Pseudo-Differential Operators
Theory and Applications
8

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# Pseudodifferential Analysis, Automorphic Distributions in the Plane and Modular Forms

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To the memory of Paul Malliavin

# Contents

# Introduction

Most practitioners of pseudo-differential analysis and of analytic number theory would probably regard the two fields as being as far apart as conceivable. However, we wish here to convey the idea that, if deepened in its appropriate aspects, pseudo-differential analysis (mostly, but not only, one-dimensional pseudo-differential analysis in this book) may find a place in the bag of tools of modular form theory. To our PDE colleagues, we shall simply offer the apology that doing some export cannot hurt: more seriously, we have written this book under the assumption that some readers with very little, or no previous knowledge of automorphic function theory, might wish to find a reason to approach this fascinating domain, in which exact formulas of much aesthetic appeal are often the reward of spectral-theoretic questions. Analysts may also find, in the first chapter, aspects of pseudo-differential analysis unknown to them.

Few things in mathematics are duller than a linear form, such as the action of testing a distribution on functions. However, if you can make an operator from your distribution, you may then test it against pairs of functions, endowing it as a result with a more interesting hermitian structure. If you are lucky, you will obtain an explicit sum of squares: examples will occur in Chapter 7. How to make in a useful way, from a distribution in $\mathbb{R}^2$, an operator on functions on the real line, say from Schwartz's space $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$, is the starting point of pseudo-differential analysis. The simplest, and most successful way to do so, is the so-called Weyl calculus, or symbolic calculus, of operators: the symbol of an operator is the distribution it is built from.

This will lead to one first reason to let pseudo-differential analysis enter modular form theory which, or so we hope, number theorists may find compelling. When considering eigenfunctions, possibly generalized (Eisenstein series or Maass forms), of the modular Laplacian $\Delta$, it is always the pair of arguments $\pm\nu$ (pure imaginary numbers in the second case), rather than the eigenvalue $\frac{1-\nu^2}{4}$, that enters functions on the spectrum. Of course, functions of $\frac{1-\nu^2}{4}$ are just the same as even functions of $\nu$, but the fact remains that, more often than not, one has to deal with products of a function of $\nu$ by the same function taken at $-\nu$. This is especially clear when dealing with such composite objects as functions of type $L(s, f \times g)$ [4, p. 72].

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Now, there is a very natural transformation $\mathfrak{S} \mapsto f_0$ (Theorem 1.1.3) from distributions on $\mathbb{R}^2$ to functions in the hyperbolic half-plane $\Pi = SL(2,\mathbb{R})/SO(2)$ for which the operator $\Delta - \frac{1}{4}$ appears as the image of the square of a first-order differential operator, to wit the Euler operator in $\mathbb{R}^2$. Such a transformation is best defined in terms of pseudo-differential analysis, since it is none other than the result of testing the operator with Weyl symbol $\mathfrak{S}$ on a diagonal pair of (Gaussian) functions on the line canonically parametrized by $z \in \Pi$. There is another way to let this transformation, or associates of it, enter the picture: one can link it to the so-called dual Radon transformation from one homogeneous space of $G = SL(2,\mathbb{R})$ (the space $G/MN$ with the standard notation for the Iwasawa decomposition $G = NAK$, and $M = \{\pm I\}$) to the homogeneous space $G/K$. Needless to say, the Weyl calculus benefits from all desirable so-called covariance properties, so that the transformation under examination does not destroy automorphy properties relative to any arithmetic group (we here limit ourselves to the case of $\Gamma = SL(2,\mathbb{Z})$) one may have in mind. Automorphic objects in $\Pi = G/K$ are automorphic functions of the usual kind, i.e., functions invariant under the group of fractional-linear transformations (in the complex coordinate) of the hyperbolic half-plane associated to matrices in $\Gamma$, while automorphic distributions in $\mathbb{R}^2$ are by definition distributions invariant under the linear changes of real coordinates associated to the same group of matrices.

Automorphic pseudo-differential analysis is just pseudo-differential analysis, in which one restricts one's interest in symbols which are automorphic distributions. Using Weil's metaplectic representation [66], it amounts to the same to say that one considers only operators from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ which commute with all operators from the metaplectic representation lying above elements of $\Gamma$: to make this explicit, it means that they commute with the operator of multiplication of a function of $x$ by $e^{i\pi x^2}$, as well as with the Fourier transformation. This pseudo-differential analysis [61] has considerable specificity: on one hand, its structure relies on most features from the spectral theory of the modular Laplacian, including Hecke's theory and $L$-functions, sometimes of a composite kind; on the other hand, since automorphic distributions are very singular, usual methods of pseudo-differential analysis, for instance boundedness theorems or composition theorems of the usual species, are not applicable. The development of composition formulas in automorphic pseudo-differential analysis leads to an original approach towards the analysis of bilinear operations in non-holomorphic modular form theory. We shall not review automorphic pseudo-differential analysis in depth in the present book — though we shall give more than a few hints — but a byproduct of this approach, going beyond known results regarding series of Kloosterman sums, will play a crucial role in the construction and study, in Chapter 4, of a certain class of automorphic functions.

Pseudo-differential analysis will have an obviously central role in the last two chapters of the book. In a greater part of the book, it will be felt, in an indirect way, by the fact that automorphic distribution theory (on $\mathbb{R}^2$) will have the upper hand in comparison to automorphic function theory (in $\Pi$): this will have considerable advantages, for instance, in Chapter 5. In each of the chapters 4 to 7 (which contain the main results of possible arithmetic significance), the zeros of the zeta and, possibly, other $L$-functions, or those lying on any given parallel to the critical line, occur in some important role: we do not believe that the results obtained constitute a new approach towards the major questions relative to these functions. Each of these chapters would call for complements and generalisations, or deepening: this is especially true of the adelic Chapter 7, which is yet mostly an outlook towards future developments.

We now turn to a more detailed description of the structure of the book. Chapter 1 provides the necessary background about the Weyl calculus. It differs in an essential way from other introductions to pseudo-differential analysis, and it is tailored not for the needs in PDE, but for those in number theory. On one hand, there is much emphasis on representation-theoretic properties; on the other hand, the popular (Moyal-type) rule of composition of symbols bears no relation to the one useful here. Theorem 1.1.3 gives the main properties of the map $\mathfrak{S} \mapsto f_0$ from distributions in $\mathbb{R}^2$ to functions on $\Pi$ alluded to before. With more detail, this map is defined by the equation $f_0(z) = (\phi_z^0 | \mathrm{Op}(\mathfrak{S}) \phi_z^0)$, where $\mathrm{Op}(\mathfrak{S})$ is the operator with symbol $\mathfrak{S}$, and $\phi^0_{-\frac{1}{z}}$ is a normalized version of the function $x \mapsto e^{-i\pi \bar{z} x^2}$. The map $\mathfrak{S} \mapsto f_0$ intertwines the two actions of $SL(2,\mathbb{R})$ in both spaces of distributions or functions, by linear or fractional-linear changes of coordinates: this makes it possible to restrict it to automorphic distributions, getting automorphic functions as a result. The other fundamental property of this map is that it transfers the operator $\pi^2 \mathcal{E}^2$ in the plane, where $2i\pi\mathcal{E} = x\frac{\partial}{\partial x} + \xi \frac{\partial}{\partial \xi} + 1$, to the operator $\Delta - \frac{1}{4} = (z-\bar{z})^2 \frac{\partial^2}{\partial z \partial \bar{z}} - \frac{1}{4}$. We consider only even distributions here (i.e., distributions invariant under the map $(x,\xi) \mapsto (-x,-\xi)$). Such a distribution $\mathfrak{S}$, automorphic or not, is not characterized by its image $f_0$: what is needed to this effect is to complete $f_0$ into a pair $\Theta\mathfrak{S} = (f_0, f_1)$, where the function $f_1$ is defined in a comparable way, using in place of $\phi_z^0$ the next simplest (odd) function $\phi_z^1$ attached to $z$. An important operator acting on symbols, to be denoted as $\mathcal{G}$, is defined by the fact that, given any distribution $\mathfrak{S} \in \mathcal{S}'(\mathbb{R}^2)$, the pseudo-differential operator with symbol $\mathcal{G}\mathfrak{S}$ is the composition of the operator $\mathrm{Op}(\mathfrak{S})$ with the operator $u \mapsto \check{u}$, with $\check{u}(x) = u(-x)$. The operator $\mathcal{G}$, a simple rescaling of the symplectic Fourier transformation in $\mathbb{R}^2$, plays a surprisingly central role, even in $p$-adic pseudo-differential analysis. A distribution $\mathfrak{S}$ is characterized by its image $f_0$ (rather than by the pair $(f_0, f_1)$) if and only if it is $\mathcal{G}$-invariant. From the relation between the Euler operator $2i\pi\mathcal{E}$ and the hyperbolic Laplacian $\Delta$, one sees in particular that automorphic distribution theory (in $\mathbb{R}^2$) is slightly subtler than automorphic function theory (in $\Pi$), since every eigenfunction $f$ (possibly generalized) of $\Delta$ for the eigenvalue $\frac{1-\nu^2}{4}$ is "covered" by exactly two distributions, automorphic if $f$ is, homogeneous of degrees $-1-\nu$ and $-1+\nu$, the images of each other under $\mathcal{G}$.

