of class $\mathcal{C}^{r-1}$ with vector bundle charts $\theta_\alpha \colon \mathrm{T}X|_{U_\alpha} \to U_\alpha \times \mathbb{R}^n$ and the transition cocycle $g_{\alpha,\beta} = (\phi_{\alpha,\beta})' \circ \phi_\beta$, where $(\phi_{\alpha,\beta})'$ is the Jacobian matrix of $\phi_{\alpha,\beta}$. The cotangent bundle T^*X is the dual bundle of $\mathrm{T}X$. For complex manifolds see §1.6 – §1.7. □

Example 1.5.3. (*The universal bundle.*) Let $G_{k,n}$ be the Grassmann manifold whose points are k-dimensional subspace of $\mathbb{C}^n$ (Example 1.2.8), and set

$$U_{k,n} = \{(\lambda, z) \in G_{k,n} \times \mathbb{C}^n \colon z \in \lambda\}.$$

The projection $\pi \colon U_{k,n} \to G_{k,n}$, $\pi(\lambda, z) = \lambda$, admits the structure of a holomorphic vector bundle (a holomorphic vector subbundle of the trivial bundle $G_{k,n} \times \mathbb{C}^n$), called the universal bundle over $G_{k,n}$.

In particular, $U_{1,n+1} \to G_{1,n+1} = \mathbb{P}^n$ is a holomorphic line bundle over the projective space $\mathbb{P}^n$. This bundle is trivial over every coordinate neighborhood $V_j = \{[z_0 \colon \cdots \colon z_n] \colon z_j \neq 0\} \cong \mathbb{C}^n$ in $\mathbb{P}^n$; a local vector bundle chart is given by

$$\theta_j([z_0 \colon \cdots \colon z_n], (v_0, \ldots, v_n)) = ([z_0 \colon \cdots \colon z_n], v_j) \in V_j \times \mathbb{C}.$$

The colinearity condition $v \in [z_0 \colon \cdots \colon z_n]$ defining $U_{1,n+1}$ implies $v_i z_j = v_j z_i$, which shows that the transition maps equal $g_{i,j}\big([z_0 \colon \cdots \colon z_n]\big) = \frac{z_i}{z_j}$. □

Definition 1.5.4. *Let $\pi: E \to X$ and $\pi': E' \to X$ be holomorphic vector bundles. A* holomorphic morphism *of (E, π, X) to (E', π', X) is a holomorphic map $\Phi: E \to E'$ such that $\pi' \circ \Phi = \pi$, $\Phi_x: E_x \to E'_x$ is $\mathbb{C}$-linear for every $x \in X$. If in addition* $\dim\ker \Phi_x$ *is independent of the point $x \in X$ then Φ is called a* holomorphic vector bundle map*. Such Φ is a* isomorphism *if $\Phi_x: E_x \to E'_x$ is an isomorphism of $\mathbb{C}$-vector spaces for every $x \in X$. A $\mathcal{C}^r$* morphism *is a $\mathcal{C}^r$ map $\Phi: E \to E'$ that is $\mathbb{C}$-linear on every fiber.*

The *kernel* and *cokernel* of a morphism $\Phi: E \to E'$ are defined by

$$\ker \Phi = \bigcup_{x \in X} \ker \Phi_x \subset E, \quad \operatorname{im} \Phi = \bigcup_{x \in X} \operatorname{im} \Phi_x \subset E'.$$

Definition 1.5.5. *Let $\pi: E \to X$ be a holomorphic vector bundle of rank n. A* holomorphic vector subbundle *of rank $m \in \{0, 1, \dots, n\}$ of (E, π, X) is a complex submanifold $E' \subset E$, with the restricted projection $\pi' = \pi|_{E'}: E' \to X$ onto X, such that every point $x_0 \in X$ admits an open neighborhood $U \subset X$ and a holomorphic vector bundle chart $\theta: E|_U \xrightarrow{\cong} U \times \mathbb{C}^n$ satisfying the condition*

$$\theta(E'|_U) = U \times (\mathbb{C}^m \times \{0\}^{n-m}). \tag{1.16}$$

Any such chart θ is said to be *adapted* to E'. Denote by

$$pr_1: \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m} \to \mathbb{C}^m, \quad pr_2: \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m} \to \mathbb{C}^{n-m}$$

the projections onto the first and the second factor, respectively. For every θ as above the map $pr_1 \circ \theta: E'|_U \to U \times \mathbb{C}^m$ is a vector bundle chart on E', and the collection of all such charts is a holomorphic vector bundle atlas on E'. In this structure the inclusion map $E' \hookrightarrow E$ is a holomorphic vector bundle morphism. We have the following elementary result.

