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Signatures, Multi-state Systems and Statistical Inference

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Editors

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Signatures, Multi-state Systems and Statistical Inference

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The book is in the honor of Prof. Ilya Gertsbakh and Prof. Igor Ushakov, who have made a great pioneering contribution to all areas of reliability and, in particular, to the topics of signatures and multi-state systems.

Preface

This book covers the recent developments in modern reliability theory, mainly in such important areas as signatures and multi-state systems and their influence on statistical inference. Research in these areas is growing rapidly due to many successful applications in very diverse problems. As the result, many industries have benefited from adopting the corresponding methods.

These methods have attracted increasing attention in recent years for solving many complex problems which were inspired by nature and technology. New methods have been successfully applied to solving many complex problems where traditional problem-solving methods have failed.

This book presents new theoretical issues that were not previously presented in the literature, as well as the solutions of important practical problems and case studies illustrating the application methodology.

The book provides an overview of the recent developments in the theory of signatures and demonstrates their role in the study of dynamic reliability and nonparametric inference for lifetime distribution of monotone systems. New properties of system signatures (D-spectra) and component importance D-spectra have been investigated. It was demonstrated how component Birnbaum importance measures can be expressed via these spectra and how bounds on lifetime variances for coherent and mixed systems can be found by using signatures. In addition, it was pointed out on the connection between several aspects of probability-signature and structure-signature.

Concerning multi-state system (MSS) reliability, the book introduces a special transform for a discrete-states continuous-time Markov process, so-called L_Z-transform and demonstrates the benefits of its applications. In MSS context, there issues such as practical availability modeling, a case-study for supermarket refrigerating system, finding optimal reserve structure for power generating system, determination of vital activities in reliability program, optimal incomplete maintenance, optimal multi-objective reliability allocation, importance analysis based on multiple-valued logic methods, and optimal replacement and protection strategy were also considered. A separate chapter is devoted to the novel issue of continuous-state system reliability. Absorbing controllable Markov processes were considered as the models of aging and degradation for some technical and/or biological objects, as well as a semi-Markov model of MSS operation reliability.

The book aims to be repository for modern theoretical methods and their applications in real-world reliability analysis and optimization. Recent advances in statistical inference are presented in this volume by reliability analysis of redundant systems with unimodal hazard rate functions, nonparametric estimation of marginal temporal functionals, frailty models in survival analysis and reliability, goodness of fit tests for reliability modeling and nonparametric estimators of the transition probabilities for three-state Markov model.

All chapters are written by leading experts in the corresponding areas. This book will be useful to postgraduate and doctoral students, researchers, reliability practitioners, engineers and industrial managers with interest in reliability theory and its applications.

We wish to thank all the authors for their insights and excellent contributions to this book. We would like to acknowledge the assistance of all involved in the review process of the book, without whose support this book could not have been successfully completed. We want to thank all who participated in the reviewing process: Prof. Somnath Datta, University of Louisville, USA, Prof. Ilya Gertsbakh, Ben Gurion University of the Negev, Israel, Dr. Gregory Gurevich, SCE–Shamoon College of Engineering, Israel, Prof. Alex Karagrigoriou, University of Cyprus, Cyprus, Prof. Ron S. Kenett, KPA Ltd., Israel, Dr. Edward Korczak, Telecommunications Research Institute, Poland, Prof. Michael Peht, University of Maryland, USA, Prof. Dmitrii Silvestrov, Stockholm University, Sweden, Dr. Armen Stepanyants, Institute of Control Science, RAS, Russia, Prof. Guram Tsitsiashvili, Institute for Applied Mathematics, Eastern Branch of RAS, Russia, Dr. Valentina Victorova, Institute of Control Science, RAS, Russia, Prof. Ilia Vonta, National Technical University of Athens, Greece, Prof. David Zucker, Hebrew University of Jerusalem, Israel.

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Israel, May 2011

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Chapter 1
Signature Representation and Preservation Results for Engineered Systems and Applications to Statistical Inference

N. Balakrishnan, Jorge Navarro and Francisco J. Samaniego

Abstract The aim of this article is to provide an overview of some of the recent developments relating to the theory of signatures and their role in the study of dynamic reliability, systems with shared components and nonparametric inference for a component lifetime distribution. Some new results and interpretations are also presented in the process.

Keywords Coherent system · Mixed system · Signature · *k*-out-of-*n* system · Exchangeability · Order statistics · Stochastic ordering · Hazard rate ordering · Likelihood ratio ordering · Dynamic reliability · Dynamic signature · Systems with shared components · Burned-in systems · *D*-spectrum · Nonparametric inference · Parametric inference · Best linear unbiased estimator · Proportional hazard rate model

1.1 A Brief Overview of Signature Theory

Most work on reliability theory focuses on the study of coherent systems; see, for example, (Barlow and Proschan 1975). A reliability system is said to be a *coherent system* if

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- it is monotone in its components (i.e., replacing a failed component by a working one cannot make the system worse), and
- every component is relevant (i.e., every component influences the functioning or failure of the system).

Suppose a coherent system has n components whose lifetimes $X_1,\ldots, X_n$ are independent and identically distributed (i.i.d.) continuous random variables with distribution function F. Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics obtained by arranging the component lifetimes X_i's in increasing order of magnitude. Then, the system lifetime T will coincide with an order statistic $X_{i:n}$ for some $i \in \{1,\ldots, n\}$, which led to the following notion of *system signature* in a natural way. Let s_i, for $i = 1,\ldots, n$, be such that $P(T = X_{i:n}) = s_i$. Then, the system signature is simply the vector $\mathbf{s} = (s_1,\ldots, s_n)$, as introduced by Samaniego (1985). Signatures are most useful in the comparison of system designs as amply demonstrated in the books by Samaniego (2007) and Gertsbakh and Shpungin (2010). The system signature is a pure distribution-free measure of a system's design, and it is important to recognize that it dissociates the quality or reliability of the components from that of the system. The comparison of systems otherwise becomes very difficult since a series system with 4 good components can outperform a parallel system with 4 poor components. Thus, signature vectors enable us to compare the performance characteristics of different systems in a complete nonparametric way without reference to the lifetime distribution of the components.

The signature vector facilitates the following representation for the system reliability.