The Weyl calculus benefits from two distinct covariance properties, in connection with the Heisenberg representation and the metaplectic representation: to each of these, one can associate a composition formula. Only the first (Moyal-type) one is generally known — it is the one useful for applications of pseudo-differential analysis to PDE — but it is the second one, introduced in [61, section 17], that is more important for number-theoretic applications: both types will be considered here. The last section of Chapter 1 deals with the totally radial Weyl calculus: this is obtained when only operators in $\mathcal{S}(\mathbb{R}^n)$ commuting with the action of rotations, and sending every function to a radial distribution, are considered. Looking for an efficient symbolic calculus of such operators, one is led in a natural way, again, to using the hyperbolic half-plane, at least as a first step.

Associates of the map $\mathfrak{S} \mapsto f_0$ are obtained as a result of composing this map by functions, in the spectral-theoretic sense, of $\Delta$ on the left, or by functions of $2i\pi\mathcal{E}$ on the right: note that an operation of the second kind is identical to one of the first kind if and only if it involves an even function of $2i\pi\mathcal{E}$. A standard such associate, defined with the help of considerations of harmonic analysis only, is the dual Radon transformation already alluded to from even functions on the plane or, what amounts to the same, functions on the homogeneous space $G/MN$, to functions on the half-plane $G/K$. Chapter 2 is devoted to a study of the Radon transformation: in particular, we make the relation between the map $\mathfrak{S} \mapsto f_0$ and the dual Radon transformation explicit, obtaining at the same time a few formulas useful in the sequel. The latter half of this chapter is concerned with the analysis of a certain function $\chi_{\rho,\nu}$ of one variable, built with the help of the hypergeometric function. The results obtained will be applied in Chapter 4, and our reasons for studying the function $\chi_{\rho,\nu}$ will be given presently with more profit.

In Chapter 3, we provide the necessary background regarding automorphic functions, recalling such notions as Eisenstein series, Hecke eigenforms, $L$-functions ... (in the case of the full modular group only, for simplicity): the first two notions have analogues which are automorphic distributions, the map $\mathfrak{S} \mapsto f_0$ defined before providing the correspondence, together with an obvious terminology. We spend some more time on Roelcke-Selberg expansions, and on matters related to Kloosterman sums. The analytic continuation of certain series of Kloosterman sums has been much studied, as a consequence of investigations (in particular [21]) originating with Selberg's work [45]. It is necessary for our purposes, however, to go beyond these results, finding when $\mathrm{Re}\, s$ and $\mathrm{Re}\, t$ are positive and $|\mathrm{Re}\,(s-t)| < 1$ the analytic continuation of the Dirichlet series in two variables defined when $\mathrm{Re}\, s > 1, \mathrm{Re}\, t > 1$ by the equation, in which $k \in \mathbb{Z}$,

$$\zeta_k(s,t) = \frac{1}{4} \sum_{\substack{m_1 m_2 \neq 0 \\ (m_1,m_2)=1}} |m_1|^{-s} |m_2|^{-t} \exp\left(2i\pi k \frac{\overline{m_2}}{m_1}\right), \tag{0.1}$$

with $m_2 \overline{m}_2 \equiv 1 \bmod m_1$. The solution to this problem is rather lengthy: its main features will be expounded in Section 3.6. It was obtained in some previous work [60], as a byproduct of the spectral analysis of the pointwise product of two Eisenstein series, which will be detailed in Section 3.5. This latter problem was mainly solved as a preparation towards the following basic problem of automorphic pseudo-differential analysis [61]: compute the symbol of the composition (the definition of which requires some care) of two operators the symbols of which are Eisenstein distributions. Though we shall not come back to this problem in any detail, we shall give a short survey, in Section 3.4, of some of its features.

In Chapter 4, we analyze the properties of a new class of automorphic functions. Such functions are formally easy to construct, quite generally, with the help of Poincaré series, starting from a function $\psi$ in $\Pi$, and setting $f = \sum_{g\in\Gamma/\Gamma_\psi} \psi \circ g^{-1}$, where $\Gamma_\psi$ is the subgroup of $\Gamma$ under which $\psi$ remains invariant: summing over the quotient set prevents one from repeating infinitely many times the same term, in the case when $\Gamma_\psi$ is infinite. Two examples, in which the group $\Gamma_\psi$ is the group $\Gamma \cap N$ consisting of matrices $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$ with $b \in \mathbb{Z}$, are well-known. In the first one, one takes $\psi(z) = (\operatorname{Im} z)^{\frac{1-\nu}{2}}$ with $\operatorname{Re}\nu < -1$, getting as a result the Eisenstein series $f = E_{\frac{1-\nu}{2}}$. In the second one, one takes $\psi(z) = (\operatorname{Im} z)^{\frac{1-\nu}{2}} e^{2i\pi kz}$ for some $k \in \mathbb{Z}$, obtaining a special case of Selberg's series [45]. In Chapter 4, we shall start, instead of an $N$-invariant function $\psi$, from an $A$-invariant function (where $A$ is the set of matrices $\begin{pmatrix} a^{\frac12} & 0 \\ 0 & a^{-\frac12} \end{pmatrix}$ with $a > 0$), or more generally from a function such that $\psi(az) = a^{\frac{\rho-1}{2}} \psi(z)$ for some fixed number $\rho$ and every $a > 0$. In general, $\Gamma_\psi$ then reduces to $\{\pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$, so that new convergence problems arise. Poincaré-style series can also be built in the realm of automorphic distribution (in the plane) theory: one such example will be given in Section 4.1, as its study will demand proving a few geometric estimates needed in the sequel.

The function $(\operatorname{Im} z)^{\frac{1-\nu}{2}}$ which gives rise, under the Poincaré summation process, to the Eisenstein series $E_{\frac{1-\nu}{2}}$, is the image, up to multiplication by a constant, of the function $(x,\xi) \mapsto |\xi|^{\nu-1}$ by the dual Radon transformation. It is therefore tempting to generalize Eisenstein series by starting, instead of a function as simple as $|\xi|^{\nu-1}$, from a function of the two variables $x, \xi$ separately homogeneous in each, to wit a function such as

$$\operatorname{hom}_{\rho,\nu}(x,\xi) = |x|^{\frac{\rho+\nu-2}{2}} |\xi|^{\frac{\nu-\rho}{2}} : \tag{0.2}$$

this function will also occur quite naturally from our study of composition formulas in the Weyl calculus. The role of the two parameters $\rho, \nu$ is quite distinct. The second one refers to the global degree of homogeneity $\nu - 1$ of $\operatorname{hom}_{\rho,\nu}$, i.e., to the fact that this is an eigenfunction of $2i\pi\mathcal{E}$ for the eigenvalue $\nu$: the corresponding "spectral line" is defined by $\operatorname{Re}\nu = 0$, since $\mathcal{E}$ is formally self-adjoint in $L^2(\mathbb{R}^2)$. The parameter $\rho - 1$ corresponds to an eigenvalue of the operator (the product of which by $i$ is formally self-adjoint) $x\frac{\partial}{\partial x} - \xi\frac{\partial}{\partial \xi}$, and the appropriate spectral line is defined by $\operatorname{Re}\rho = 1$: we chose $\rho$, rather than $\rho - 1$, as a parameter, to help not making any confusion between $\rho$ and $\nu$; also, $\frac{\rho}{2}$ will have to move throughout the critical strip for the zeta function, and the spectral line $\operatorname{Re}\rho = 1$ will correspond to the critical line. If one starts from a distribution in $\mathbb{R}^2$ homogeneous of degree $-1-\nu$, or from an eigenfunction of $\Delta$ (in $\Pi$) for the eigenvalue $\frac{1-\nu^2}{4}$, the Poincaré-type series (in the automorphic distribution environment), assumed to be convergent, built from such an object, will have the same property. Nothing comparable can hold relative to the parameter $\rho$: but, as will be seen, something important will remain from it after the summation has been performed.

The case when $\rho = 1$ is a special one on several accounts. One can verify that the image, under the dual Radon transform, of the function $\mathrm{hom}_{1,\nu}$, is a multiple of the function $\psi(z) = \mathfrak{P}_{\frac{\nu-1}{2}}\left(-i\frac{\operatorname{Re} z}{\operatorname{Im} z}\right) + \mathfrak{P}_{\frac{\nu-1}{2}}\left(i\frac{\operatorname{Re} z}{\operatorname{Im} z}\right)$ involving Legendre functions. There is nothing one can do with the Poincaré series $\frac{1}{2}\sum_{g\in\Gamma}\psi(g.z)$, as it converges for no value of $\nu$. One can trace the reason for this as lying in the invariance of the function $\psi$ under the change $\nu \mapsto -\nu$: to recover convergence, it is necessary to break the function $\psi$ into two parts, the transforms of each other under the symmetry $\nu \mapsto -\nu$.

