Progress in Mathematics
294

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Birkhäuser

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Volume 294

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2010 Mathematical Subject Classification: 37D35, 37D20, 37D25, 37C45

ISBN 978-3-0348-0205-5 e-ISBN 978-3-0348-0206-2
DOI 10.1007/978-3-0348-0206-2

Library of Congress Control Number: 2011936516

Printed on acid-free paper

Springer Basel AG is part of Springer Science+Business Media

www.birkhauser-science.com

To Claudia

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Preface

This monograph gives a unified exposition of the *thermodynamic formalism* and some of its main extensions, with emphasis on their relation to *dimension theory* and *multifractal analysis* of dynamical systems. Not only are these natural playgrounds for nontrivial applications of the thermodynamic formalism, but are also major sources of inspiration for further developments of the theory.

In particular, we present the main results and main techniques in the interplay between the thermodynamic formalism, symbolic dynamics, dimension theory, and multifractal analysis. We also discuss selected topics of current research interest that until now were scattered in the literature (incidentally, more than two thirds of the material appears here for the first time in book form). This includes the discussion of some of the most significant recent results in the area as well as some of its open problems, in particular concerning dimension estimates for repellers and hyperbolic sets, dimension estimates or even formulas for the dimension of limit sets of geometric constructions, and the multifractal analysis of entropy and dimension spectra, in particular associated to nonconformal repellers. Undoubtedly, this selection, although quite conscious, also reflects a personal taste.

The dimension theory and the multifractal analysis of dynamical systems have progressively developed into an independent field of research during the last three decades. Nevertheless, despite a large number of interesting and nontrivial developments, only the case of *conformal* dynamics is completely understood. In the case of repellers this corresponds to assuming that the derivative of the map is a multiple of an isometry at every point. This property allowed Bowen in 1979 (in the particular case of quasi-circles) and then Ruelle in 1982 (in full generality) to develop a fairly complete theory for the dimension of repellers of conformal maps. Their work is strongly based on the thermodynamic formalism, earlier developed by Ruelle in 1973 for expansive transformations, and then by Walters in 1976 in full generality.

On the other hand, the study of the dimension of invariant sets of *nonconformal* maps unveiled several new phenomena, but it still lacks today a satisfactory general approach. In particular, we are often only able to establish dimension estimates instead of giving formulas for the dimension of the invariant sets. Thus, somemes the emphasis is on how to obtain sharp dimension estimates, starting essentially with the seminal work of Douady and Oesterlé in 1980, who devised an approach to cover an invariant set in a more optimal manner. Furthermore, it was early recognized, notably by Pesin and Pitskel' in 1984 (with the notion of topological pressure for noncompact sets) and by Falconer in 1988 (with his subadditive version of the thermodynamic formalism), that it would also be desirable to have an appropriate extension of the thermodynamic formalism in order to consider more general classes of invariant sets, and in particular invariant sets of nonconformal transformations. Most certainly, this is not foreign to the fact that virtually all known equations used to compute or estimate dimensions are appropriate versions of an equation introduced by Bowen in his study of quasi-circles that involves topological pressure, which is the most basic notion of the thermodynamic formalism.

The exposition is organized in four parts. The first part gives an introduction to the classical thermodynamic formalism and its relations to symbolic dynamics. Although everything is proven, we develop the theory in a pragmatic manner, only as much as needed for the following parts. The remaining three parts consider three different versions of the thermodynamic formalism, namely nonadditive, subadditive, and almost additive. In each of these parts we detail generously not only the most significant results in the area, some of them quite recent, but also some of the most substantial applications of the corresponding thermodynamic formalism to dimension theory and multifractal analysis of dynamical systems.

The nonadditive thermodynamic formalism, which is a considerable extension of the classical thermodynamic formalism, provides the most general setting and has a unifying role. The subadditive and the almost additive formalisms successively consider more special situations. As always in mathematics, when one makes further hypotheses, one can often establish additional results. Thus, it is not surprising that the nonadditive, subadditive, and almost additive thermodynamic formalisms are progressively richer. On the other hand, and this is a major motivation for such developments, the new hypotheses are still sufficiently general to allow a large number of nontrivial applications. This includes dimension estimates for nonconformal repellers, nonconformal hyperbolic sets, and limit sets of geometric constructions, as well as a multifractal analysis of entropy and dimension spectra of a large class of nonconformal repellers.

The book is directed to researchers as well as graduate students who wish to have a global view of the main results and main techniques in the area. It can also be used for graduate courses on the thermodynamic formalism and its extensions, with the optional discussion of some applications to dimension theory and multifractal analysis, or for graduate courses on special topics of dimension theory and multifractal analysis, with the discussion of the strictly necessary material from the thermodynamic formalism. We emphasize that with the exception of a few sections of survey type, the text is self-contained and all the results are included with detailed proofs. In particular, it can also be used for independent study.

There are no words that can adequately express my gratitude to Claudia Valls for her help, patience, encouragement, and inspiration during the preparation of this book. I acknowledge the support by FCT through the Center for Mathematical Analysis, Geometry, and Dynamical Systems of Instituto Superior Técnico.