Proposition 1.5.6. *Let $\Phi: E \to E'$ be a holomorphic morphism of holomorphic vector bundles $E \to X$, $E' \to X$. If* $\dim\ker \Phi_x$ *is independent of the point $x \in X$, then the kernel* $\ker \Phi$ *is a holomorphic vector subbundle of E and the image* $\operatorname{im} \Phi$ *is a holomorphic vector subbundle of E'.*

We give a description of morphisms in local charts, beginning with the simplest case of product bundles. Let $\operatorname{Hom}(\mathbb{C}^n, \mathbb{C}^m)$ denote the set of all $\mathbb{C}$-linear maps $\mathbb{C}^n \to \mathbb{C}^m$. With respect to any pair of complex bases on the two space this equals $M_{m,n}(\mathbb{C}) \cong \mathbb{C}^{mn}$, the set of all complex $m \times n$ matrices. A morphism $\Phi: X \times \mathbb{C}^n \to X \times \mathbb{C}^m$ of product bundles is of the form $(x, v) \to (x, \varphi(x)v)$ for a holomorphic map $\varphi: X \to M_{m,n}(\mathbb{C})$. In particular, an automorphism of $X \times \mathbb{C}^n$ is given by a map $X \to GL_n(\mathbb{C}) \subset M_{n,n}(\mathbb{C})$.

Assume now that $E \to X$ and $E' \to X$ are holomorphic vector bundles of rank n, m, respectively. Choose holomorphic vector bundle atlases $\{(U_\alpha, \theta_\alpha)\}$, $\{(U_\alpha, \theta'_\alpha)\}$ for E, E', with the transition maps $g_{\alpha,\beta}$, $g'_{\alpha,\beta}$, respectively. A morphism $\Phi: E \to E'$ is given by a collection of maps $\varphi_\alpha: U_\alpha \to M_{m,n}(\mathbb{C})$ satisfying the compatibility conditions

$$\varphi_\alpha\, g_{\alpha,\beta} = g'_{\alpha,\beta}\, \varphi_\beta \quad \text{on } U_{\alpha,\beta}.$$

If Φ is an isomorphism, then $\varphi_\alpha: U_\alpha \to GL_n(\mathbb{C})$ and we can write

$$g'_{\alpha,\beta} = \varphi_\alpha\, g_{\alpha,\beta}\, \varphi_\beta^{-1}. \tag{1.17}$$

We say that the 1-cocycle $g' = (g'_{\alpha,\beta})$ is obtained by *twisting the 1-cocycle* $g = (g_{\alpha,\beta})$ *by the* 0*-cochain* $\varphi = (\varphi_\alpha)$, and we write $g' = \varphi \,\square\, g$. This leads to the observation that the isomorphism classes of holomorphic vector bundles of rank n over X are given by elements of the cohomology group $H^1(X; \mathcal{O}^{GL_n(\mathbb{C})})$ with coefficients in the multiplicative sheaf of germs of holomorphic maps $X \to GL_n(\mathbb{C})$. (See §7.1 for a further discussion of this topic.)

The group $H^1(X; \mathcal{O}^{GL_n(\mathbb{C})})$ is Abelian only for $n = 1$ when it equals $H^1(X; \mathcal{O}^*)$. The multiplicative group $H^1(X; \mathcal{O}^*) = \mathrm{Pic}(X)$ of equivalence classes of holomorphic line bundles on X is called the *Picard group* of X. The product on $\mathrm{Pic}(X)$ corresponds to the tensor product of line bundles.