Theorem 1 (Samaniego 1985) *Let $X_1,\ldots, X_n \sim F$ be the i.i.d. component lifetimes of a coherent system of order n, and let T be the system lifetime and* $\mathbf{s}$ *its signature. Then,*

$$\begin{aligned}\overline{F}_T(t) &= \Pr(T > t) = \sum_{i=1}^{n} s_i \Pr(X_{i:n} > t) \\ &= \sum_{i=1}^{n} s_i \sum_{j=0}^{i-1} n\binom{n}{j}\{F(t)\}^j\{1 - F(t)\}^{n-j}, \quad \textit{for} \quad t > 0. \end{aligned} \tag{1.1}$$

From representation (1.1), we also readily see, for example, that

$$E(T) = \sum_{i=1}^{n} s_i E(X_{i:n}).$$

Signatures also become useful in determining the reliability of one coherent system relative to another as demonstrated in the following representation result.

Theorem 2 (Hollander and Samaniego 2008) *Let T_1 and T_2 represent the lifetimes of coherent systems of orders n and m, with respective signatures $s_1 = (s_{1,1},\ldots, s_{1,n})$ and $s_2 = (s_{2,1},\ldots, s_{2,m})$ and ordered component lifetimes* $\{X_{1:n},\ldots, X_{n:n}\}$ and $\{Y_{1:m},\ldots, Y_{m:m}\}$ *drawn from independent i.i.d. samples from a common continuous distribution F. Then,*

$$\Pr(T_1 \le T_2) = \sum_{i=1}^{n}\sum_{j=1}^{m} s_{1,i}s_{2,j} \Pr\left(X_{i:n} \le Y_{j:m}\right). \tag{1.2}$$

Since the precedence probability $\Pr\left(X_{i:n} \le Y_{j:m}\right)$ is known to be

$$\Pr\left(X_{i:n} \le Y_{j:m}\right) = \frac{1}{\binom{m+n}{n}} \sum_{\ell=i}^{n} \binom{j+\ell-1}{\ell}\binom{m+n-j-\ell}{n-\ell},$$

readily representation (1.2) shows that the relative reliability of two systems can be readily determined from their respective signatures. More on such representation results and also preservation results in terms of signatures are presented in the subsequent sections.

The above representations can be stated in a more general way through mixed systems. A *mixed system*, as defined by Boland and Samaniego (2004) is a stochastic (ST) mixture of coherent systems; so, any probability vector $\mathbf{s}$ in the simplex

$$\left\{\mathbf{s} \in [0,1]^n : \sum_{i=1}^{n} s_i = 1\right\}$$

is the signature of a mixed system. Although mixed systems are not physical systems and are realized in practice only by using a randomization device which chooses a coherent system according to a fixed discrete probability distribution, it facilitates the study of reliability characteristics of systems in a general framework.

With this brief overview of signature theory, we are ready to proceed to the main discussions of this paper. The rest of the paper is organized as follows. In Sect. 1.2, we describe some general representation and preservation results based on system signatures. In Sect. 1.3, we discuss the concept of dynamic reliability and the notion of dynamic signature and associated representation and preservation results. Next, in Sect. 1.4, we consider the situation of two systems sharing some components and introduce the idea of joint signatures and then describe some distributional results and properties associated with them. Finally, in Sect. 1.5, we describe both nonparametric and parametric methods of inference for component lifetime distributions based on system lifetime data under the assumption that the system signature is known.

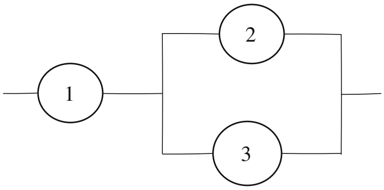

Fig. 1.1 Coherent system with lifetime $T = \min\,[X_1, \max\,(X_2, X_3)]$

1.2 Signature-Based Representation and Preservation Results

We refer the readers to the book by Barlow and Proschan (1975) for basic details on the coherent system reliability theory. There, it is stated (see page 12) that if T is the lifetime of a coherent system with component lifetimes $X_1, \ldots, X_n$ and minimal path sets $P_1, \ldots, P_k$, then

$$T = \max_{j=1,\ldots,k} X_{P_j}, \tag{1.3}$$

where $X_{P_j} = \min_{i \in P_j} X_i$ is the lifetime of the series system with components in P_j, for $j = 1, \ldots, k$. A set $P \subseteq \{1, \ldots, n\}$ is a *path set* of a coherent system if the system works when all the components in P work. A path set P is a *minimal path set* if it does not contain other path sets (see Barlow and Proschan 1975, p. 9 or Gertsbakh and Shpungin 2010, p. 38). For example, the minimal path sets of the series–parallel system $T = \min[X_1, \max(X_2, X_3)]$ (see Fig. 1.1) are $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$. A k-out-of-n system is a system which works if at least k of its n component works. Thus, the minimal path sets of a k-out-of-n, for $k = 1, \ldots, n$, system are all k-element subsets of $\{1, \ldots, n\}$. It is evident that the lifetimes of the k-out-of-n systems are the ordered component lifetimes $X_{1:n} < \cdots < X_{n:n}$, respectively.

From (1.3), it is easy to prove by means of inclusion–exclusion formula that the reliability function $\overline{F}_T(t) = \Pr(T > t)$ of the system can be expressed as

$$\overline{F}_T(t) = \sum_{j=1}^{k} \overline{F}_{P_j}(t) - \sum_{i<j} \overline{F}_{P_i \cup P_j}(t) + \cdots + (-1)^{k+1} \overline{F}_{P_1 \cup \cdots \cup P_k}(t), \tag{1.4}$$

where $\overline{F_P}$ stands for the reliability function of the series system lifetime $X_P = \min_{i \in P} X_i$ for $P \subseteq \{1, \ldots, n\}$. This representation can be traced back to (Agrawal and Barlow 1984). Note that the system reliability function is a linear combination of reliability functions of series systems with positive and negative coefficients (that sum to one). Such representations are called *generalized mixtures* since mixtures usually involve only positive coefficients. It may also be noted that representation (1.4) holds for general coherent systems (without any assumption about the components). For example, the reliability function of the system depicted in Fig. 1.1 can be expressed as

$$\overline{F}_T(t) = \overline{F}_{\{1,2\}}(t) + \overline{F}_{\{1,3\}}(t) - \overline{F}_{\{1,2,3\}}(t). \tag{1.5}$$