Luis Barreira
Lisbon, May 2011

Chapter 1

Introduction

This book is dedicated to the thermodynamic formalism, its extensions, and its applications, with emphasis on the study of the relation to dimension theory and multifractal analysis of dynamical systems. We describe briefly in this chapter the historical origins and the principal elements of the research areas considered in the book. We also describe its contents. Finally, we recall in a pragmatic manner all the notions and results from dimension theory and ergodic theory that are needed later on.

1.1 Thermodynamic formalism and dimension theory

We describe in this section the historical origins of the thermodynamic formalism as well as of dimension theory and multifractal analysis of dynamical systems. In particular, we illustrate the rich interplay between these areas.

1.1.1 Classical thermodynamic formalism

The (mathematical) thermodynamic formalism has its roots in thermodynamics. For example, quoting from Gallavotti's foreword to Ruelle's book [166]:

> "Thermodynamics is still, as it always was, at the center of physics, the standard-bearer of successful science. As happens with many a theory, rich in applications, its foundations have been murky from the start and have provided a traditional challenge on which physicists and mathematicians alike have tested their latest skills."

Essentially, the thermodynamic formalism (following Ruelle's original expression) can be described as a rigorous study of certain mathematical structures inspired in thermodynamics. To differentiate it from the various extensions that are described in the book, we shall call it *classical thermodynamic formalism*.

L. Barreira, *Thermodynamic Formalism and Applications to Dimension Theory*,
Progress in Mathematics 294, DOI 10.1007/978-3-0348-0206-2_1, © Springer Basel AG 2011

The notion of topological pressure, which is the most basic notion of the thermodynamic formalism, was introduced by Ruelle [164] for expansive transformations and by Walters [194] in the general case. For a continuous transformation $f\colon X \to X$ of a compact metric space, the *topological pressure* of a continuous function $\varphi\colon X \to \mathbb{R}$ (with respect to f) is defined by

$$P(\varphi) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log \sup_E \sum_{x\in E} \exp \sum_{k=0}^{n-1} \varphi(f^k(x)),$$

where the supremum is taken over all (n,ε)-separated sets $E \subset X$ (see Section 2.1 for details). For example, taking $\varphi = 0$ we recover the notion of topological entropy.

The theory has progressively developed into a broad independent field, with many promising directions of research. In particular, the variational principle relating topological pressure to Kolmogorov–Sinai entropy was established by Ruelle [164] for expansive transformations and by Walters [194] in the general case. It says that

$$P(\varphi) = \sup_\mu \left(h_\mu(f) + \int_X \varphi \, d\mu \right), \tag{1.1}$$

where the supremum is taken over all f-invariant probability measures μ in X, and where $h_\mu(f)$ is the entropy with respect to the measure μ. The theory also includes a discussion of the existence and uniqueness of equilibrium and Gibbs measures, with the latter having a privileged relation to the Gibbs distributions of thermodynamics. For further developments of the thermodynamic formalism as well as a detailed discussion of its historical origins, we refer to the books [39, 108, 109, 149, 166, 195]. These developments also include directions of research that apparently are unrelated to the original motivation stemming from thermodynamics. We emphasize that it is entirely out of the scope of this book to provide any comprehensive exposition of the theory.

1.1.2 Dimension theory and multifractal analysis

We emphasize that in this book we are mainly concerned with the relation of the thermodynamic formalism and its extensions to the dimension theory of dynamical systems, which includes in particular the subfield of multifractal analysis. In other words, we do not consider topics of dimension theory that are not of a dynamical nature, of course independently of their importance. Roughly speaking, the main objective of the dimension theory of dynamical systems is to measure the complexity, from the dimensional point of view, of objects that remain invariant under the dynamics, such as invariant sets and measures. The first monograph that clearly took this point of view was Pesin's book [152], which describes the state-of-the-art up to 1997. We refer to the book [7] for a detailed description of many of the more recent results in the area.

The existence of a privileged relation between the thermodynamic formalism and the dimension theory of dynamical systems is due to the following fact. The unique solution s of the equation

$$P(s\varphi) = 0, \tag{1.2}$$

where φ is a certain function associated to a given invariant set, is often related to the Hausdorff dimension of the set. This equation was introduced by Bowen in [40] (in his study of quasi-circles) and is usually called Bowen's equation. It is also appropriate to call it the Bowen–Ruelle equation, taking not only into account the fundamental role of the thermodynamic formalism developed by Ruelle, but also his article [167] with a study of the Hausdorff dimension of the repellers of a conformal dynamics (this corresponds to assuming that the derivative of the map is a multiple of an isometry at every point). To a certain extent, the study of the dimension of hyperbolic sets is analogous. Indeed, assuming that the derivatives of the map along the stable and unstable directions are multiples of isometries, starting with the work of McCluskey and Manning in [133] it was possible to develop a sufficiently complete corresponding theory. However, there are nontrivial differences between the theory for repellers and the theory for hyperbolic sets. For example, each conformal repeller has a unique invariant measure of full dimension. On the other hand, unless some cohomology relations hold, there are no invariant measures of full dimension concentrated on a given conformal hyperbolic set.

Let us emphasize that virtually all known equations used to compute or to estimate the dimension of an invariant set, either of an invertible or a noninvertible dynamics, are particular cases of equation (1.2) or of an appropriate generalization. Nevertheless, despite these and many other significant developments, only the case of *conformal* dynamics is completely understood. In particular, many of the developments towards a nonconformal theory depend on each particular class of dynamics. On the other hand, this drawback of the theory is also a principal motivation for further developments and in particular for the extensions of the thermodynamic formalism that are presented in the book.