Example 1.5.7. (*Line bundles and divisors.*) A *divisor* D on a complex manifold X is determined by an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of X and a collection of meromorphic functions $f_\alpha \in \mathcal{M}(U_\alpha)$ that are not identically zero on any connected component of U_α such that for any pair of indexes $\alpha, \beta \in A$ there exists a nowhere vanishing holomorphic function $f_{\alpha,\beta} \in \mathcal{O}^*(U_{\alpha,\beta})$ satisfying

$$f_\alpha = f_{\alpha,\beta} f_\beta \quad \text{on } U_{\alpha,\beta}. \tag{1.18}$$

The 1-cocycle $(f_{\alpha,\beta})$ determines a holomorphic line bundle $E = [D]$ over X, and the collection (f_α) is a meromorphic section of $[D]$ in view of (1.18). In particular, a meromorphic function $f \in \mathcal{M}(X)$ that is not identically zero on any connected component determines a trivial line bundle on X. Conversely, if a line bundle E is presented over an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ of X by a 1-cocycle $(f_{\alpha,\beta})$ with coefficients in the sheaf $\mathcal{O}_X^*$, then each meromorphic section f of E is given in the respective holomorphic trivializations of $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$ by a collection of meromorphic functions $f_\alpha \in \mathcal{M}(U_\alpha)$ satisfying (1.18). If D is the divisor determined by (f_α) then clearly $E \cong [D]$.

A complex hypersurface $V \subset X$ determines a divisor D given by a collection of local defining functions for V. Conversely, every divisor D on X can be represented by a locally finite formal combination $D = \sum_i a_i V_i$ of irreducible complex hypersurfaces $V_i \subset X$ with integer coefficients $a_i \in \mathbb{Z}$ [233, p. 130]. The divisors on X form an Abelian group, $\mathrm{Div}(X)$, and by the above discussion we have a natural homomorphism $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. This homomorphism is surjective on any quasi-projective manifold. The line bundle $[D]$ determined by a divisor D is trivial if and only if D is given by a global meromorphic function on X. (In this connection see the discussion in

§5.2 concerning the *second Cousin problem.*) Two divisors D, D' are *linearly equivalent* if $D = D' + (f)$ for some $f \in \mathcal{M}(X)$; thus linear equivalence of a pair of divisors corresponds to holomorphic equivalence of the corresponding line bundles $[D]$ and $[D']$. For a sheaf theoretic interpretation of divisors and linear equivalence see [233]. □

Definition 1.5.8. *Given holomorphic vector bundles* $\pi\colon E \to X$, $\pi'\colon E' \to X'$, *a morphism of the first to the second bundle is a pair of holomorphic maps* $f\colon X \to X'$, $F\colon E \to E'$ *such that* $\pi' \circ F = f \circ \pi$ *and* F *is* $\mathbb{C}$*-linear on fibers.*

An example is the tangent map $F = \mathrm{T}f\colon \mathrm{T}X \to \mathrm{T}X'$ of a holomorphic map $f\colon X \to X'$; in this case $F_x\colon \mathrm{T}_x X \to \mathrm{T}_{f(x)} X'$ is the differential $\mathrm{d}f_x$ of f at x. The analogous definition applies to $\mathcal{C}^r$ vector bundles.

Given a holomorphic vector bundle $\pi\colon E \to X$ and a holomorphic map $f\colon W \to X$, the *pull-back bundle* $f^*E \to W$ (see (1.12) on p. 11) is a holomorphic vector bundle over W. We have a natural morphism $f^*E \to E$ over f which maps each fiber $(f^*E)_x$ isomorphically onto the fiber $E_{f(x)}$.

Assume that $E \to X$ and $F \to X$ are complex (or holomorphic) vector bundles. Using standard functors on complex vector spaces we obtain the following derived complex (resp. holomorphic) vector bundles over X:

(a) $E \oplus F = \cup_{x\in X} E_x \oplus F_x$, the *direct sum* or the *Whitney sum*,
(b) $E \otimes F = \cup_{x\in X} E_x \otimes F_x$, the *tensor product*,
(c) $E^* = \cup_{x\in X} E_x^*$, the *dual bundle* of E,
(d) $\mathrm{Hom}(E,F) = \cup_{x\in X} \mathrm{Hom}(E_x, F_x) = E^* \otimes F$,
(e) $\wedge^k E = \cup_{x\in X} \wedge^k E_x$, the *k-th exterior power* of E,
(f) $S^k(E) = \cup_{x\in X} S^k(E_x)$, the *k-th symmetric power* of E.