No longer specializing the parameter $\rho$, but assuming that $0 < \operatorname{Re}\rho < 2$, we define with the help of the hypergeometric function, if $\nu \notin \mathbb{Z}$ and $\rho \pm \nu \notin 2\mathbb{Z}$, a certain function $\chi_{\rho,\nu}$ of one real variable (2.3.31) (already alluded to when discussing Chapter 2); we consider then the function

$$\psi_{\rho,\nu}(z) = (\operatorname{Im} z)^{\frac{\rho-1}{2}}\chi^{\mathrm{even}}_{\rho,\nu}\left(\frac{\operatorname{Re} z}{\operatorname{Im} z}\right), \tag{0.3}$$

where $\chi^{\mathrm{even}}_{\rho,\nu}$ is the even part of $\chi_{\rho,\nu}$, and make the following observations. First, and most important, the dual Radon transform of $\mathrm{hom}_{\rho,\nu}$ is a multiple of the sum $\psi_{\rho,\nu} + \psi_{\rho,-\nu}$. From the two eigenvalue equations expressing the bihomogeneity of $\mathrm{hom}_{\rho,\nu}$, it follows by general properties that its dual Radon transform undergoes a multiplication by $a^{\frac{\rho-1}{2}}$ under any change of variable $z \mapsto az$ with $a > 0$, and that it is an eigenfunction of $\Delta$ for the eigenvalue $\frac{1-\nu^2}{4}$. The first of these two properties is also, obviously, satisfied by the function $\psi_{\rho,\nu}$. So far as the second eigenvalue equation is concerned, it is still satisfied by $\psi_{\rho,\nu}$ in the complement of the hyperbolic line from 0 to $i\infty$: however, the $\frac{\partial}{\partial x}$-derivative of this function (which is continuous in $\Pi$) has a jump at points on this line. Making this discontinuity explicit, one obtains that, in the distribution sense, one has

$$\left(\Delta - \frac{1-\nu^2}{4}\right)\psi_{\rho,\nu} = C(\rho,\nu)(\operatorname{Im} z)^{\frac{\rho-1}{2}}\delta_{(0,i\infty)}, \tag{0.4}$$

where $C(\rho,\nu)$ is an explicit constant important in the theory and $\delta_{(0,i\infty)}$ is the measure supported in the line under consideration, coinciding with $\frac{dy}{y}$ in terms of the coordinate $y = \operatorname{Im} z$. Chapter 2 ends with an intrinsic distinction, in spectral-theoretic terms, between the functions $\psi_{\rho,\nu}$ and $\psi_{\rho,-\nu}$ when $\operatorname{Re}\nu \neq 0$.

Chapter 4 is concerned, for the essential, with the construction and analysis of the series

$$f_{\rho,\nu}(z) = \frac{1}{2}\sum_{g\in\Gamma} \psi_{\rho,\nu}(g.z). \tag{0.5}$$

It converges when $\mathrm{Re}\,\nu < -1 - |\mathrm{Re}\,\rho - 1|$, though this is somewhat more difficult to establish than the corresponding convergence of Eisenstein series: indeed, there are "more" terms since one cannot divide here the group $\Gamma$ by any subgroup larger than $\{\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\}$, and geometric estimates are more involved. After having defined $f_{\rho,\nu}$ in the initial domain $\mathrm{Re}\,\nu < -1 - |\mathrm{Re}\,\rho - 1|$, we need to continue it analytically to the domain $\mathrm{Re}\,\nu < 1 - |\mathrm{Re}\,\rho - 1|$. All the difficulties concentrate on the continuation, in this latter domain, of the function

$$z \mapsto \frac{1}{2}\sum_{\substack{m\in\mathbb{Z}^\times,\ n\in\mathbb{Z}\\ m|n(n+1)}} \left|\frac{m_1}{m_2}\right|^{\frac{1-\rho}{2}} |m|^{\frac{1-\nu}{2}} \left(\frac{|n+\frac{1}{2}-mz|^2}{|m|y}\right)^{\frac{\nu-1}{2}}, \tag{0.6}$$

where the pair $m_1, m_2$ is characterized by the conditions $m = m_1 m_2$ and $1 \leq m_1 | n+1, m_2 | n$. A Fourier expansion substitutes for this problem the equivalent one, already mentioned, of continuing analytically the function $\zeta_k(s,t)$ in (0.1).

A summary of the main results regarding $f_{\rho,\nu}$ is as follows. Fixing $\rho$ with $\frac{\rho}{2}$ in the critical strip, the function $f_{\rho,\nu}$ extends as a meromorphic function of $\nu$ for $\mathrm{Re}\,\nu < 1 - |\mathrm{Re}\,\rho - 1|$, with the following poles: the non-trivial zeros of zeta, and the points $i\lambda_p$, with $\frac{1+\lambda_p^2}{4}$ in the even part of the discrete spectrum of $\Delta$, to wit the part for which there exist cusp-forms invariant under the symmetry $z \mapsto -\bar{z}$. Next, one has the equation

$$f_{\rho,\nu} + f_{\rho,-\nu} = -\frac{C(\rho,\nu)}{\nu}\frac{\zeta^*(\frac{\rho-\nu}{2})\zeta^*(\frac{\rho+\nu}{2})}{\zeta^*(\nu)} E_{\frac{1+\nu}{2}}, \tag{0.7}$$

involving the Eisenstein series $E_{\frac{1+\nu}{2}}$ and the "full zeta function" $\zeta^*(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \cdot\zeta(s)$. On the other hand, the function $(C(\rho,\nu))^{-1} f_{\rho,\nu}$ is invariant under the symmetry $\rho \mapsto 2-\rho$. Finding the asymptotic expansion, as $\mathrm{Im}\,z \to \infty$, of $f_{\rho,\nu}(z)$, makes it possible, finally, to obtain the complete (Roelcke-Selberg) spectral decomposition of this function: it does not lie in $L^2(\Gamma\backslash\Pi)$, but it does so after one has subtracted from it a certain linear combination of the Eisenstein series $E_{\frac{1+\rho}{2}}$ and $E_{\frac{3-\rho}{2}}$. An essential property, a consequence of (0.4), is the following.

Denote as $\Sigma$ the one-dimensional subset of $\Pi$ consisting of the (disjoint) union of all lines congruent, under elements of $\Gamma$, to the hyperbolic line from 0 to $i\infty$: making from the measure $\frac{1}{2}\left[(\mathrm{Im}\,z)^{\frac{\rho-1}{2}} + (\mathrm{Im}\,z)^{\frac{1-\rho}{2}}\right]\delta_{(0,i\infty)}$, in an obvious way, an automorphic measure $ds_\Sigma^{(\rho)}$ supported in $\Sigma$, one has the identity, in the distribution sense,

$$\left(\Delta - \frac{1-\nu^2}{4}\right) f_{\rho,\nu} = 2C(\rho,\nu) ds_{\Sigma}^{(\rho)}. \tag{0.8}$$

Making a careful (not quite standard) analysis of the resolvent of $\Delta$, one obtains the more precise result

$$(2C(\rho,\nu))^{-1} f_{\rho,\nu} = \left(\Delta - \frac{1-\nu^2}{4}\right)^{-1} ds_{\Sigma}^{(\rho)} \quad \text{if } \operatorname{Re}\nu < 0, \tag{0.9}$$

which makes a clear distinction between $f_{\rho,\nu}$ and $f_{\rho,-\nu}$ possible. It is just as well to give, in place of the Roelcke-Selberg expansion of $f_{\rho,\nu}$, that of the one-dimensional automorphic object $ds_{\Sigma}^{(\rho)}$, which is of course to be regarded in some appropriate weak sense,

$$\begin{aligned} ds_{\Sigma}^{(\rho)} = \frac{1}{2}\left(E_{\frac{1+\rho}{2}} + E_{\frac{3-\rho}{2}}\right) + \frac{1}{16\pi}\int_{-\infty}^{\infty} \frac{\zeta^*(\frac{\rho-i\lambda}{2})\zeta^*(\frac{\rho+i\lambda}{2})}{\zeta^*(1+i\lambda)} E_{\frac{1-i\lambda}{2}}\, d\lambda \\ + \frac{1}{4}\sum_{p,j \text{ even}} L^*(\frac{\rho}{2}, \mathcal{M}_{p,j})\mathcal{M}_{p,j} : \end{aligned} \tag{0.10}$$

the functions $\mathcal{M}_{p,j}$ are Hecke eigenforms (only the ones invariant under the symmetry $z \mapsto -\bar{z}$ are to be considered here) and, again, the "full $L$-series" $L^*(s,\mathcal{M})$ is the $L$-series $L(s,\mathcal{M})$ completed by the Archimedean factor (a product of two Gamma functions) which makes its functional equation simple. In view of Dunford's integral formula, one may also consider the images of the measure $ds_{\Sigma}^{(\rho)}$ under all operators of the kind $H\left(2\sqrt{\Delta - \frac{1}{4}}\right)$ where we assume that $H = H(\mu)$ is an even holomorphic function in some strip $|\operatorname{Im}\mu| < \beta_0$, such that $\int_{\operatorname{Im}\mu=\beta} |\mu|^2 |H(\mu)|^2 d\mu < \infty$ for every $\beta$ with $|\beta| < \beta_0$. The spectral density of every automorphic function so defined is a $C^\infty$ function of $\lambda$: within any appropriate subspace of $L^2(\Gamma\backslash\Pi)$ making this extra condition valid, let us consider the closure $S_\rho$ of the linear space of all functions $H\left(2\sqrt{\Delta - \frac{1}{4}}\right) ds_{\Sigma}^{(\rho)}$. It is clear that from the knowledge of $S_\rho$, one can determine whether $\operatorname{Re}\rho = 1$ or not and, if such is the case, the value of $\rho$. However, whether $S_\rho$ is independent of $\rho$ when $\operatorname{Re}\rho \neq 1$ cannot be answered at present: the continuous part (relative to the spectral decomposition of the modular Laplacian) of this space is so if and only if the Riemann hypothesis is true for zeta, while the discrete part is independent of $\rho$ if and only if the Riemann hypothesis is true for all $L$-functions attached to cusp-forms of even type (relative to the symmetry $z \mapsto -\overline{z}$), and all eigenvalues of the even part of $\Delta$ are simple. Needless to say, even though (0.10) gives some interpretation of the zeros of zeta lying on any given parallel to the critical line, we do not believe that this indicates any possible line of attack on any of these deep conjectures.