Now we turn to the theory of multifractal analysis. This is a subfield of the dimension theory of dynamical systems. Briefly, multifractal analysis studies the complexity of the level sets of any invariant local quantity obtained from a dynamical system. For example, one can consider Birkhoff averages, Lyapunov exponents, pointwise dimensions, or local entropies. These functions are usually only measurable and thus their level sets are rarely manifolds. Hence, in order to measure their complexity it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension. The concept of multifractal analysis was suggested by Halsey, Jensen, Kadanoff, Procaccia and Shraiman in [84]. The first rigorous approach is due to Collet, Lebowitz and Porzio in [45] for a class of measures that are invariant under one-dimensional Markov maps. In [122], Lopes considered the measure of maximal entropy for hyperbolic Julia sets, and in [162], Rand studied Gibbs measures for a class of repellers. We refer the reader to the books [7, 152] for further references and for detailed expositions of parts of the theory. We also note that Morán [138] proposed a quite interesting approach to multifractal analysis in terms of multifractal decompositions obtained from multiplicative set functions.

1.1.3 Attractors in infinite-dimensional spaces

We discuss briefly in this section some motivations for the study of dimension in the context of the theory dynamical systems, mostly in connection with the theory of attractors in infinite-dimensional spaces.

The longtime behavior of many dynamical systems, such as those coming from delay differential equations and partial differential equations, can essentially be described in terms of a global attractor (see [3, 83, 189]). An important question, particularly in the context of infinite-dimensional systems, is how many degrees of freedom are necessary to specify the dynamics on the attractor. It turns out that a large class of attractors have finite Hausdorff dimension and even finite box dimension. Hence, the dynamics on the attractor is essentially finite-dimensional (see [3, 83, 189] for related discussions). In particular, Mañé [127] obtained the following result.

Theorem 1.1.1. *Let $f\colon E \to E$ be a C^1 map of a Banach space such that for each $x \in E$ the derivative $d_x f$ is the sum of a compact map and a contraction. Then every compact f-invariant set in E has finite upper box dimension.*

An analogous statement for the Hausdorff dimension was obtained earlier by Mallet-Paret [126] in the particular case of Hilbert spaces.

Moreover, particularly in the experimental study of attractors one often considers their projection into an Euclidean space. It is also possible to give conditions for the invertibility of the projection. In particular, the following result is also due to Mañé [127].

Theorem 1.1.2. *Let E be a Banach space and let $F \subset E$ be a p-dimensional subspace with $p < \infty$. For a residual set of the space of all continuous projections of E onto F (with respect to the topology induced by the operator norm), each projection is injective on a compact set $\Lambda \subset E$ provided that the product $\Lambda \times \Lambda$ has Hausdorff dimension less then $p - 1$.*

For an arbitrary projection of a compact subset of a Banach space, Hunt and Kaloshin [95] showed that typically (in the sense of prevalence in [96]) the projection is injective and has Hölder continuous inverse. Earlier results on the Hölder continuity of the inverse are due to Ben-Artzi, Eden, Foias and Nikolaenko [29] in $\mathbb{R}^n$ and to Foias and Olson [70] in Hilbert spaces. These results estimate how much the dimension of the set can decrease under the projection.

1.2 Extensions and applications

We describe in this section the main elements of the three extensions of the classical thermodynamic formalism that are discussed in the book, namely the nonadditive, the subadditive, and the almost additive thermodynamic formalisms. We also describe briefly some of the nontrivial applications of each extension, in particular to the dimension of repellers and hyperbolic sets, the dimension of limit sets of geometric constructions, and the multifractal analysis of entropy and dimension spectra.

1.2.1 Nonadditive formalism and dimension estimates

The nonadditive thermodynamic formalism was introduced by Barreira in [5]. It is a generalization of the classical thermodynamic formalism, in which the topological pressure $P(\varphi)$ of a continuous function φ (with respect to a given dynamics on a compact metric space) is replaced by the topological pressure $P(\Phi)$ of a sequence of continuous functions $\Phi = (\varphi_n)_{n\in\mathbb{N}}$. The nonadditive thermodynamic formalism contains as a particular case a new formulation of the subadditive thermodynamic formalism earlier introduced by Falconer in [56]. For additive sequences and arbitrary sets, it recovers the notion of topological pressure introduced by Pesin and Pitskel' in [153], and the notions of lower and upper capacity topological pressures introduced by Pesin in [151]. It also gives an equivalent description of the notion of topological pressure for compact sets introduced by Ruelle in [164] in the case of expansive maps, and by Walters in [194] in the general case.

Among the main motivations for the nonadditive thermodynamic formalism are certain applications to a much more general class of invariant sets in the context of the dimension theory of dynamical systems. Indeed, while the study of the dimension of invariant sets of *nonconformal* maps unveiled several new phenomena, it still lacks today a satisfactory general approach, both for repellers and for hyperbolic sets. In particular, most authors make additional assumptions that essentially avoid two main types of difficulties. The first difficulty is the lack of a clear separation between different Lyapunov directions, together with a possible small regularity of the associated distributions (or the associated holonomies). Typically, these distributions are only Hölder continuous, which causes that in general it is impossible to add the dimensions along various distributions. This strongly contrasts to what happens for hyperbolic sets of a conformal dynamics, in which case the stable and unstable holonomies are Lipschitz. The second difficulty is the existence of number-theoretical properties that may cause a variation of the Hausdorff dimension with respect to a certain typical value (such as that obtained by Falconer in [55]; see Theorem 5.3.5). Other authors have obtained results not for a specific invariant set, but instead for almost all invariant sets in a given parameterized family. Unfortunately, sometimes it is quite difficult to determine what happens for each specific value of the parameter, if at all possible.