If the random vector $(X_1,\ldots, X_n)$ has exchangeable component lifetimes (i.e., $(X_1,\ldots, X_n)$ is equal in distribution to $[X_{\sigma(1)},\ldots, X_{\sigma(n)}]$ for any permutation σ of the set $\{1,\ldots, n\}$), then $\overline{F}_P = \overline{F}_Q$ whenever P and Q have the same cardinality. In this case, representation (1.4) can be simplified to

$$\overline{F}_T(t) = \sum_{j=1}^{n} a_j \overline{F}_{1:j}(t), \tag{1.6}$$

where $\overline{F}_{1:j}(t) = \Pr(X_{1:j} > t)$ is the reliability function of the series system lifetime $X_{1:j} = \min\,(X_1,\ldots, X_j)$ and $a_1,\ldots, a_n$ are some coefficients that depend only on the structure of the system. Note that some coefficients can be negative and so here again we have a generalized mixture representation. The vector $\mathbf{a} = (a_1,\ldots, a_n)$ has been termed the *minimal signature* in Navarro et al. (2007). An analogous representation can also be presented by using the reliability functions of parallel systems, which has been termed the *maximal signature* in Navarro et al. (2007). For example, if the system depicted in Fig. 1.1 has exchangeable components, then representation (1.5) simplifies to

$$\overline{F}_T(t) = 2\overline{F}_{1:2}(t) - \overline{F}_{1:3}(t), \tag{1.7}$$

which gives the minimal signature of T to be (0, 2, -1). The minimal signatures of all coherent systems with 1–5 components have been tabulated in Navarro and Rubio (2010a, b).

In particular, if the component lifetimes are i.i.d., then representation (1.6) can be simplified to

$$\overline{F}_T(t) = \sum_{j=1}^{n} a_j \overline{F}^j(t) = p\big(\overline{F}(t)\big), \tag{1.8}$$

where $\overline{F}(t) = \Pr(X_1 > t)$ is the common reliability function of the component lifetimes and $p(x) = \sum_{j=1}^{n} a_j x^j$ is the *reliability polynomial* of the system. This representation for the i.i.d case was obtained in (Birnbaum et al. 1966, Esary and Proschan 1963, Satyarananaya and Prabhakar 1978) and the coefficients in this polynomial are called *dominations*. For example, if the components of the system in Fig. 1 are independent, then representation (1.7) simplifies to

$$\overline{F}_T(t) = 2\overline{F}^2(t) - \overline{F}^3(t) = p\big(\overline{F}(t)\big),$$

where the reliability polynomial is $p(x) = 2x^2 - x^3$.

Evidently, the minimal signature of the series system $X_{1:n}$ is (0, 0,…,0, 1). The lifetime of the $(n-1)$-out-of-n system is $X_{2:n}$ and its minimal path sets are $P_i = \{1,\ldots, n\} - \{i\}$, for $i = 1,\ldots, n$. Hence, from (1.3), its minimal signature is (0, 0,…, 0, n, $1-n$). In general, the minimal path sets of $X_{i:n}$ are all $(n-i+1)$-element subsets of $\{1,\ldots, n\}$. It is then easy to see from (1.3) that its minimal signature $\mathbf{a} = (a_1,\ldots, a_n)$ satisfies $a_1 = \cdots = a_{n-i} = 0$ and $a_{n-i+1} = \binom{n}{n-i+1}$.

The complete set of coefficients can be obtained from the expressions given in David and Nagaraja (2003), p. 46. Therefore, if the component lifetimes are exchangeable, then the vector $\overline{\mathbf{F}}_{\mathrm{OS}} = (\overline{F}_{1:n}, \ldots, \overline{F}_{n:n})$ of reliability functions of lifetimes of k-out-of-n systems (order statistics) can be obtained from the vector $\overline{\mathbf{F}}_{\mathrm{SER}} = (\overline{F}_{1:1}, \ldots, \overline{F}_{1:n})$ of reliability functions of lifetimes of series systems as $\overline{\mathbf{F}}_{\mathrm{OS}} = \overline{\mathbf{F}}_{\mathrm{SER}} A_n$, where A_n is a non-singular $n \times n$ triangular matrix. Conversely, $\overline{\mathbf{F}}_{\mathrm{SER}}$ can also be obtained from $\overline{\mathbf{F}}_{\mathrm{OS}}$ as $\overline{\mathbf{F}}_{\mathrm{SER}} = \overline{\mathbf{F}}_{\mathrm{OS}} A_n^{-1}$, where A_n^{-1} is the inverse matrix of A_n. Hence, in this exchangeable case, upon replacing $\overline{F}_{1:1}, \ldots, \overline{F}_{1:n}$ by $\overline{F}_{1:n}, \ldots, \overline{F}_{n:n}$ in (1.6), we obtain

$$\overline{F}_T(t) = \sum_{j=1}^{n} s_j \overline{F}_{j:n}(t), \tag{1.9}$$

where the coefficients $s_1, \ldots, s_n$ depend only on the structure of the system. Hence, these coefficients should be the same as those in (1.1) for the i.i.d. continuous case. This shows that the coefficients in (1.9) are non-negative and consequently (1.9) is indeed a mixture representation. In fact, these coefficients are such that $s_i = \Pr\,(T = X_{i:n})$ for $i = 1, \ldots, n$, whenever the random vector $(X_1, \ldots, X_n)$ has a joint absolutely continuous distribution (see Navarro and Rychlik 2007). However, this is not necessarily the case when $(X_1, \ldots, X_n)$ has an arbitrary exchangeable joint distribution. For example, if we consider the series system $T = X_{1:2} = \min\,(X_1, X_2)$, evidently

$$\overline{F}_{1:2}(t) = 1 \cdot \overline{F}_{1:2}(t) + 0 \cdot \overline{F}_{2:2}(t),$$

and so (1, 2) is its signature vector. However, if X_1 and X_2 are i.i.d. with a common Bernoulli distribution with parameter $p \in (0, 1)$ (i.e., $\Pr\,(X_i = 1) = p$ and $\Pr\,(X_i = 0) = 1 - p$ for $i = 1, 2$), then $\Pr\,(T = X_{2:2}) = p^2 + (1 - p)^2 \neq 0$. Consequently, representation (1.9) obtained in Navarro et al. 2008a, b) extends representation (1.1) for coherent systems having arbitrary exchangeable components, but by using the signature vector obtained in the i.i.d. continuous case. The coefficients s_i can be obtained from the domination coefficients a_i by using the matrix A_n or through the general expressions presented in (Boland et al. 2003).