At the end of Section 4.7, we raise questions regarding possible generalizations of the function $f_{\rho,\nu}$, as one would come to when replacing $\Sigma$ by another set of lines, and refer to links, tenuous or strong, with previous work on quadratic extensions of the rationals, by Hejhal [16], Zagier[69] and ourselves [60, Sec. 19–20].

Automorphic distribution theory is again crucial in Chapter 5, another central chapter of the book. Let us denote as $\mathfrak{E}_\nu$ "the" automorphic distribution which is the analogue of the Eisenstein series $E_{\frac{1-\nu}{2}}$, and as $\mathfrak{M}_{p,j}$ "the" analogue, in the same spirit, of the Maass-Hecke form $\mathcal{M}_{p,j}$. Actually, as already mentioned, automorphic distribution theory (on $\mathbb{R}^2$) is more precise than automorphic function theory in the upper half-plane (this is why an automorphic distribution is characterized by *a pair* of automorphic functions) and, as will be seen in Section 3.2, there are two modular distributions, one the image of the other under the symplectic Fourier transformation, corresponding to just one non-holomorphic modular form. Given any pair $h, f$ of functions lying in the image of $\mathcal{S}_{\text{even}}(\mathbb{R}^2)$ under the operator $2i\pi\mathcal{E}(1+2i\pi\mathcal{E})$, the series

$$\langle \mathfrak{P}, h \otimes f \rangle = \sum_{g \in \Gamma} \int_{\mathbb{R}^2} (h \circ g)(x,\xi) f(x,\xi) dx d\xi \tag{0.11}$$

is convergent. The main result of Chapter 5 is the identity

$$\langle \mathfrak{P}, \bar{h} \otimes h \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\langle \mathfrak{E}_{i\lambda}, h \rangle|^2 |\zeta(i\lambda)|^{-2} d\lambda + 2 \sum_{p \neq 0} \sum_j \left| \Gamma\left( \frac{i\lambda_p}{2} \right) \right|^2 \left| \langle \mathfrak{M}_{p,j}, h \rangle \right|^2 : \tag{0.12}$$

note that it is not a priori obvious, from its definition, that the left-hand side is non-negative. The proof of this identity is quite delicate: besides automorphic distribution theory, it relies on the theory of series of Kloosterman sums, especially the version, based on the automorphic Green's operator for the modular Laplacian, as developed by Iwaniec [21].

Another part of the book (Chapter 6) deals with arithmetic questions involved in connection with the totally radial Weyl calculus. A comparison with automorphic pseudo-differential analysis, in which non-holomorphic modular form theory enters the structure of the symbols under study, may clarify things at this point: here, arithmetic enters, instead, the functions, or rather discretely supported measures, operators are applied to; besides, it is an extension of holomorphic modular form theory that is now relevant. For the main part, Chapter 6 consists in a quotation of arithmetic results from a recent book [63] of ours: but the symbolic calculus, or quantization process (called the "soft" calculus), which has to be used is introduced here in a quite natural way, while in the quoted work it appeared as a branch in a forest of assorted symbolic calculi, in which the reader had probably no desire to venture. From an arithmetic point of view, the main features of this chapter consist in a necessary extension of the Rankin-Selberg unfolding method, and in an application of puzzling results of Shimura [46] and Iwaniec [22], involving the critical zeros of zeta, to operator theory.

The last part of the book (Chapter 7), still in a quite underdeveloped stage, raises the question whether pseudo-differential analysis should be generalized to an adelic setting: what we have in mind, here, is a pseudo-differential analysis of operators acting on *complex-valued* functions on adeles. Our main point, in this direction, is the following. Starting from a certain problem in automorphic distribution theory, one is led to asking for a version of pseudo-differential analysis in which Planck's constant would depend on the prime $p$ under consideration. This is not possible while staying within Archimedean analysis: at this point, calling for an adelic substitute seems to be required.

Let us stress that we certainly do not consider the present chapter (half of which deals with Archimedean analysis anyway) as an introduction to adelic pseudo-differential analysis: this, on the number-theoretic side of the question, would require another author. We have been satisfied, here, with recalling some concepts of $p$-adic analysis, and making a few calculations, in this context, specifically related to the problem in automorphic distribution theory we started with. Even though the modification demanded, for each $p$, by the change of Planck's constant, has been addressed, it is unclear, to us, how the various $p$-adic pseudo-differential analyses so defined should be pieced together: the usual restricted direct product machinery does not seem to provide the right answer. On the other hand, which may be some justification for this chapter or so we hope, the Archimedean developments which precede in this book — the Weyl calculus, the Radon transformation, elements of representation theory — may be helpful in providing some guidelines for a future possible adelic theory: but one should certainly not limit oneself to adeles of the field $\mathbb{Q}$, and one reason not to do so has been indicated in Remark 7.2.b.(iv).

The chapter starts with the construction of a certain automorphic distribution (in the Archimedean sense) $\mathfrak{T}_\infty - \mathcal{G}\mathfrak{M}_\infty$, where $\mathfrak{T}_\infty$ and $\mathfrak{M}_\infty$ have a purely arithmetic character in that their definition involves series with arithmetic coefficients but no analytic factor such as a Gamma function: the essential property of this distribution is that it coincides with a certain series of Eisenstein distributions $\mathfrak{E}_{-\mu}$ and, possibly, some of their $\frac{d}{d\mu}$-derivatives, taken over the set of non-trivial zeros $\mu$ of zeta. If one agrees with the point of view regarding pseudo-differential analysis expounded in the very beginning of this introduction, one is led to the conviction that part of the deeper structure of the automorphic distribution $\mathfrak{T}_\infty$ may be hidden in that of the operator of which it is a symbol. While the operator with symbol $\mathcal{G}\mathfrak{M}_\infty$ can be perfectly understood in a classical distribution setting (7.1.40), and turns out to have an interesting structure, truly understanding the operator $\mathrm{Op}\,(\mathfrak{T}_\infty)$ seems to be a quite difficult task. The distribution $\mathfrak{T}_\infty$ is the limit, as the integer $N$ goes to $\infty$ while absorbing all primes, of a sequence $(\mathfrak{T}_N)$ with the following property. If not the operator with symbol $\mathfrak{T}_N$, that with the rescaled symbol $N^{i\pi\mathcal{E}}\mathfrak{T}_N$ has a completely clear structure: it is a finite-rank operator from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ associated to a finite family, depending on $N$, of discretely supported measures $\mathfrak{d}_\rho$ on the line (here, $\rho \in (\mathbb{Z}/N\mathbb{Z})^\times$ has nothing to do with the number $\rho$ from Chapters 1 and 2), with interesting properties. As a consequence of this, the first component $f_0$ of the $\Theta$-transform (*cf. supra*) of $\mathfrak{T}_N$ can be expressed as a nice sum of squares with, however, one crucial minus sign (Theorem 7.2.1).

The necessity to rescale the symbol $\mathfrak{T}_N$ with the help of the operation $N^{i\pi\mathcal{E}}$ prevents one from finding a good interpretation of the operator with such a symbol, even more so of the operator with symbol $\mathfrak{T}_\infty$. This takes us to the question we started the current discussion with, of making Planck's constant depend on $p$. The adelic point of view, even when insufficiently developed, might have good heuristic value in the search for fundamentally new useful Hilbert space structures on appropriate spaces of automorphic distributions: this may, or not, help understanding the automorphic distribution, a series of Eisenstein distributions, which was the starting point of this last chapter.

We dedicate this book to the memory of Paul Malliavin. It is thanks to him that, almost half a century ago, we had our first contact both with singular integral operators (which were soon to become pseudo-differential operators) and with (holomorphic) modular form theory. We have never ceased, in the intervening decades, marveling at his mathematical accomplishments.