These difficulties cause that in the general case of nonconformal maps, at the present stage of the theory we are often only able to establish dimension estimates instead of giving formulas for the dimension of an invariant set. Thus, sometimes the emphasis is on how to obtain sharp lower and upper dimension estimates. There are however some notable exceptions. In particular, we have included in the book a description of all the preeminent results concerning lower and upper dimension estimates, both for repellers and for hyperbolic sets. The nonadditive thermodynamic formalism plays not only a unifying role but also allows one to consider much more general classes of invariant sets. This includes repellers and hyperbolic sets for maps that are not differentiable.

When more complete geometric information is available, one can often obtain sharper estimates for the dimension or even compute its value. On the other hand, this often requires a more elaborate approach, starting essentially with the seminal work of Douady and Oesterlé in [49], who devised an approach to cover the invariant set in a more optimal manner. Incidentally, sharp lower dimension estimates are in general more difficult to obtain than sharp upper dimension estimates. Moreover, in some cases these estimates are either unknown or are only known to occur for almost *all* parameters in some specific classes of invariant sets of nonconformal maps. For completeness, we also give in the book a sufficiently broad panorama of the existing results concerning dimension estimates for repellers of smooth dynamical systems, with emphasis on the relation to the thermodynamic formalism. Among other topics, we consider self-affine repellers, their nonlinear generalizations, and repellers of nonuniformly expanding maps. In particular, Falconer [55, 58] studied a class of limit sets obtained from the composition of affine transformations that are not necessarily conformal.

1.2.2 Subadditive formalism and entropy spectra

We consider in this section the subadditive version of the thermodynamic formalism. We recall that a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *subadditive* if there is a constant $C > 0$ such that

$$\varphi_{n+m} \le C + \varphi_n + \varphi_m \circ f^n$$

for every $n, m \in \mathbb{N}$. Among the main motivations for the subadditive thermodynamic formalism is the lack of a nonadditive theory of equilibrium measures.

The nonadditive thermodynamic formalism also includes a variational principle for the topological pressure but with a restrictive assumption on the sequence Φ. Namely, consider a sequence of continuous functions $\varphi_n \colon X \to \mathbb{R}$, and assume that there is a continuous function $\varphi \colon X \to \mathbb{R}$ such that

$$\varphi_{n+1} - \varphi_n \circ f \to \varphi \quad \text{uniformly when} \quad n \to \infty. \tag{1.3}$$

Then the nonadditive topological pressure of the sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ with respect to the map f satisfies the variational principle

$$P(\Phi) = \sup_{\mu} \left(h_{\mu}(f) + \int_X \varphi \, d\mu \right), \tag{1.4}$$

where the supremum is taken over all f-invariant probability measures μ in X. We notice that the classical variational principle for the topological pressure in (1.1) is a particular case of the variational principle in (1.4). Nevertheless, condition (1.3) is a strong requirement, certainly also caused by considering arbitrary sequences.

On the other hand, it is well-known that equilibrium and Gibbs measures play a prominent role in the dimension theory and in the multifractal analysis of dynamical systems. These often provide natural measures sitting on the corresponding invariant sets, that at the same time carry some "dynamical" information (we note that both equilibrium and Gibbs measures depend on the dynamics). For example, they can be measures of full dimension or measures of full entropy. It is sometimes possible to develop the dimension theory or the multifractal analysis of a given dynamics without a variational principle for the topological pressure, and thus without the possibility of looking for equilibrium and Gibbs measures, but the corresponding proofs tend to be much more technical. Moreover, the theory tends to be less rich, although it may be applicable to more general classes of maps and potentials. Overall, it would be desirable to continue using equilibrium and Gibbs measures even when the classical thermodynamic formalism cannot be used.

This justifies the interest in looking for a more general class of sequences of functions for which it is still possible to establish a variational principle, without further hypotheses, and to develop a corresponding theory of equilibrium measures. Somewhat recently, it was shown by Feng and Huang [66] that a natural class is that of subadditive sequences. In fact, they considered the more general class of asymptotically subadditive sequences (see Definition 7.1.1), and established the variational principle

$$P(\Phi) = \sup_{\mu} \left(h_{\mu}(f) + \lim_{n\to\infty} \int_X \frac{\varphi_n}{n} \, d\mu \right), \tag{1.5}$$

where the supremum is taken over all f-invariant probability measures μ in X. Identity (1.5) was obtained earlier by Cao, Feng and Huang [42] in the particular case of subadditive sequences, and its generalization to arbitrary asymptotically subadditive sequences follows from a minor modification of their proof. Incidentally, one can show that any sequence satisfying (1.3) is asymptotically subadditive.