Finally, we show how a system with n exchangeable components can also be represented as a mixture of ordered lifetimes from m similar components. This property will enable us to compare systems of different sizes. Let T be the lifetime of a coherent system with component lifetimes $X_1, \ldots, X_n$, and let $(X_1, \ldots, X_m)$ be an exchangeable random vector (with $m \geq n$) comprising component lifetimes. Now, recall from (1.6) that the reliability function of the system is a linear combination of $\overline{F}_{1:1}, \ldots, \overline{F}_{1:n}$, and consequently it is also a linear combination of $\overline{F}_{1:1}, \ldots, \overline{F}_{1:m}$ $(m \geq n)$. Then, by using the fact that $\overline{F}_{1:1}, \ldots, \overline{F}_{1:m}$ can be obtained from $\overline{F}_{1:m}, \ldots, \overline{F}_{m:m}$ (using a matrix A_m), we readily have

$$\overline{F}_T(t) = \sum_{j=1}^{m} s_j^{(m)} \overline{F}_{j:m}(t), \tag{1.10}$$

where the coefficients $s_1^{(m)},\ldots, s_m^{(m)}$ depend only on the structure of the system. These coefficients can be computed either by using the triangle rule of order statistics or by using the general formulas presented in Samaniego (2007), Navarro et al. (2008a, b). In fact, it can be proved that if $(X_1,\ldots, X_m)$ has an absolutely continuous joint distribution, then the coefficients $s_j^{(m)}$ are such that $s_j^{(m)} = \Pr(T = X_{j:m})$. Thus, $s_j^{(m)}$ are non-negative and (1.10) shows that T is equal in distribution to a mixture of k-out-of-m systems with component lifetimes $X_1,\ldots, X_m$. The vector $\mathbf{s}^{(m)} = [s_1^{(m)},\ldots, s_m^{(m)}]$ is called *signature of order m* for the system. Of course, the signature of order $m = n$ coincides with the usual signature presented in (1.1).

The mixture representation results can be used to carry out ST comparisons of systems by using signatures. The first result in this direction was obtained by Kochar et al. (1999) for coherent systems with i.i.d. components, which is stated below in Theorem 3. We refer the readers to the book by Shaked and Shanthikumar (2007) for definitions and various properties of the ST, hazard rate (HR), mean residual life (MRL) and likelihood ratio (LR) orders that are pertinent to subsequent discussions.

Theorem 3 (Kochar et al. 1999) *If* $T_1 = \phi_1(X_1,\ldots, X_n)$ *and* $T_2 = \phi_2(X_1,\ldots, X_n)$ *are the lifetimes of two coherent systems with respective signatures* $\mathbf{s}_1 = (s_{1,1},\ldots, s_{1,n})$ *and* $\mathbf{s}_2 = (s_{2,1},\ldots, s_{2,n})$ *and* $X_1,\ldots, X_n$ *are i.i.d. with a common continuous distribution F, then the following properties hold*:

- *If* $\mathbf{s}_1 \le_{\mathrm{ST}} \mathbf{s}_2$, *then* $T_1 \le_{\mathrm{ST}} T_2$;
- *If* $\mathbf{s}_1 \le_{\mathrm{HR}} \mathbf{s}_2$, *then* $T_1 \le_{\mathrm{HR}} T_2$;
- *If F is absolutely continuous and* $\mathbf{s}_1 \le_{\mathrm{LR}} \mathbf{s}_2$, *then* $T_1 \le_{\mathrm{LR}} T_2$.

These preservation results were extended to the exchangeable case by Navarro et al. (2008a, b). Moreover, their results, presented below in Theorem 4, can also be used to compare systems of different sizes by using the concept of signature of order m described above.

Theorem 4 (Navarro et al. 2008a, b) *If* $T_1 = \phi_1(Y_1,\ldots,Y_{n_1})$ *and* $T_2 = \phi_2(Z_1,\ldots,Z_{n_2})$ *are the lifetimes of two coherent systems with respective signatures of order n* [*for* $n \ge \max(n_1, n_2)$] $\mathbf{s}_1 = (s_{1,1},\ldots, s_{1,n})$ *and* $\mathbf{s}_2 = (s_{2,1},\ldots, s_{2,n})$, $\{Y_1,\ldots,Y_{n_1}\}$ *and* $\{Z_1,\ldots,Z_{n_2}\}$ *are contained in* $(X_1,\ldots, X_n)$ *with* $(X_1,\ldots, X_n)$ *having an exchangeable joint distribution, then the following properties hold*:

- *If* $\mathbf{s}_1 \le_{\mathrm{ST}} \mathbf{s}_2$, *then* $T_1 \le_{\mathrm{ST}} T_2$;
- *If* $\mathbf{s}_1 \le_{\mathrm{HR}} \mathbf{s}_2$ *and*

$$X_{1:n} \le_{\mathrm{HR}} \cdots \le_{\mathrm{HR}} X_{n:n}, \tag{1.11}$$

then $T_1 \le_{\mathrm{HR}} T_2$;
- *If* $\mathbf{s}_1 \le_{\mathrm{HR}} \mathbf{s}_2$ *and*

$$X_{1:n} \leq_{\mathrm{MRL}} \cdots \leq_{\mathrm{MRL}} X_{n:n}, \tag{1.12}$$

then $T_1 \leq_{\mathrm{MRL}} T_2$*;*

- *If* $(X_1,\ldots,X_n)$ *has an absolutely continuous joint distribution,* $\mathbf{s}_1 \leq_{\mathrm{LR}} \mathbf{s}_2$ *and*

$$X_{1:n} \leq_{\mathrm{LR}} \cdots \leq_{\mathrm{LR}} X_{n:n}, \tag{1.13}$$

- *then* $T_1 \leq_{\mathrm{LR}} T_2$.

These properties are proved by using the representation (1.10) and the mixture preservation results in Shaked and Shanthikumar (2007). It should be noted that in Theorem 4 we need the ordering properties (1.11), (1.12) and (1.13) for the order statistics which need not hold for exchangeable distributions (see Navarro and Shaked 2006). However, since these ordering properties hold in the i.i.d. case, they can be dropped from the statement of the theorem in this case. Also, observe that the condition $\mathbf{p} \leq_{\mathrm{HR}} \mathbf{q}$ is required for the MRL ordering property.