# Chapter 1

# The Weyl calculus

We start with a description of the basic features of the Weyl pseudo-differential analysis, to be used throughout the book: the emphasis is on group-theoretic properties, which is what is needed in the sequel. In the first section of the chapter, we shall show in which way the analysis of operators from the Weyl calculus by means of their diagonal matrix elements against appropriate families, parametrized by points of the hyperbolic half-plane $\Pi$, of functions of Gaussian type, establishes a link between function theory on the plane and on the hyperbolic half-plane. This $\Theta$-transformation will appear in most parts of the book. Later, in Chapter 3, it will specialize as a correspondence from automorphic distribution theory in the plane to automorphic function theory in $\Pi$. In Section 1.2, we discuss two quite different composition formulas, meaning by this two analyses of the (partially defined) bilinear map $\#$ such that the composition $\mathrm{Op}(h_1)\mathrm{Op}(h_2)$ of the operators with symbols $h_1$ and $h_2$ should agree with the operator $\mathrm{Op}(h_1 \# h_2)$: most practitioners of pseudo-differential analysis will only be familiar with the first one. In Section 1.3, we derive from a restriction of the $n$-dimensional Weyl calculus an efficient symbolic calculus of totally radial operators in $\mathbb{R}^n$: remarkably, this demands that the appropriate species of symbols should live, again, on $\Pi$.

## 1.1 An introduction to the usual Weyl calculus

A symbolic calculus of operators is a linear one-to-one way of associating operators, say on $L^2(\mathbb{R}^n)$, to functions of $2n$ variables: with the exception of the totally radial calculus, we shall be mostly concerned, in this book, with the one-dimensional case. One of the best-known ways of doing this, that which consists in associating with an operator its integral kernel, fails on two major accounts. The first one has to do with the fact that, under such a correspondence, the composition of operators has nothing to do whatever, even on an approximate level, with the pointwise multiplication of integral kernels; the second one is that this correspondence does

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not benefit from a large group of *visible* symmetries, in a sense to be made clear shortly.

The function $h$ on $\mathbb{R}^{2n}$ associated to some operator $A$ under a given symbolic calculus is called the *symbol* of $A$ while, in the reverse direction, one generally writes $A = \mathrm{Op}(h)$. Since operators on $L^2(\mathbb{R}^n)$ very seldom commute, the composition of operators can never correspond, under any symbolic calculus, to the pointwise multiplication of functions, which is a commutative operation. Still, considering for instance two differential operators $A_1$ and $A_2$ of orders $m_1$ and $m_2$ respectively, the top-order part of $A_1A_2$, which can be defined as the equivalence class of the product when differential operators of order $\leq m_1 + m_2 - 1$ are neglected, is the same as the top-order part of $A_2A_1$. Pseudo-differential analysis was developed, towards the needs of partial differential equations, as a symbolic calculus (several possibilities have been considered), in which the two bilinear operations under consideration, to wit the composition of operators and the pointwise multiplication of symbols, would roughly correspond to each other, modulo error terms of "lower order". We shall not, here, approach this domain of applications, in which hundreds of papers and a few major books [56, 53, 30, 19, 47, 35] have been written. So as to prevent misunderstanding, let us make it clear, however, that PDE people are certainly not interested in symbolic calculi of *differential* operators: the point is that good symbolic calculi (e.g. Weyl's) make it possible to construct auxiliary operators needed for the solution of PDE problems; the simplest instance concerns the construction of parametrices, i.e., approximate inverses, of elliptic operators.

Possibly the most obvious pair of non-commuting bounded operators on $L^2(\mathbb{R})$ is the pair $\tau_{y,0}, \tau_{0,\eta}$, defined by the equations

$$(\tau_{y,0}u)(x) = u(x-y) \quad \text{and} \quad (\tau_{0,\eta}u)(x) = u(x)e^{2i\pi\eta x}. \tag{1.1.1}$$

One can combine these two operations into an operation $\tau_{y,\eta}$ (almost a product of the two): coming back to the $n$-dimensional case, we assume that $y$ and $\eta$ lie in $\mathbb{R}^n$ and set

$$(\tau_{y,\eta}u)(x) = u(x-y)e^{2i\pi<x-\frac{y}{2},\eta>}. \tag{1.1.2}$$

If one introduces an extra real parameter $t$, one notes the identity

$$\begin{aligned}\frac{1}{2i\pi}\frac{d}{dt}&\left(\tau_{ty,t\eta}u\right)(x)\\ &= \left[-\frac{1}{2i\pi}\sum y_j u_j'(x-ty) + \langle x-ty,\eta\rangle u(x-ty)\right] e^{2i\pi\langle x-\frac{ty}{2},t\eta\rangle}\\ &= \left(\sum \eta_j x_j - \frac{1}{2i\pi} y_k \frac{\partial}{\partial x_k}\right)\left(\tau_{ty,t\eta}u\right)(x).\end{aligned} \tag{1.1.3}$$

It is thus natural to think of the operator $\tau_{ty,t\eta}$ as being the exponential $\exp(2i\pi tD)$, where $D$ is the differential operator (on functions of $x$) $D = \sum\left(\eta_j x_j - \frac{1}{2i\pi} y_k \frac{\partial}{\partial x_k}\right)$. Setting

$$(Q_j u)(x) = x_j u(x), \quad (P_j u)(x) = \frac{1}{2i\pi} \frac{\partial u}{\partial x_j}, \tag{1.1.4}$$

we shall write, assuming without loss of generality that $t = 1$,

$$\tau_{y,\eta} = \exp(2i\pi(\langle \eta, Q \rangle - \langle y, P \rangle)). \tag{1.1.5}$$

To give this equation a more than formal meaning, we must refer to Stone's theorem on one-parameter groups of unitary operators, to be found in many places, for instance [42]. On many occasions, we shall be dealing with *explicit* one-parameter groups $(U_t)_{t\in\mathbb{R}}$ of unitary operators in some Hilbert space $H$, and we shall only need the easy part of Stone's theorem, to wit that such a group has a well-defined self-adjoint generator

$$D = \frac{1}{2i\pi} \frac{d}{dt}\bigg|_{t=0} U_t : \tag{1.1.6}$$

recall that the operator $D$ is not, generally, a bounded operator in $H$, and that the notion of self-adjoint operator has a precise meaning, which demands defining its domain; in this case, it is just the set of vectors $u$ in $H$ such that $t^{-1}(U_t u - u)$ has a limit in $H$ as $t \to 0$.

Given two pairs $(y, \eta)$ and $(y', \eta')$, one has the formula, of immediate verification,

$$\tau_{y,\eta} \tau_{y',\eta'} = e^{i\pi[(y,\eta),(y',\eta')]} \tau_{y+y',\eta+\eta'} \tag{1.1.7}$$

if one introduces the *symplectic form* $[\,,]$ on $\mathbb{R}^n \times \mathbb{R}^n$, by definition the (alternate) bilinear form such that

$$[(y, \eta), (y', \eta')] = -\langle y, \eta' \rangle + \langle y', \eta \rangle. \tag{1.1.8}$$

We assume that the reader is familiar with the basic language of representation theory. From (1.1.7), it is easy to define with the help of the symplectic form a group structure on the set-theoretic product $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, then a unitary representation $\pi$ of the group obtained in $H = L^2(\mathbb{R}^n)$, such that $\pi(y, \eta; 0) = \tau_{y\eta}$. The group and representation so defined are called the Heisenberg group and Heisenberg representation. Alternatively, one can weaken the notion of representation to that of projective representation, which consists, given a topological group $G$ and a Hilbert space $H$, in defining, for every $g \in G$, the operator $\pi(g)$ only up to multiplication by an indeterminate constant $\omega(g) \in \mathbb{C}$ of absolute value 1 (such an indeterminate factor will be called, generally, a *phase factor*), weakening of necessity the basic property of a representation to the relation $\pi(g)\pi(g') = \omega(g, g')\pi(gg')$. Then, (1.1.7) shows that the map $(y, \eta) \mapsto \tau_{y,\eta}$ is a projective representation of the additive group $\mathbb{R}^n \times \mathbb{R}^n$ in $L^2(\mathbb{R}^n)$.