Feng and Huang also established the existence of equilibrium measures for continuous transformations with upper semicontinuous entropy, without further hypotheses on the asymptotically subadditive sequence. These are measures μ at which the supremum in (1.5) is attained, that is, they satisfy

$$P(\Phi) = h_{\mu}(f) + \lim_{n\to\infty} \int_X \frac{\varphi_n}{n} \, d\mu.$$

As an application of these results, one can obtain a detailed multifractal analysis of the entropy spectra of generalized Birkhoff averages of an asymptotically subadditive sequence. More precisely, for an asymptotically subadditive sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ we consider the level sets $E(\alpha)$ composed of the points x such that $\varphi_n(x)/n \to \alpha$ when $n \to \infty$. The associated entropy spectrum $\mathcal{E}$ is obtained from computing the topological entropy of the level sets $E(\alpha)$ as a function of α, and its multifractal analysis corresponds to describe the properties of the function $\mathcal{E}$ in terms of the thermodynamic formalism.

In another direction, again taking advantage of the subadditive thermodynamic formalism, one can also give a detailed description of the dimension of a large class of limits sets of geometric constructions, with more explicit formulas when the associated sequences are subadditive. Roughly speaking, a geometric construction corresponds to the geometric structure provided by the rectangles of any Markov partition of a repeller, although now not necessarily determined by an underlying dynamics. More precisely, geometric constructions are defined in terms of certain decreasing sequences of compact sets, such as the intervals of decreasing size in the construction of the middle-third Cantor set. Moreover, even when one can define naturally an induced map for which the limit set of the geometric construction is an invariant set, this map need not be expanding.

1.2.3 Almost additive formalism and Gibbs measures

The almost additive thermodynamic formalism considers a more specific class of sequences, for which it is possible to construct not only equilibrium measures but also Gibbs measures. We recall that a sequence $\Phi = (\varphi_n)_{n\in\mathbb{N}}$ is said to be *almost additive* if there is a constant $C > 0$ such that

$$-C + \varphi_n + \varphi_m \circ f^n \le \varphi_{n+m} \le C + \varphi_n + \varphi_m \circ f^n$$

for every $n, m \in \mathbb{N}$. Clearly, any additive sequence $\varphi_n = \sum_{k=0}^{n-1} \varphi \circ f^k$ is almost additive, but there is a large class of nontrivial examples, in particular related to the study of Lyapunov exponents of nonconformal transformations (see Chapter 11 for details).

More precisely, the main objective of the almost additive thermodynamic formalism developed by Barreira in [6], building on earlier work with Gelfert in [10], is not only to establish a variational principle, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures. The notion of Gibbs measure mimics the corresponding notion in the classical thermodynamic formalism. Among other results, the formalism establishes the uniqueness of equilibrium measures for an almost additive sequence Φ with bounded variation as well as some regularity properties of the topological pressure. In addition, the unique equilibrium and Gibbs measures for a given almost additive sequence coincide and are mixing.

The almost additive thermodynamic formalism allows one to develop a new approach to the multifractal analysis of entropy spectra obtained from the level sets of the Lyapunov exponents, for a class of *nonconformal* repellers. The relation can be described as follows. The Lyapunov exponents are naturally associated to the limits of subadditive sequences, obtained from the norms of some products of matrices. Nevertheless, for the class of nonconformal repellers satisfying a cone condition these sequences of functions are almost additive. In particular, this includes repellers with a strongly unstable foliation and repellers with a dominated splitting (see Chapter 11). We are thus able to apply the almost additive thermodynamic formalism to effect a complete multifractal analysis of the entropy spectra. A priori one could also use the subadditive thermodynamic formalism, but we need Gibbs measures and these are only provided by the almost additive thermodynamic formalism.

Further applications of the almost additive thermodynamic formalism include a conditional variational principle for the spectra of almost additive sequences, and a complete description of the dimension spectra of the generalized Birkhoff averages of an almost additive sequence in a conformal hyperbolic set (we refer to Chapter 12 for details and references). We emphasize that we consider simultaneously averages into the future and into the past. More precisely, the dimension spectra are obtained by computing the Hausdorff dimension of the level sets of the generalized Birkhoff averages both for positive and negative time.

1.3 Contents of the book

In this section we describe systematically the contents of the book. The exposition is divided into four parts:

1. classical thermodynamic formalism;
2. nonadditive thermodynamic formalism, with applications to the dimension of repellers and hyperbolic sets;
3. subadditive thermodynamic formalism, with applications to the dimension of limits sets and the multifractal analysis of entropy spectra;
4. almost additive thermodynamic formalism, with applications to the spectra of Lyapunov exponents and the multifractal analysis of dimension spectra.

The first part is of introductory nature and gives a pragmatic introduction to the classical thermodynamic formalism and its relations to symbolic dynamics. Although everything is proven, we develop the theory only as much as needed for the following chapters. Certainly, a large part of the material is available in other sources, but mostly mixed with other topics. In Chapter 2, we introduce the notion of topological pressure, and after establishing its variational principle, we show that there exist equilibrium measures for any expansive transformation. We also present the characterization of the topological pressure as a Carathéodory dimension, which will be very useful later on. Chapter 3 considers the particular case of symbolic dynamics, which plays an important role in many applications of dynamical systems. After presenting a more explicit formula for the topological pressure with respect to the shift map, we construct equilibrium and Gibbs measures avoiding on purpose Perron–Frobenius operators, and using instead a more elementary approach that is sufficient and in fact convenient for our purposes.