1.3 Dynamic Reliability with Representation and Preservation Results

A system that is working at time t has a profile at that point in time which would differ from its profile at time 0 when the system was new. This may be due to the aging of its components which, if not for the exponential lifetime distribution, will typically result in poorer performance than when the components were new. But there may also be a change in the system itself which, while still working, may be operating with one or more failed components. In this section, we will describe some recent work aimed at characterizing the lifetime characteristics of working used systems at a particular inspection time t at which some information about the system and its components may have become available. There are many different formulations possible for this problem, and here we shall discuss three of them. For a detailed discussion on these problems and their solutions, one may refer to the recent works of Navarro et al. (2008a, b) and Samaniego et al. (2009).

Before embarking on our intended survey, it is useful to make some remarks on a slightly broader applicability of system signatures than has been typical in the existing literature on this topic. The typical definition of the signature of a system begins with the assumption that the system is *coherent*, as described, for example, in Sect. 1.1. However, if several components have failed by time t during which the system has been in service, the used system may in fact no longer be coherent. The monotonicity of the original system will of course be inherited by the used system, but it is no longer true that every component is necessarily relevant. This is apparent from the following simple example. Suppose the 4-component coherent system depicted in Fig. 1.2 is put into service.

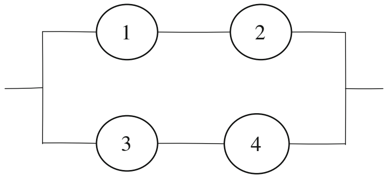

Fig. 1.2 The 4-component parallel-series coherent system with lifetime $T = \max[\min(X_1, X_2), \min(X_3, X_4)]$

Suppose, at time t, it is noted that the system is still working, but that Component 1 has failed. It is clear that Component 2 is now irrelevant to the functioning of the used system, and that the system now behaves exactly as a 2-component series system with Components 3 and 4. Thus, at time t, such a used system is no longer coherent, but is still a monotone 3-component system. The natural question that arises then is whether the notion of system signature is applicable to a used incoherent system, and fortunately the answer to this question is *affirmative*. In the above example, by denoting the failure times of the three working components at time t as Y_1, Y_2 and Y_3, and their ordered values by $Y_{1:3}$, $Y_{2:3}$ and $Y_{3:3}$, one may easily compute the signature vector $\mathbf{s}$, with $s_i = \Pr(T = Y_{i:3})$ for $i = 1, 2, 3$, as $\mathbf{s} = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$. Indeed, it is straightforward to show that the representation and preservation theorems for system signatures described in Sect. 1.2 apply in the broader context of monotone systems. In what follows, we will use the standard definition and notation for system signatures without making specific reference to whether the system in question is coherent or simply monotone.

We now turn our attention to the ST behavior of working used systems. We focus, first, on the case in which the system is inspected at time t and is simply noted to be working. This is simply equivalent to studying the system residual lifetime $T–t$, given that it is known that $T > t$. Navarro et al. (2008a, b) established that the following representation of the conditional residual reliability of a system, given $T > t$, holds:

$$\Pr(T - t > x | T > t) = \sum_{i=1}^{n} p_i(t) \Pr(X_{i:n} - t > x | X_{i:n} > t), \tag{1.14}$$

where $p_i(t) = s_i \Pr(X_{i:n} > t | T > t)$, with $\mathbf{s}$ being the signature of the system when new and $X_{i:n}$ being the ith order statistic (for $i = 1,\ldots, n$) of the lifetimes of the n i.i.d components of the original system. It should be noted that the vector $\mathbf{p}(t) = [p_1(t),\ldots, p_n(t)]$ depends on the system design (through s) as well as the common component lifetime distribution F. In this way, unlike the signature of a new system, this signature vector is a not a distribution-free measure. In this case, the following preservation theorem has been established in Navarro et al. (2008a, b).

Theorem 5 (Navarro et al. 2008a, b) *Let $p_1(t)$ and $p_2(t)$ be the vectors of the coefficients in representation (1.14) of two coherent systems, both based on n components with i.i.d. lifetimes distributed according to a common continuous distribution F, and let T_1 and T_2 be their respective lifetimes. Then, the following properties hold*:

- *If* $p_1(t) \leq_{\mathrm{ST}} p_2(t)$, *then* $(T_1 - t | T_1 > t) \leq_{\mathrm{ST}} (T_2 - t | T_2 > t)$;
- *If* $p_1(t) \leq_{\mathrm{HR}} p_2(t)$, *then* $(T_1 - t | T_1 > t) \leq_{\mathrm{HR}} (T_2 - t | T_2 > t)$;
- *If F is absolutely continuous and* $p_1(t) \leq_{\mathrm{LR}} p_2(t)$, *then*

$$(T_1 - t | T_1 > t) \leq_{\mathrm{LR}} (T_2 - t | T_2 > t).$$

Navarro et al. (2008a, b) also discussed the behavior of a working used system at time t, but with the information that at least i components have failed by time t. They obtained the following representation theorem in this case.

Theorem 6 (Navarro et al. 2008a, b) *If T is the lifetime of a coherent system with n i.i.d components having a common continuous distribution function F and* $i \in \{1, 2, \ldots, n-1\}$ *such that* $P(T > t, X_{i:n} < t) > 0$, *then there exist coefficients* $p_1(t, i), \ldots, p_n(t, i)$ (*that depend on F*) *such that* $\sum_{j=1}^{n} p_j(t, i) = 1$ *and*

$$\Pr(T - t > x | T > t, X_{i:n} < t) = \sum_{j=1}^{n} p_j(t, i) \Pr(X_{j:n} - t > x | X_{j:n} > t) \qquad (1.15)$$

for all $x \geq 0$.

Some coefficients in (1.15) can be negative and it is therefore a generalized mixture representation. One may refer to (Navarro et al. 2008a, b) for some comments on the interpretation and computation of the coefficients $p(t, i)$, as well as a preservation result for the ST order.

A detailed study of *dynamic system signatures* has been made by Samaniego et al. (2009). While the coefficients in representations (1.14) and (1.15) depend on both the system design and the underlying component distribution F, it is of natural interest to obtain the signature of a used system working at the inspection time t that is distribution-free and is therefore a measure on the design, just as the signature of a new system is. This becomes possible under a different form of conditioning than those considered above. The following concept of the dynamic signature of a used system working at time t, given that exactly i components of the system have failed by time t, has been formulated in Samaniego et al. (2009).