Of course, all concepts or proofs based on infinitesimal elements will disappear from the more arithmetic parts of the book, in particular the sections devoted to extending the Weyl calculus to a $p$-adic setting. Coming back to our present environment, the symplectic form on $\mathbb{R}^n \times \mathbb{R}^n$ gives rise to the *symplectic group* $\mathrm{Sp}(n, \mathbb{R})$, by definition the group of linear automorphisms $g$ of $\mathbb{R}^n \times \mathbb{R}^n$ preserving the symplectic form: this means, given any two vectors $Y = (y, \eta)$ and $Y' = (y', \eta')$, that one has the identity

$$[gY, gY] = [Y, Y']. \tag{1.1.9}$$

When $n = 1$ (the case that will occur most frequently here), the symplectic group coincides with $SL(2, \mathbb{R})$. There is a notion of *symplectic* Fourier transformation $\mathcal{F}^{\mathrm{symp}}$ on $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ (no such notion exists on odd-dimensional spaces, and in the one-dimensional case $\mathcal{F}$ will denote the usual Fourier transform, normalized in the way comparable to (1.1.11) below), defined by the equation

$$(\mathcal{F}^{\mathrm{symp}} h)(y, \eta) = \int_{\mathbb{R}^n \times \mathbb{R}^n} h(x, \xi) e^{2i\pi(\langle y, \xi\rangle - \langle x, \eta\rangle)} dx d\xi : \tag{1.1.10}$$

it may look not very different from the usual Euclidean Fourier transformation

$$(\mathcal{F}^{\mathrm{euc}} h)(y, \eta) = \int_{\mathbb{R}^n \times \mathbb{R}^n} h(x, \xi) e^{-2i\pi(\langle x, y\rangle + \langle \xi, \eta\rangle)} dx d\xi, \tag{1.1.11}$$

but it has the fundamental property (the verification of which is trivial) that it commutes with all transformations $h \mapsto h \circ g^{-1}$ with $g \in \mathrm{Sp}(n, \mathbb{R})$, whereas the Euclidean Fourier transformation commutes with such transformations for $g$ in the orthogonal group of $\mathbb{R}^{2n}$: these two groups cannot be compared generally, but when $n = 1$, the group $SO(2)$ is a proper subgroup of $SL(2, \mathbb{R})$. Since $(\mathcal{F}^{\mathrm{euc}})^2$ is the operator which transforms a symbol $h$ into the symbol $(x, \xi) \mapsto h(-x, -\xi)$, the symplectic Fourier transformation is an involution, i.e., $(\mathcal{F}^{\mathrm{symp}})^2 = I$.

One of several (equivalent) ways of introducing the Weyl calculus Op is based on this property, and leads to the definition

$$\mathrm{Op}(h) = \int_{\mathbb{R}^n \times \mathbb{R}^n} (\mathcal{F}^{\mathrm{symp}} h)(y, \eta) \exp\left(2i\pi(\langle \eta, Q\rangle - \langle y, P\rangle)\right) dy d\eta \tag{1.1.12}$$

if $h \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$, the Schwartz space of $C^\infty$ functions on $\mathbb{R}^{2n}$ rapidly decreasing at infinity: in this way, it is immediate that, for any function $\tilde{h} \in \mathcal{S}(\mathbb{R}^{2n})$, the operator with symbol $(x, \xi) \mapsto \int \tilde{h}(y, \eta) e^{2i\pi(\langle \eta, x\rangle - \langle y, \xi\rangle)} dy d\eta$ is the operator $\int \tilde{h}(y, \eta) \exp\left(2i\pi(\langle \eta, Q\rangle - \langle y, P\rangle)\right) dy d\eta$. Thus, in one sense, the Weyl symbolic calculus is the correspondence obtained when substituting the pair of (vector-valued) operators $(Q, P)$ to the pair of $\mathbb{R}^n$-valued functions $(x, \xi)$: but this (which could not make sense for arbitrary functions of $(x, \xi)$ for reasons of non-commutativity) is only true after the symbol has been expanded as a superposition of exponentials with linear exponents.

From (1.1.5) and (1.1.2), the integral kernel of the operator $\exp(2i\pi(\langle \eta, Q\rangle - \langle y, P\rangle))$ is the function $(x_1, y_1) \mapsto \delta(y_1 - x_1 + y) e^{2i\pi\langle x_1 - \frac{y}{2}, \eta\rangle}$, from which it follows that the integral kernel $k$ of the operator $\mathrm{Op}(h)$ is the function

$$k(x,y) = \left(\mathcal{F}_2^{-1}h\right)\left(\frac{x+y}{2}, x-y\right), \tag{1.1.13}$$

where $\mathcal{F}_2^{-1}h$ denotes the inverse Fourier transform of $h$ with respect to the second variable in $\mathbb{R}^n$, i.e., the function defined by the equation $(\mathcal{F}_2^{-1}h)(x,z) = \int h(x,\xi)e^{2i\pi\langle z,\xi\rangle}d\xi$. From this equation, it follows that, just as the map which associates an operator to its integral kernel, the map Op extends as an isometry from $L^2(\mathbb{R}^{2n})$ onto the Hilbert space of all Hilbert-Schmidt endomorphisms of $L^2(\mathbb{R}^n)$. Equation (1.1.13) leads to the more traditional way of defining the Weyl calculus, by means of the equation

$$\left(\mathrm{Op}(h)u\right)(x) = \int_{\mathbb{R}^n\times\mathbb{R}^n} h\left(\frac{x+y}{2}, \eta\right) e^{2i\pi\langle x-y,\eta\rangle}u(y)dyd\eta. \tag{1.1.14}$$

We now come to the all-important concept of *Wigner function*. Given a pair $(u,v)$ of functions in $L^2(\mathbb{R}^n)$, their Wigner function $W(v,u)$ is the function on $\mathbb{R}^n \times \mathbb{R}^n$ which makes the identity

$$(v|\mathrm{Op}(h)u) = \int_{\mathbb{R}^n\times\mathbb{R}^n} h(x,\xi)W(v,u)(x,\xi)dxd\xi \tag{1.1.15}$$

valid for every symbol $h \in L^2(\mathbb{R}^{2n})$: note that we define the scalar product $(|)$ on $L^2(\mathbb{R}^n)$ by the equation

$$(v|u) = \int_{\mathbb{R}^n} \bar{v}(x)u(x)dx, \tag{1.1.16}$$

as an object antilinear with respect to the variable on the left side. The function $W(v,u)$ can be obtained for instance by a computation of the transpose of the map $h \mapsto k$ in (1.1.13), applying the result to the function $u \otimes \bar{v}$: we obtain

$$W(v,u)(x,\xi) = 2^n \int_{\mathbb{R}^n} \bar{v}(x+t)u(x-t)e^{4i\pi\langle t,\xi\rangle}dt. \tag{1.1.17}$$

It is immediate, with the help of an integration by parts in order to treat extra powers of $\xi$, that this function lies in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ if both $u$ and $v$ lie in $\mathcal{S}(\mathbb{R}^n)$. A consequence, using (1.1.15), is that the operator $\mathrm{Op}(h)$ still makes sense as a linear operator from $\mathcal{S}(\mathbb{R}^n)$ to its dual space $\mathcal{S}'(\mathbb{R}^n)$ as soon as $h \in \mathcal{S}'(\mathbb{R}^{2n})$. On the other hand, if $h \in \mathcal{S}(\mathbb{R}^{2n})$, the operator $\mathrm{Op}(h)$ extends as a linear operator from the whole of $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$: the simplest way to see this is to observe that, in view of (1.1.13), the integral kernel of $\mathrm{Op}(h)$ also lies in $\mathcal{S}(\mathbb{R}^{2n})$ in this case. Spaces resembling the spaces $\mathcal{S}$ or $\mathcal{S}'$, on the line or on $\mathbb{R}^2$, will play an important role everywhere: the consideration of singular species of symbols, or the application of operators with smooth symbols to rather general measures on the line, is essential in applications of pseudo-differential analysis to arithmetic.

The Wigner function has another, dual, role: given $\phi, \psi$ in $L^2(\mathbb{R}^n)$, the Wigner function $W(\psi, \phi)$ is the Weyl symbol of the rank-one operator $u \mapsto (\psi|u)\phi$. This could be seen immediately from (1.1.13), since the integral kernel of the operator under consideration is the function $k(x,y) = \bar{\psi}(y)\phi(x)$. However, we prefer, since this is a general phenomenon, to remark that the fact that the same concept of Wigner function plays the two roles under consideration is a consequence of the isometry property (from $L^2(\mathbb{R}^{2n})$ to the space of Hilbert-Schmidt operators) of the Weyl calculus: it suffices indeed to apply to two rank-one operators $A_j = \mathrm{Op}(h_j)$ the polarized version $\mathrm{Tr}(A_1^* A_2) = \int_{\mathbb{R}^{2n}} \bar{h}_1(x,\xi) h_2(x,\xi)\, dx d\xi$ of the isometry property of the calculus, concluding with the help of the fact that a total subspace of the space of Hilbert-Schmidt operators consists of all rank-one operators.

The Weyl symbolic calculus benefits from two species of symmetries, or more precisely *covariance* properties. The first one is expressed in the formula

$$\tau_{y,\eta} \mathrm{Op}(h) \tau_{y,\eta}^{-1} = \mathrm{Op}((x,\xi) \mapsto h(x-y, \xi-\eta)), \quad (y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n : \qquad (1.1.18)$$

it is valid whenever $h \in \mathcal{S}'(\mathbb{R}^{2n})$, after it has been observed, of course, that the operators $\tau_{y,\eta}$ preserve both the space $\mathcal{S}(\mathbb{R}^n)$ and the space $\mathcal{S}'(\mathbb{R}^n)$. The proof is immediate, with the help of (1.1.14) and (1.1.7), together with (1.1.5).