In each of the remaining three parts, we discuss the foundations, main results, and main techniques in the interplay between the particular thermodynamic formalism under consideration (either nonadditive, subadditive, or almost additive), and the dimension theory of dynamical systems. Namely, after an initial chapter in which the core of each thermodynamic formalism is presented in detail, we describe several nontrivial applications of that formalism. The following is a systematic description of each part.

In Part II, we discuss the nonadditive thermodynamic formalism and its applications to the dimension theory of repellers and hyperbolic sets. In Chapter 4, after introducing the notion of nonadditive topological pressure as a Carathéodory dimension, we establish some of its basic properties. We also present nonadditive versions of the variational principle for the topological pressure and of Bowen's equation. As an application, Chapter 5 considers the dimension of repellers, which are invariant sets of a hyperbolic noninvertible dynamics. After describing how Markov partitions can be used to model repellers, we present several applications of the nonadditive thermodynamic formalism to the study of their dimension. This includes lower and upper dimension estimates for a large class of repellers, in particular for maps that need not be differentiable. Chapter 6 is dedicated to the dimension of hyperbolic sets, which are invariant sets of a hyperbolic invertible dynamics. The main aim is to develop to a large extent a corresponding theory to that for repellers in the former chapter.

Part III is dedicated to the subadditive thermodynamic formalism and its applications both to dimension theory and multifractal analysis. We consider in Chapter 7 the particular class of asymptotically subadditive sequences, and we develop the theory in several directions. In particular, we present a variational principle for the topological pressure of an arbitrary asymptotically subadditive sequence, and we establish the existence of equilibrium measures for maps with upper semicontinuous entropy. Chapter 8 is dedicated to the study of limits sets of geometric constructions, from the point of view of the dimension theory of dynamical systems. Our main aim is to describe how the theory for repellers developed in Chapter 5 can be extended to this more general setting, with emphasis on the case when the associated sequences are subadditive. In Chapter 9, for the class of asymptotically subadditive sequences, we describe a multifractal analysis of the entropy spectra of the corresponding generalized Birkhoff averages. We consider the general cases when the Kolmogorov–Sinai entropy is not upper semicontinuous and when the topological pressure is not differentiable. We also consider multidimensional sequences, that is, vectors of asymptotically subadditive sequences.

In Part IV, we discuss the almost additive thermodynamic formalism and its application to multifractal analysis. We consider in Chapter 10 the class of almost additive sequences and we develop to a larger extent the nonadditive thermodynamic formalism in this setting. This includes a discussion of the existence and uniqueness of equilibrium and Gibbs measures, both for repellers and for hyperbolic sets. In order to avoid unnecessary technicalities, we first develop the theory for repellers. We then explain how the proofs of the corresponding results for hyperbolic sets and more generally for continuous maps with upper semicontinuous entropy can be obtained from the proofs for repellers. We also describe some regularity properties of the topological pressure for continuous maps with upper semicontinuous entropy. Chapter 11 considers a class of nonconformal repellers to which one can apply the almost additive thermodynamic formalism developed in the former chapter. Namely, we consider the class of repellers satisfying a cone condition, which includes for example repellers with a strongly unstable foliation and repellers with a dominated splitting. In particular, we describe a multifractal analysis of the entropy spectrum of the Lyapunov exponents of a nonconformal repeller. Further applications to multifractal analysis are described in Chapter 12. In particular, we establish a conditional variational principle for the spectra of an almost additive sequence and we give a complete description of the dimension spectra of the corresponding generalized Birkhoff averages in a conformal hyperbolic set, considering simultaneously averages into the future and into the past. We also consider the general case of multidimensional sequences, that is, vectors of almost additive sequences.

1.4 Basic notions

This section collects in a pragmatic manner all the notions and results from dimension theory and ergodic theory that are needed in the book.

1.4.1 Dimension theory

We introduce in this section the notions of Hausdorff dimension and of lower and upper box dimensions, both for sets and measures. We also introduce the notions of lower and upper pointwise dimensions. We refer to the books [7, 60, 152] for details.

The *diameter* of a set $U \subset \mathbb{R}^m$ is defined by

$$\operatorname{diam} U = \sup\{d(x,y) : x, y \in U\},$$

where d is the distance in $\mathbb{R}^m$, and the *diameter* of a collection $\mathcal{U}$ of subsets of $\mathbb{R}^m$ is defined by

$$\operatorname{diam} \mathcal{U} = \sup\{\operatorname{diam} U : U \in \mathcal{U}\}.$$

Given $Z \subset \mathbb{R}^m$ and $\alpha \in \mathbb{R}$, we define the *α-dimensional Hausdorff measure* of Z by

$$m_H(Z,\alpha) = \lim_{\varepsilon \to 0} \inf_{\mathcal{U}} \sum_{U \in \mathcal{U}} (\operatorname{diam} U)^\alpha, \tag{1.6}$$

where the infimum is taken over all finite or countable covers $\mathcal{U}$ of the set Z with diameter $\operatorname{diam} \mathcal{U} \le \varepsilon$.

Definition 1.4.1. The *Hausdorff dimension* of $Z \subset \mathbb{R}^m$ is defined by

$$\dim_H Z = \inf\big\{\alpha \in \mathbb{R} : m_H(Z,\alpha) = 0\big\}.$$

The *lower* and *upper box dimensions* of $Z \subset \mathbb{R}^m$ are defined respectively by

$$\underline{\dim}_B Z = \liminf_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{-\log\varepsilon} \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{-\log\varepsilon},$$

where $N(Z,\varepsilon)$ denotes the least number of balls of radius ε that are needed to cover the set Z.