Theorem 7 (Samaniego et al. 2009) *Let* **s** *be the signature of a coherent system based on n components with i.i.d lifetimes having a common continuous distribution F. Let T be the system lifetime and let* $X_{k:n}$ (*for* $k = 1, \ldots, n$) *be the kth ordered component lifetime. Moreover, let* $E_i = \{X_{i:n} \leq t < X_{i+1:n}\}$ *for* $i \in \{0, \ldots, n-1\}$, and assume that $\Pr(E_i \cap \{T > t\}) > 0$. *Then, the dynamic signature of the system, given* $E_i \cap \{T > t\}$, *is the* $(n-i)$*-dimensional vector* $\mathbf{s}_{n-i}(n-i)$ *with its elements as*

$$s_{n-i,k}(n-i) = \Pr(T = X_{k:n} | E_i \cap \{T > t\}) = \frac{s_k}{\sum_{j=i+1}^{n} s_j}, \quad \text{for } k = i+1, \ldots, n. \qquad (1.16)$$

Remark The ratio on the RHS of (1.16) is in fact the conditional probability mass function of an integer-valued random variable Y whose distribution function is given by the cumulative signature, which is called *D-spectrum* (see Gertsbakh and Shpungin 2010). In their setting, Y stands for the random number of sequentially destroyed components needed to cause the failure of the system.

From Theorem 7, the following representation result is readily deduced (see Samaniego et al. 2009):

$$\Pr(T > t + x | E_i \cap \{T > t\}) = \sum_{j=1}^{n-i} s_{n-i,j}(n-i)\overline{G}_{j:n-i|t}(x), \quad \text{for } x > 0, \qquad (1.17)$$

where $\overline{G}_{j:n-i|t}(x)$ is the reliability function of the jth order statistic from a random sample of size $n-i$ from the population with conditional reliability function $\overline{G}(x|t) = \overline{F}(x+t)/\overline{F}(t)$ and $s_{n-i,j}(n-i)$ is the jth element of the signature vector $\mathbf{s}_{n-i}(n-i)$ of the working used system with exactly i failed components. A similar representation, but of order n, is given by

$$\Pr(T > t + x | E_i \cap \{T > t\}) = \sum_{j=1}^{n-i} s_{n,j}(n-i)\overline{G}_{j:n|t}(x), \quad \text{for } x > 0, \qquad (1.18)$$

where $\overline{G}_{j:n|t}(x)$ is the reliability function of the jth order statistic from a random sample of size n from the population with conditional reliability function $\overline{G}(x|t) = \overline{F}(x+t)/\overline{F}(t)$. The vector $\mathbf{s}_n(n-i)$ with elements $s_{n,j}(n-i)$ in representation (1.18) is called the *dynamic signature of order n*. Under an i.i.d. assumption ($\sim$F) on component lifetimes, Samaniego (2007, p.32) proved that, for any coherent system of size k, and for $n>k$, there is a coherent system of size n with the same lifetime distribution. In the notation used here, $s_k(k)$ is the signature of the system of size k and $s_n(k)$ is the signature of the system of size n that is equivalent to it. Using this vector, the following preservation theorem has been established in Samaniego et al. (2009).

Theorem 8 (Samaniego et al. 2009) *Let* $\mathbf{s}_n^{(1)}(n-i)$ *and* $\mathbf{s}_n^{(2)}(n-j)$ *be the dynamic signatures of order n of two coherent systems, both based on n components with i.i.d. lifetimes having a common continuous distribution F. Let* T_1 *and* T_2 *be their respective lifetimes and suppose both systems are working at the inspection time t and have exactly i and j failed components, respectively, by time t. Then, the following properties hold*:

- *If* $\mathbf{s}_n^{(1)}(n-i) \leq_{ST} \mathbf{s}_n^{(2)}(n-j)$, *then* $(T_1 - t | T_1 > t, E_i) \leq_{\mathrm{ST}} (T_2 - t | T_2 > t, E_j)$;
- *If* $\mathbf{s}_n^{(1)}(n-i) \leq_{\mathrm{HR}} \mathbf{s}_n^{(2)}(n-j)$, *then* $(T_1 - t | T_1 > t, E_i) \leq_{\mathrm{HR}} (T_2 - t | T_2 > t, E_j)$;
- *If F is absolutely continuous and* $\mathbf{s}_n^{(1)}(n-i) \leq_{\mathrm{LR}} \mathbf{s}_n^{(2)}(n-j)$, *then* $(T_1 - t | T_1 > t, E_i) \leq_{\mathrm{LR}} (T_2 - t | T_2 > t, E_j)$.

Signature-based necessary and sufficient conditions for various orderings of the residual lifetimes of the systems compared in the above theorem have also been given by Samaniego et al. (2009).

The representation of the residual reliability of a used system working at time t, given $X_{i:n} \leq t < X_{i+1:n}$, facilitates a novel study of certain well-known notions of aging. In the definitions below, drawn from Samaniego et al. (2009), the notions of *conditional New Better than Used* (i-NBU) lifetime distributions and *Uniformly* NBU (UNBU) lifetime distributions are introduced.

Definition 1 Consider a coherent system based on n components with i.i.d. lifetimes $X_1, \ldots, X_n \sim F$, where F is a continuous distribution with support $(0, \infty)$. Let T be the system's lifetime, and let $E_i = \{X_{i:n} \leq t < X_{i+1:n}\}$, where $X_{0:n} \equiv 0$. For fixed $i \in \{0, \ldots, n-1\}$, T is conditionally NBU, given i failed components, (denoted by i-NBU) if for all $t > 0$, either

- $\Pr(E_i \cap \{T > t\}) = 0$ or
- $\Pr(E_i \cap \{T > t\}) > 0$ and

$$\Pr(T > x) \geq \Pr(T > x + t | E_i \cap \{T > t\}) \quad \text{for all} \quad x > 0. \tag{1.19}$$

Definition 2 An n component coherent system is said to be UNBU if it is i-NBU for $i \in \{0, 1, \ldots, n-1\}$.

Sufficient conditions on the common component lifetime distribution F and on the system's dynamic signatures to ensure that the system is UNBU have also been given by Samaniego et al. (2009), and their result is as follows.

Theorem 9 (Samaniego et al. 2009**)** *Let* $\mathbf{s}_n(n-i)$, $i = 0, \ldots, n-1$, *be the dynamic signatures and T be the lifetime of a coherent system based on n components whose lifetimes are i.i.d. with common continuous distribution F. Assume that F is NBU and that*

$$\mathbf{s}_n(n) \geq_{ST} \mathbf{s}_n(n-i) \quad \text{for } i = 1, \ldots, n-1. \tag{1.20}$$

Then, the system is UNBU.