This formula is of constant use, even though, for applications to modular form theory, the second covariance property, associated to the *metaplectic* representation, is more fundamental. This representation, defined in full generality (including the $p$-adic and adelic situations) in [66], is in the present Archimedean environment a genuine (as opposed to projective only) unitary representation Met in $L^2(\mathbb{R}^n)$ of the *metaplectic group*, by definition the twofold cover of $\mathrm{Sp}(n,\mathbb{R})$ (a connected group, the fundamental group of which is $\mathbb{Z}$). It is linked (*loc.cit.*) to the Heisenberg representation by the formula

$$\mathrm{Met}(\tilde{g}) \exp(2i\pi(\langle \eta, Q\rangle - \langle y, P\rangle)) \mathrm{Met}(\tilde{g})^{-1} = \exp(2i\pi(\langle \eta', Q\rangle - \langle y', P\rangle)), \quad (1.1.19)$$

in which $\tilde{g}$ is an arbitrary element of the metaplectic group, the canonical image of which in $\mathrm{Sp}(n,\mathbb{R})$ is $g$ (one then says that $\tilde{g}$ lies *above* $g$), and the vectors $\begin{pmatrix} y' \\ \eta' \end{pmatrix}$ and $\begin{pmatrix} y \\ \eta \end{pmatrix}$ are linked by the relation $\begin{pmatrix} y' \\ \eta' \end{pmatrix} = g \begin{pmatrix} y \\ \eta \end{pmatrix}$. Using this equation together with the definition (1.1.14) of the operator $\mathrm{Op}(h)$, it is immediate that one has, for every $h \in \mathcal{S}'(\mathbb{R}^{2n})$, the covariance formula

$$\mathrm{Met}(\tilde{g}) \mathrm{Op}(h) \mathrm{Met}(\tilde{g})^{-1}) = \mathrm{Op}(h \circ g^{-1}) : \qquad (1.1.20)$$

the symbol $h \circ g^{-1}$ is of course the one obtained from $h$ after one has applied it the linear change of coordinates on $\mathbb{R}^{2n}$ associated with $g^{-1}$. Again, the left-hand side of (1.1.20) only makes sense after it has been observed, as was done in [66], that operators in the image of the metaplectic representation preserve the space $\mathcal{S}(\mathbb{R}^n)$ and extend as automorphisms of the dual space $\mathcal{S}'(\mathbb{R}^n)$.

For our present purposes, it will be sufficient to make the metaplectic representation explicit up to an indeterminacy factor $\pm 1$ (in particular, this will make it well-defined as a projective representation): it then becomes possible to regard it as defined on the group $\mathrm{Sp}(n,\mathbb{R})$, rather than on the twofold cover of that group. In this sense, one can list the unitary transformations $\mathrm{Met}(g)$ for $g$ lying in an appropriate set of generators of $\mathrm{Sp}(n,\mathbb{R})$, as follows: (i) if $g = \left(\begin{smallmatrix} A & 0 \\ 0 & A'^{-1} \end{smallmatrix}\right)$ with $A \in GL^+(n,\mathbb{R})$ and $A'$ denoting the transpose of $A$, $\mathrm{Met}(g)$ is (plus or minus) the transform $u \mapsto v$ with $v(x) = (\det A)^{-\frac{1}{2}} u(A^{-1}x)$; (ii) if $g = \left(\begin{smallmatrix} I & 0 \\ C & I \end{smallmatrix}\right)$, where $C$ is a symmetric $(n \times n)$-matrix, the same holds with $v(x) = u(x) e^{i\pi\langle Cx,x\rangle}$; (iii) finally, if $g = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, the same holds with $v = e^{-\frac{i\pi n}{4}} \mathcal{F}^{\mathrm{euc}}$.

Note that the covariance equation (1.1.20) makes sense even if $\mathrm{Met}(\tilde{g})$ is only defined up to an arbitrary phase factor. From its (almost) explicit definition on generators, the metaplectic representation is not irreducible, but acts within $L^2_{\mathrm{even}}(\mathbb{R}^n)$ and $L^2_{\mathrm{odd}}(\mathbb{R}^n)$ separately: the two terms can then be shown to be acted upon in an irreducible way. This puts forward the role of the involution ch on $\mathcal{S}'(\mathbb{R}^n)$ such that

$$(\mathrm{ch}\, u)(x) = \check{u}(x) = u(-x), \quad x \in \mathbb{R}^n, \tag{1.1.21}$$

and of the two operations on symbols which correspond to composing an operator on both sides, or on one side only, with the operator ch. The first identity, to wit

$$\mathrm{ch}\,\mathrm{Op}(h)\mathrm{ch} = \mathrm{Op}((x,\xi) \mapsto h(-x,-\xi)), \tag{1.1.22}$$

though trivial to verify in a direct way, can also be regarded as a consequence of item (iii) in the presentation above of the metaplectic representation: using the relation $(\mathcal{F}^{\mathrm{euc}})^2 = \mathrm{ch}$, so that the operator $\pm e^{-\frac{i\pi n}{2}}\mathrm{ch}$ is an element of the metaplectic representation lying above the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, one can derive (1.1.22) from (1.1.20). This equation implies, in particular, that only even symbols must be used if one is interested only in operators on $\mathcal{S}(\mathbb{R}^n)$ preserving the parity of functions. The second identity, not a covariance equation, demands that we should compute the operation $\mathcal{G}$ on symbols making the identity

$$\mathrm{Op}(h)\mathrm{ch} = \mathrm{Op}(\mathcal{G}h), \tag{1.1.23}$$

or $\mathrm{Op}(\mathcal{G}h)u = \mathrm{Op}(h)\check{u}$, valid. It is of course easy to make this computation, for instance by using the link (1.1.13) between the Weyl symbol $h$ and the integral kernel $k$ of the same operator, together with the fact that if $k$ is the integral kernel of an operator $A$, that of $A\,\mathrm{ch}$ is the function $(x,y) \mapsto k(x,-y)$. One obtains the formula, valid in any dimension,

$$(\mathcal{G}h)(x,\xi) = 2^n \int_{\mathbb{R}^{2n}} h(y,\eta) e^{4i\pi(\langle x,\eta\rangle - \langle y,\xi\rangle)} dy d\eta. \tag{1.1.24}$$

Note that $\mathcal{G}$ is just a rescaled version (by a factor 2) of the symplectic Fourier transformation (1.1.10): when we have defined the Euler operator $2i\pi\mathcal{E}$ (1.1.39), we can connect the two transformations by the equation $\mathcal{G} = 2^{2i\pi\mathcal{E}}\mathcal{F}^{\text{symp}}$ (*cf.* (7.1.1)). The operator $\mathcal{G}$ and its $p$-adic variants will play an important role in applications of pseudo-differential analysis to number theory. In particular, we shall often use the fact that the symbol of the operator $u \mapsto \check{u}$ is $2^{-n}\delta$, where $\delta$ is the unit mass at $0 \in \mathbb{R}^{2n}$.

One can break the space of operators in $L^2(\mathbb{R}^n)$ into four parts (even-even, even-odd, odd-even and odd-odd), a self-explaining notion after we have made it clear that even-odd operators, for instance, are those which send even functions to odd ones and kill odd functions. The corresponding symbols are characterized as those being even and $\mathcal{G}$-invariant, odd and $\mathcal{G}$-invariant, odd and $\mathcal{G}$-anti-invariant, finally even and $\mathcal{G}$-anti-invariant. A remark pertinent to quantization theory as well as to applications of pseudo-differential analysis to modular form theory is that the Weyl calculus has a much nicer behaviour than any of its four parts. For instance, the Weyl symbol of an operator on the line as simple as the multiplication by $x^2$ is just, as will be seen shortly, the function $h(x,\xi) = x^2$, while that of the *even-even* part of this operator is the complicated distribution

$$\frac{1}{2}(h + \mathcal{G}h)(x,\xi) = \frac{1}{2}[x^2 - \frac{1}{16\pi^2}\delta(x)\delta''(\xi)]. \tag{1.1.25}$$

Facing this situation will have consequences throughout the book: in particular, it will explain why, from a certain point of view, it is better to let symbols live on the homogeneous space $G/N \sim \mathbb{R}^2\backslash\{0\}$ of $G = SL(2,\mathbb{R})$ (with $N = \{\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$, $b \in \mathbb{R}\}$) than on the space $G/K$ (with $K = \{\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right), \theta \bmod 2\pi\}$), a model of which is the hyperbolic half-space, or Poincaré half-space $\Pi = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$. It will also explain why dealing with appropriate pairs of non-holomorphic modular forms, rather than individual ones, in a way connected to the Lax-Phillips scattering theory for the automorphic wave equation [34], has important advantages.

Specializing from now on in this section in the one-dimensional case, let us explain our last comment, relying for this on the notion of *family of coherent states*. Forgetting the reason, having to do with Physics, which gave this notion its name, we only retain the representation-theoretic part of it: given a topological group $G$ and a unitary representation $\pi$ of $G$ in some Hilbert space $H$, a related family of coherent states will be just a family of elements of $H$ making up a total subset, permuted with one another, up to phase factors, under any operator $\pi(g)$, $g \in G$.