The transition maps in these bundles are obtained by applying the respective functor fiberwise to the transition maps of the original bundles. For example, if E and E' are given by cocycles $g_{\alpha,\beta}$, $g'_{\alpha,\beta}$ over the same open cover $\mathcal{U} = \{U_\alpha\}$ of X then the direct sum $E \oplus E'$ is given by the cocycle

$$\begin{pmatrix} g_{\alpha,\beta} & 0 \\ 0 & g'_{\alpha,\beta} \end{pmatrix}.$$

Given a subbundle E' of E, the *quotient bundle* $E/E' \to X$ is defined by $E/E' = \cup_{x\in X} E_x/E'_x$. For any vector bundle chart θ on E satisfying (1.16) the map $pr_2 \circ \theta\colon E|_U \to U \times \mathbb{C}^{n-m}$ factors through $(E/E')|_U$ and induces a bijective map $\widetilde{\theta}\colon (E/E')|_U \to U \times \mathbb{C}^{n-m}$. The collection of all such maps is a complex (resp. holomorphic) vector bundle atlas on E/E'. If $E = E' \oplus E''$ is a direct sum of its subbundles $E', E'' \subset E$ then the projection $\tau\colon E \to E''$ with the kernel $\ker\tau = E'$ induces an isomorphism of E/E' onto E''.

A sequence of vector bundle maps over X,

$$\cdots \longrightarrow E_{k-1} \xrightarrow{\sigma_{k-1}} E_k \xrightarrow{\sigma_k} E_{k+1} \longrightarrow \cdots$$

is a *complex* if $\sigma_k \circ \sigma_{k-1} = 0$ (equivalently, $\mathrm{im}\, \sigma_{k-1} \subset \ker \sigma_k$) for every k. The sequence is *exact* at E_k if $\mathrm{im}\, \sigma_{k-1} = \ker \sigma_k$. A *short exact sequence* is an exact sequence of the form

$$0 \longrightarrow E' \xrightarrow{\sigma} E \xrightarrow{\tau} E'' \longrightarrow 0. \tag{1.19}$$

This means that σ is injective, τ is surjective, and $\mathrm{im}\, \sigma = \ker \tau$. Hence τ induces an isomorphism of the quotient bundle $E/\sigma(E')$ onto E''.

A short exact sequence (1.19) *splits* if there exists a vector bundle homomorphism $\rho\colon E'' \to E$ such that $\tau \circ \rho$ is the identity on E''. Such ρ is called a *splitting map* for the sequence. In this case E is isomorphic to the Whitney sum $E = \sigma(E') \oplus \rho(E'')$ of its subbundles $\sigma(E')$ and $\rho(E'')$. Note that every short exact sequence splits locally over small open subsets of the base, and any convex linear combination of splittings is again a splitting. By patching local splittings with a partition of unity one gets the following.

Proposition 1.5.9. *Every short exact sequence (1.19) of complex vector bundle maps of class $\mathcal{C}^r$ ($r \in \{0, 1, \ldots, \infty\}$) admits a $\mathcal{C}^r$ splitting. In particular, we have $E \cong E' \oplus E''$ as complex vector bundles of class $\mathcal{C}^r$.*

The analogous result for holomorphic vector bundles over Stein spaces follows from Cartan's Theorem B; see Corollary 2.4.5 on p. 54.

1.6 The Tangent Bundle

We assume that the reader is familiar with the construction of the real tangent bundle $\mathrm{T}X$ of a smooth manifold X (Example 1.5.2). A tangent vector $V_x \in \mathrm{T}_x X$ is viewed as a derivation $\mathcal{C}^\infty_x \ni f \mapsto V_x(f) \in \mathbb{R}$ on the algebra of germs of smooth functions at x. Sections $X \to \mathrm{T}X$ are called *vector fields* on X. The complexification $\mathbb{C}\mathrm{T}X = \mathrm{T}X \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathrm{T}X$ is the *complexified tangent bundle* of X; its sections are called *complex vector fields* on X.

Assume now that X is a complex manifold. There is a unique real linear endomorphism $J \in \mathrm{End}_{\mathbb{R}} \mathrm{T}X$, called the *almost complex structure operator*, which is given in any local holomorphic coordinate system $z = (z_1, \ldots, z_n)$ ($z_j = x_j + \mathrm{i} y_j$) on X by

$$J \frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}, \qquad J \frac{\partial}{\partial y_j} = -\frac{\partial}{\partial x_j}. \tag{1.20}$$

The operator J extends to $\mathbb{C}\mathrm{T}X$ by $J(v \otimes \alpha) = J(v) \otimes \alpha$ for $v \in \mathrm{T}X$ and $\alpha \in \mathbb{C}$. From $J^2 = -\mathrm{Id}$ we infer that the eigenvalues of J are $+\mathrm{i}$ and $-\mathrm{i}$. Hence we have a decomposition