One can show that

$$\dim_H Z \le \underline{\dim}_B Z \le \overline{\dim}_B Z. \tag{1.7}$$

Now we introduce corresponding notions for measures. Let μ be a finite measure in $X \subset \mathbb{R}^m$.

Definition 1.4.2. The *Hausdorff dimension* and the *lower* and *upper box dimensions* of μ are defined respectively by

$$\begin{aligned}
\dim_H \mu &= \inf\{\dim_H Z : \mu(X \setminus Z) = 0\},\\
\underline{\dim}_B \mu &= \lim_{\delta\to 0} \inf\{\underline{\dim}_B Z : \mu(Z) \ge \mu(X) - \delta\},\\
\overline{\dim}_B \mu &= \lim_{\delta\to 0} \inf\{\overline{\dim}_B Z : \mu(Z) \ge \mu(X) - \delta\}.
\end{aligned}$$

One can show that

$$\dim_H \mu = \lim_{\delta\to 0} \inf\{\dim_H Z : \mu(Z) \ge \mu(X) - \delta\},$$

and thus, it follows from (1.7) that

$$\dim_H \mu \le \underline{\dim}_B \mu \le \overline{\dim}_B \mu.$$

We also introduce the notions of lower and upper pointwise dimensions.

Definition 1.4.3. The *lower* and *upper pointwise dimensions* of the measure μ at the point $x \in X$ are defined by

$$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

The following result relates the Hausdorff dimension with the lower pointwise dimension.

Theorem 1.4.4. *The following properties hold:*

1. *if* $\underline{d}_\mu(x) \ge \alpha$ *for* μ*-almost every* $x \in X$*, then* $\dim_H \mu \ge \alpha$*;*

2. *if* $\underline{d}_\mu(x) \le \alpha$ *for every* $x \in Z \subset X$, *then* $\dim_H Z \le \alpha$;
3. *we have*

$$\dim_H \mu = \operatorname{ess\,sup}\{\underline{d}_\mu(x) : x \in X\}.$$

We also recall a criterion established by Young in [199] for the coincidence between the Hausdorff and box dimensions of a measure.

Theorem 1.4.5. *If* μ *is a finite measure in* $X \subset \mathbb{R}^m$ *and there exists* $d \ge 0$ *such that*

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = d$$

for μ*-almost every* $x \in X$, *then*

$$\dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu = d.$$

For any finite measure μ invariant under a $C^{1+\varepsilon}$ diffeomorphism with nonzero Lyapunov exponents almost everywhere, it was shown by Barreira, Pesin and Schmeling in [17] that $\underline{d}_\mu(x) = \overline{d}_\mu(x)$ for μ-almost every x.

1.4.2 Ergodic theory

We recall in this section a few basic notions and results from ergodic theory, including Birkhoff's ergodic theorem, the notion of Kolmogorov–Sinai entropy, and the Shannon–McMillan–Breiman theorem. We refer to the books [108, 128, 195] for details.

We first introduce the notion of invariant measure. Let X be a space with a σ-algebra.

Definition 1.4.6. Given a measurable transformation $f\colon X \to X$, a measure μ in X is said to be f*-invariant* if

$$\mu(f^{-1}A) = \mu(A)$$

for every measurable set $A \subset X$.

The study of the transformations with an invariant measure is the main theme of ergodic theory. We denote by $\mathcal{M}_f$ the set of all f-invariant probability measures in X. A measure $\mu \in \mathcal{M}_f$ is said to be *ergodic* if for any f-invariant measurable set $A \subset X$ (this means that $f^{-1}A = A$) either $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

The following is a basic result from ergodic theory. We denote by $L^1(X,\mu)$ the space of all measurable functions $\varphi\colon X \to \mathbb{R}$ with $\int_X |\varphi|\, d\mu < \infty$.

Theorem 1.4.7 (Birkhoff's ergodic theorem [30]). *Let* $f\colon X \to X$ *be a measurable transformation. For each* $\mu \in \mathcal{M}_f$ *and* $\varphi \in L^1(X,\mu)$ *the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x))$$

exists for μ-almost every $x \in X$. If in addition μ is ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int_X \varphi \, d\mu$$

for μ-almost every $x \in X$.

More generally, we have the following result.

Theorem 1.4.8 (see [128]). *Let $f \colon X \to X$ be a measurable transformation and let $\mu \in \mathcal{M}_f$. For each sequence $(\varphi_n)_{n\in\mathbb{N}} \subset L^1(X,\mu)$ converging μ-almost everywhere and in $L^1(X,\mu)$ to a function $\varphi \in L^1(X,\mu)$, the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi_{n-k} \circ f^k$$

exists μ-almost everywhere and in $L^1(X,\mu)$.

Now we recall the notion of entropy. Given $\mu \in \mathcal{M}_f$, let ξ be a *measurable partition* of X, that is, a finite or countable family of measurable subsets of X such that:

1. $\mu(\bigcup_{C\in\xi} C) = 1$;
2. $\mu(C \cap D) = 0$ for every $C, D \in \xi$ with $C \neq D$.