In order to build useful families of coherent states for each of the two irreducible parts of the metaplectic representation, we start — number theorists will remember at this point the usual Poincaré construction of modular forms — from a function already invariant, up to phase factors, under all operators $\operatorname{Met}(\tilde{g})$ for $\tilde{g}$ above any element of some "large" subgroup of $SL(2,\mathbb{R})$, in the present case the subgroup $K = SO(2)$. We first show that the pair of (normalized) functions

$$\phi_i^0(x) = 2^{\frac{1}{4}}e^{-\pi x^2}, \quad \phi_i^1(x) = 2^{\frac{3}{4}}\pi^{\frac{1}{2}}xe^{-\pi x^2} \tag{1.1.26}$$

satisfies the required invariance property (note that the superscript 0 or 1 refers to parity, and that, as will be apparent when generalized later, the subscript $i$ denotes the base-point of $\Pi$). This demands considering the all-important harmonic oscillator

$$L = \pi(Q^2 + P^2) = \pi x^2 - \frac{1}{4\pi}\frac{d^2}{dx^2}. \tag{1.1.27}$$

This operator is consistently treated in elementary Physics textbooks, with the help of the so-called creation and annihilation operators: we shall not give reminders here, and a full set of eigenfunctions of $L$ will only be needed in Section 7.2. Let us just recall that $L$ has a purely discrete spectrum without multiplicity, which implies that its full spectral resolution is caught in the list of its square-integrable eigenfunctions: the spectrum is the set $\frac{1}{2} + \mathbb{N} = \{\frac{1}{2}, \frac{3}{2}, \dots\}$, and the eigenfunctions corresponding to the two lowest eigenvalues $\frac{1}{2}$ and $\frac{3}{2}$ are the functions $\phi_i^0$ and $\phi_i^1$. One does not even need Stone's theorem in order to define the unitary group $t \mapsto \exp(-itL)$: one has in particular

$$\exp(-itL)\phi_i^0 = e^{-\frac{it}{2}}\phi_i^0, \quad \exp(-itL)\phi_i^1 = e^{-\frac{3it}{2}}\phi_i^1. \tag{1.1.28}$$

Now, it was found by Mehler (the complete reference seems, unfortunately, to have disappeared from the contemporary literature; we shall give, in a moment, a proof of the formula based on the Weyl calculus) that the operator $\exp(-itL)$ has, for $0 < t < \pi$, an explicit integral kernel $k_t$, given as

$$k_t(x, y) = e^{-\frac{i\pi}{4}}(\sin t)^{-\frac{1}{2}} \exp\left(\frac{i\pi}{\sin t}[(x^2 + y^2)\cos t - 2xy]\right): \tag{1.1.29}$$

looking at $k_{\frac{\pi}{2}}$, one obtains the equation $\exp(-\frac{i\pi}{2}L) = e^{-\frac{i\pi}{4}}\mathcal{F}$, from which the way to extend the map $t \mapsto k_t$ as a group homomorphism follows. Using the list of metaplectic unitaries given between (1.1.20) and (1.1.21), one sees after a change of variable $y \mapsto y \sin t$ that the operator with integral kernel $(x, y) \mapsto e^{-\frac{i\pi}{4}}(\sin t)^{-\frac{1}{2}} \exp\left(-\frac{2i\pi xy}{\sin t}\right)$ is one of the two metaplectic operators (one the negative of the other) lying above the matrix $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\begin{pmatrix} \frac{1}{\sin t} & 0 \\ 0 & \sin t \end{pmatrix} = \begin{pmatrix} 0 & \sin t \\ -\frac{1}{\sin t} & 0 \end{pmatrix}$; using again the case (ii) of the same list, one obtains that the operator $\exp(-itL)$ is (up to multiplication by $\pm 1$ again) a metaplectic operator lying above the matrix $\begin{pmatrix} 1 & 0 \\ \frac{1}{\tan t} & 1 \end{pmatrix}\begin{pmatrix} 0 & \sin t \\ -\frac{1}{\sin t} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{1}{\tan t} & 1 \end{pmatrix} = \left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right)$. Since such a family of matrices, taken for $0 < t < \pi$, generates the group $K = SO(2)$, it follows that the functions $\phi_i^0$ and $\phi_i^1$ are indeed invariant, up to phase factors, under all metaplectic unitary transformations above $SO(2)$.

We have considered, on several occasions, the linear action $\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right) \mapsto g\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right)$ of $g \in SL(2, \mathbb{R})$ on $\mathbb{R}^2$: we consider now the action of $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $\Pi$, to be denoted as $z \mapsto g.z$, by means of fractional-linear transformations, defined in the usual way as $z \mapsto \frac{az+b}{cz+d}$. It follows from what precedes that, up to multiplication by phase factors, the function $\mathrm{Met}(\tilde{g})\phi_i^j$ (with $j = 0, 1$) only depends, if $\tilde{g}$ lies above

$g \in SL(2,\mathbb{R})$, on the class $gK$: as is well-known and immediate, the knowledge of this class is equivalent to that of the point $z = g.i$. To make the computation simple, we choose $g = \left(\begin{smallmatrix} a & 0 \\ c & a^{-1} \end{smallmatrix}\right))$, with $a > 0$, if $z = \frac{ai}{ci+a^{-1}}$, in other words if $-z^{-1} = a^{-2}i - ca^{-1}$: then, the list of metaplectic unitaries between (1.1.20) and (1.1.21) gives

$$(\mathrm{Met}(\tilde{g})\phi_i^j)(x) = 2^{\frac{i\pi cx^2}{a}} a^{-\frac{1}{2}} \phi_i^j(a^{-1}x), \tag{1.1.30}$$

hence

$$\begin{aligned}(\mathrm{Met}(\tilde{g})\phi_i^0)(x) &= a^{-\frac{1}{2}} e^{\frac{i\pi cx^2}{a}} \phi_i^0(a^{-1}x) \\ &= 2^{\frac{1}{4}} \left(\mathrm{Im}\,(-z^{-1})\right)^{\frac{1}{4}} \exp \frac{i\pi x^2}{\bar{z}}.\end{aligned} \tag{1.1.31}$$

Doing the same, starting this time from the function $\phi_i^1$, we are led to introducing the pair of functions which occur in the following theorem.

**Theorem 1.1.1.** *Given $z \in \Pi$, set*

$$\begin{aligned}\phi_z^0(x) &= 2^{\frac{1}{4}} \left(\mathrm{Im}\,(-z^{-1})\right)^{\frac{1}{4}} \exp \frac{i\pi x^2}{\bar{z}}, \\ \phi_z^1(x) &= 2^{\frac{3}{4}} \pi^{\frac{1}{2}} \left(\mathrm{Im}\,(-z^{-1})\right)^{\frac{3}{4}} x \exp \frac{i\pi x^2}{\bar{z}}.\end{aligned} \tag{1.1.32}$$

*Given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ and any $\tilde{g}$ lying above $g$ in the metaplectic group, one has for some phase factors $\omega_0, \omega_1$ depending on $z, g$ the equations*

$$\mathrm{Met}(\tilde{g})\phi_z^0 = \omega_0 \phi^0_{\frac{az+b}{cz+d}}, \quad \mathrm{Met}(\tilde{g})\phi_z^1 = \omega_1 \phi^1_{\frac{az+b}{cz+d}}. \tag{1.1.33}$$

*The set* $\{\phi_z^0 \colon z \in \Pi\}$ (resp. $\{\phi_z^1 \colon z \in \Pi\}$) *is total in* $L^2_{\mathrm{even}}(\mathbb{R})$ (resp. $L^2_{\mathrm{odd}}(\mathbb{R})$). *Any* even *distribution $h \in \mathcal{S}'(\mathbb{R}^2)$ is characterized by the pair of functions*

$$f_0(z) = (\phi_z^0 | \mathrm{Op}(h)\phi_z^0), \quad f_1(z) = (\phi_z^1 | \mathrm{Op}(h)\phi_z^1) : \tag{1.1.34}$$

*the pair $(f_0, f_1)$ will be called the $\Theta$-transform of $h$.*

*Proof.* Equations (1.1.33) follow from the definition of $\phi_z^j$ as the image of $\phi_i^j$ under $\mathrm{Met}(\tilde{g})$ for some $\tilde{g}$ with $g.i = z$. The density claims are then a consequence of Schur's lemma and of the irreducibility of the two components of the representation Met. Actually, for the odd case only, one has a more precise result, to wit the formula, the proof of which is straightforward (a simplification will occur from the measure-preserving change $z \mapsto -z^{-1}$ on the left-hand side)

$$(8\pi)^{-1} \int_\Pi |(\phi_z^1|u)|^2 dm(z) = \|u\|^2, \quad u \in L^2_{\mathrm{odd}}(\mathbb{R}), \tag{1.1.35}$$

in which $dm(z) = (\mathrm{Im}\, z)^{-2}\, d\mathrm{Re}\, z\, d\mathrm{Im}\, z$ is the usual measure on $\Pi$ invariant under all transformations $z \mapsto g.z$. Approximating functions in $\mathcal{S}(\mathbb{R})$ with the appropriate parity by linear combinations of functions $\phi_z^j$ with $j = 0, 1$ and $z \in \Pi$, one sees,