An interesting application of these dynamic signatures to evaluation of burned-in systems has also been discussed by Samaniego et al. (2009). The engineering practice of *burn-in* is widely used as a vehicle for weeding out poor systems or poor components before a product is deployed or released for sale. The testing of new computer software for bugs that might be detected and removed constitutes a prototypical example. Using a *performance per unit cost* criterion, the options of fielding a new system or fielding a system burned into the ith component failure (that is, to $X_{i:n}$) have been compared in Samaniego et al. (2009). For an n component system with i.i.d. component lifetimes ($\sim F$), three modeling scenarios have been investigated: F is exponential, F is increasing failure rate (IFR) Weibull (i.e., with shape parameter larger than 1) and F is increasing failure rate (DFR) Weibull (i.e., with shape parameter less than 1). Conditions are identified in which a system burned into the ith component failure, for some given $i \in \{1, \ldots, n-1\}$, will provide better performance per unit cost than a new system. The answers obtained are shown to depend critically on the relationship between the fixed cost A of building the system and the cost C of each of its components. We refer the readers to Samaniego et al. (2009) for further details.

1.4 Joint Signatures and Systems with Shared Components

Let us consider two coherent systems with lifetimes $T_1 = \phi_1(Y_1, \ldots, Y_{n_1})$ and $T_2 = \phi_2(Z_1, \ldots, Z_{n_2})$, where $\{Y_1, \ldots, Y_{n_1}\}$ and $\{Z_1, \ldots, Z_{n_2}\}$ are the respective sets of component lifetimes. We shall assume that $\{Y_1, \ldots, Y_{n_1}\}$ and $\{Z_1, \ldots, Z_{n_2}\}$ are contained in $\{X_1, \ldots, X_n\}$, where $X_1, \ldots, X_n$ are i.i.d. with a common distribution function F. Under this setup, note that T_1 and T_2 may share some components and thus can be dependent. The dependence can be represented by their joint distribution function

$$G(t_1, t_2) = \Pr(T_1 \le t_1, T_2 \le t_2).$$

Two cases of special interest are (a) when $T_1 = X_{i:n}$ for a fixed $i \in \{1, \ldots, n\}$ and $T_1 < T_2$, and (b) when $T_1 = X_i$ for a fixed $i \in \{1, \ldots, n\}$ and $X_i < T_2$.

The joint distribution function G of these two coherent systems can also be represented in terms of the distribution functions $F_{1:n}, \ldots, F_{n:n}$ of the associated k-out-of-n system lifetimes. This result, due to Navarro et al. (2010), is as follows.

Theorem 10 (Navarro et al. 2010) *The joint distribution function G of T_1 and T_2 can be written as*

$$G(t_1, t_2) = \sum_{i=1}^{n} \sum_{j=0}^{n} s_{i,j} F_{i:n}(t_1) F_{j:n}(t_2) \text{ for } t_1 \le t_2 \tag{1.21}$$

and

$$G(t_1, t_2) = \sum_{i=0}^{n} \sum_{j=1}^{n} s^*_{i,j} F_{i:n}(t_1) F_{j:n}(t_2) \text{ for } t_1 > t_2, \tag{1.22}$$

where $F_{0:n} = 1$ (*by convention*) *and* $\{s_{i,j}\}$ *and* $\{s^*_{i,j}\}$ *are collections of coefficients* (*which do not depend on* F) *such that* $\sum_{i=1}^{n} \sum_{j=0}^{n} s_{i,j} = \sum_{i=0}^{n} \sum_{j=1}^{n} s^*_{i,j} = 1$.

The proof given in Navarro et al. (2010) is based on the minimal cut set representation obtained in Barlow and Proschan (1975), p. 12 and representation (1.10). Observe once again that both expressions (1.21) and (1.22) are generalized mixture representations. Moreover, $\Pr(T_1 = T_2)$ can be positive and consequently G can have a singular part. For this reason, it is not possible to obtain a common mixture representation based on $F_{i:n}(t_1)$ and $F_{j:n}(t_2)$ for all t_1 and t_2. The vector of matrices $\mathbf{S} = (S, S^*)$, where $S = (s_{i,j})$ and $S^* = (s^*_{i,j})$, with the coefficients in representations (1.21) and (1.22), has been termed the *joint signature* of the systems by Navarro et al. (2010). Their procedure for computing these coefficients is illustrated in the following example. Incidentally, this example also shows that some coefficients $s_{i,\,j}$ and $s^*_{i,\,j}$ can be negative.

Example Let us consider the coherent system depited in Fig. 1.1 with lifetime $T_1 = \min\,[X_1, \max\,(X_2, X_3)]$ and $T_2 = X_{3:3}$, where X_1, X_2, X_3 are i.i.d. variables with a continuous distribution function. First of all, note that $T_1 < T_2$. The joint distribution function G of (T_1, T_2) can be written, for $t_1 \le t_2$, as

$$\begin{aligned}
G(t_1,t_2) &= \Pr(\min(X_1,\max(X_2,X_3)) \le t_1, \max(X_1,X_2,X_3) \le t_2) \\
&= \Pr(\{X_1 \le t_1\} \cup \{\max(X_2,X_3) \le t_2\}, \max(X_1,X_2,X_3) \le t_2) \\
&= \Pr(X_1 \le t_1, \max(X_1,X_2,X_3) \le t_2) \\
&\quad + \Pr(\max(X_2,X_3) \le t_1, \max(X_1,X_2,X_3) \le t_2) \\
&\quad - \Pr(X_1 \le t_1, \max(X_2,X_3) \le t_1, \max(X_1,X_2,X_3) \le t_2) \\
&= \Pr(X_1 \le t_1, X_2 \le t_2, X_3 \le t_2) + \Pr(X_2 \le t_1, X_3 \le t_1, X_1 \le t_2) \\
&\quad - \Pr(X_1 \le t_1, X_2 \le t_1, X_3 \le t_1) \\
&= F(t_1)F^2(t_2) + F^2(t_1)F(t_2) - F^3(t_1) \\
&= F_{1:1}(t_1)F_{2:2}(t_2) + F_{2:2}(t_1)F_{1:1}(t_2) - F_{3:3}(t_1).
\end{aligned}$$