The *entropy* of the partition ξ with respect to μ is defined by

$$H_\mu(\xi) = -\sum_{C\in\xi} \mu(C) \log \mu(C),$$

with the convention that $0 \log 0 = 0$. One can show that

$$H_\mu(\xi) \leq \log \operatorname{card} \xi. \tag{1.8}$$

Definition 1.4.9. The *Kolmogorov–Sinai entropy* or *metric entropy* of f with respect to a measure $\mu \in \mathcal{M}_f$ is defined by

$$h_\mu(f) = \sup \bigl\{ h_\mu(f,\xi) : H_\mu(\xi) < \infty \bigr\}, \tag{1.9}$$

where

$$h_\mu(f,\xi) = \inf_{n\in\mathbb{N}} \frac{1}{n} H_\mu(\xi_n),$$

and where $\xi_n = \bigvee_{k=0}^{n-1} f^{-k}\xi$ is the measurable partition of X composed of the sets

$$C_{i_1\cdots i_n} = \bigcap_{k=0}^{n-1} f^{-k} C_{i_{k+1}}$$

with $C_{i_1}, \ldots, C_{i_n} \in \xi$.

The notion of metric entropy is due to Kolmogorov [115, 116]. It was extended to arbitrary dynamical systems by Sinai [182], in the form (1.9).

One can show that if $(\xi_n)_{n\in\mathbb{N}}$ is a sequence of measurable partitions such that $\bigcup_{n\in\mathbb{N}}\xi_n$ generates the σ-algebra of X, and ξ_{n+1} is a refinement of ξ_n for each $n\in\mathbb{N}$ (this means that each element of ξ_{n+1} is contained in some element of ξ_n), then

$$h_\mu(f) = \lim_{n\to\infty} h_\mu(f,\xi_n). \tag{1.10}$$

We also have

$$h_\mu(f^k) = k h_\mu(f) \quad \text{for each} \quad k\in\mathbb{N}. \tag{1.11}$$

An alternative definition of metric entropy can be introduced as follows. We first note that if ξ is a measurable partition of X, then for μ-almost every $x\in X$ and each $n\in\mathbb{N}$ there exists a single element $\xi_n(x)$ of ξ_n such that $x\in\xi_n(x)$.

Theorem 1.4.10 (Shannon–McMillan–Breiman). *If $f\colon X\to X$ is a measurable transformation, $\mu\in\mathcal{M}_f$, and ξ is a measurable partition of X, then the limit*

$$h_\mu(f,\xi,x) := \lim_{n\to\infty} -\frac{1}{n}\log\mu(\xi_n(x))$$

exists for μ-almost every $x\in X$. Moreover, the function $x\mapsto h_\mu(f,\xi,x)$ is μ-integrable and

$$h_\mu(f,\xi) = \int_X h_\mu(f,\xi,x)\,d\mu(x).$$

The statement in Theorem 1.4.10 was obtained successively in more general forms by several authors. Shannon [177] considered Markov measures, although the statement was only derived rigorously by Khinchin [113] (see also [114]). McMillan [134] obtained the L^1 convergence, and Breiman [41] obtained the convergence almost everywhere.

It is also convenient to introduce the notion of conditional entropy.

Definition 1.4.11. Given measurable partitions ξ and η of X, we define the *conditional entropy* of ξ with respect to η by

$$H_\mu(\xi|\eta) = -\sum_{C\in\xi,D\in\eta}\mu(C\cap D)\log\frac{\mu(C\cap D)}{\mu(D)}.$$

One can show that $H_\mu(\xi|\eta)=0$ if and only if η is a *refinement* of ξ, that is, if and only if for every $D\in\eta$ there exists $C\in\xi$ such that $\mu(D\setminus C)=0$.

Proposition 1.4.12. *If ξ and η are measurable partitions of X, then*

$$H_\mu(\xi\vee\eta) = H_\mu(\eta)+H_\mu(\xi|\eta) \le H_\mu(\eta)+H_\mu(\xi), \tag{1.12}$$

and

$$h_\mu(f,\xi) = h_\mu(f,\eta)+H_\mu(\xi|\eta). \tag{1.13}$$

It follows from (1.12) that $H_\mu(\xi \vee \eta) \geq H_\mu(\eta)$, with equality if and only if η is a refinement of ξ.

It is sometimes possible to compute the entropy of a measurable transformation using a single partition.

Definition 1.4.13. Let $f\colon X \to X$ be a measurable transformation. A measurable partition ξ of X is said to be:

1. a *one-sided generator* (with respect to f) if the sets in $\bigcup_{k\in\mathbb{N}\cup\{0\}} f^{-k}\xi$ generate the σ-algebra of X;

2. a *two-sided generator* (with respect to f) if the sets in $\bigcup_{k\in\mathbb{Z}} f^{-k}\xi$ generate the σ-algebra of X.

When there exists a generator the entropy can be computed as follows.

Theorem 1.4.14 (Kolmogorov–Sinai). *Let $f\colon X \to X$ be a measurable transformation and let $\mu \in \mathcal{M}_f$. Then the following properties hold:*

1. *if ξ is a one-sided generator, then $h_\mu(f) = h_\mu(f, \xi)$;*

2. *if ξ is a two-sided generator and f is invertible μ-almost everywhere, then $h_\mu(f) = h_\mu(f, \xi)$.*

Part I

Classical Thermodynamic Formalism