Then, since the signatures of order 3 of $X_{1:1}$, $X_{2:2}$ and $X_{3:3}$ are $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$, $\left(0,\frac{1}{3},\frac{2}{3}\right)$ and $(0,0,1)$, respectively (see Navarro et al. 2008a, b), we obtain

$$\begin{aligned}
G(t_1,t_2) =& \frac{1}{9}F_{1:3}(t_1)F_{2:3}(t_2) + \frac{2}{9}F_{1:3}(t_1)F_{3:3}(t_2) \\
&+ \frac{1}{9}F_{2:3}(t_1)F_{1:3}(t_2) + \frac{2}{9}F_{2:3}(t_1)F_{2:3}(t_2) + \frac{3}{9}F_{2:3}(t_1)F_{3:3}(t_2) \\
&- F_{3:3}(t_1) + \frac{2}{9}F_{3:3}(t_1)F_{1:3}(t_2) + \frac{3}{9}F_{3:3}(t_1)F_{2:3}(t_2) + \frac{4}{9}F_{3:3}(t_1)F_{3:3}(t_2)
\end{aligned}$$

for $t_1 \le t_2$. Similarly, for $t_1 > t_2$, we obtain

$$\begin{aligned}
G(t_1,t_2) &= \Pr(T_1 < t_1, T_2 < t_2) \\
&= \Pr(T_2 < t_2) \\
&= F_{3:3}(t_2).
\end{aligned}$$

Therefore, the joint signature in this case is determined by

$$S = \begin{pmatrix} 0 & 0 & 1/9 & 2/9 \\ 0 & 1/9 & 2/9 & 3/9 \\ -1 & 2/9 & 3/9 & 4/9 \end{pmatrix} \text{ and } S^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is of interest in this case to note that $\Pr(T_1 = T_2) = 0$ and so the joint distribution function G is absolutely continuous.

A similar representation holds for the joint reliability function of lifetimes of two coherent systems sharing some components. Also, representations in terms of series and parallel systems can be obtained similarly (see Navarro et al. 2010). Finally, some preservation results can be obtained for the lower orthant and upper orthant bivariate orders based on some matrix-ordering properties for the joint signature of these systems. Interested readers may refer to Navarro et al. (2010) for further details.

1.5 Statistical Inference from System Lifetime Data

The first problem we will treat in this section is the problem of estimating the component lifetime distribution from a random sample of failure times of systems whose components have i.i.d. lifetimes with common distribution F. This problem is of some interest and importance in engineering reliability. Since the behavior of the components may differ under laboratory and field conditions, solving the above problem may be the only approach available for accurately estimating F. While the problem of estimating F from observed system lifetimes is treated in the reliability literature under varied assumptions, the estimator described below has the advantage of being applicable to systems of arbitrary size and design and of being fully nonparametric, that is, free of the assumption that F has a known parametric form. The solution described below has a number of desirable properties: it is the nonparametric maximum likelihood estimator (NPMLE) of F and is a consistent, asymptotically normal and nonparametrically efficient estimator of F. More details on properties of this estimator may be found in Bhattacharya and Samaniego (2010).

In what follows, we restrict our attention to the class of coherent systems and ST mixtures of them (i.e., mixed systems), and we tacitly assume again that the components of the systems considered here have i.i.d. lifetimes with a common continuous distribution F. Suppose the mixed system of interest has a fixed, known design with signature vector $\mathbf{s}$. Then, under these conditions, it is known (see Sect. 1.1) that the reliability function $\overline{F}_T$ of the system lifetime T is given by

$$\overline{F}_T(t) = \sum_{i=1}^{n} s_i \sum_{j=n-i+1}^{n} \binom{n}{j} (\overline{F}(t))^j (F(t))^{n-j}. \tag{1.23}$$

Much of what is done in the sequel will utilize the particular form of the relationship in (1.23) given in (1.8) and based on the domination coefficients (or, equivalently, in the minimal signature).

Suppose that a random sample of system failure times $T_1, \ldots, T_N \sim_{i.i.d.} F_T$ is available. The empirical reliability function of the sample system lifetimes $\widehat{\overline{F}}_{T,N}(t)$ is, of course, the NPMLE of the system reliability function $\overline{F}_T(t)$ and is a consistent, asymptotically normal and nonparametrically efficient estimator of $\overline{F}_T(t)$. Now, all that remains to be done is to solve the inverse problem, that is, to find the estimator $\widehat{\overline{F}}_N(t)$ of $\overline{F}(t)$ which solves, for all $t > 0$, the equation

$$\widehat{\overline{F}}_{T,N}(t) = \sum_{i=1}^{n} a_i \left(\widehat{\overline{F}}_N(t)\right)^i.$$

For any $t > 0$, the asymptotic distribution of $\widehat{\overline{F}}_{T,N}(t)$ may be expressed as

$$\sqrt{N}\left(\widehat{\overline{F}}_{T,N}(t) - \overline{F}_T(t)\right) \to_D U \sim N(0, F_T(t)\overline{F}_T(t)). \tag{1.24}$$

As noted above, for a given fixed t, the estimators $\widehat{\overline{F}}_N(t)$ and $\widehat{\overline{F}}_{T,N}(t)$ are explicitly related via the equation

$$\widehat{\overline{F}}_{T,N}(t) = p\left(\widehat{\overline{F}}_N(t)\right), \tag{1.25}$$

where p is the reliability polynomial defined earlier in (1.8). Since the reliability polynomial p is a strictly increasing function when its argument is in the interval [0, 1], we may obtain a well-defined estimator of $\widehat{\overline{F}}_N(t)$ by inverting the relationship in (1.25). The estimator of interest, which is, by the invariance property of maximum likelihood estimation, the NPMLE of $\overline{F}(t)$, may be expressed as

$$\widehat{\overline{F}}_N(t) = p^{-1}\left(\widehat{\overline{F}}_{T,N}(t)\right).$$

Since the asymptotic distribution of $\widehat{\overline{F}}_{T,N}(t)$ is known and is as given in (1.24), and since p is a smooth one-to-one function, we may apply the δ-method to obtain the asymptotic distribution of $\widehat{\overline{F}}_N(t)$. In the standard presentation of the δ-method, the exact expression for the asymptotic variance of the transformed variable $p^{-1}\left(\widehat{\overline{F}}_{T,N}(t)\right)$ involves the derivative of the function p^{-1}. Indeed, we may write

$$\sqrt{N}\left(\widehat{\overline{F}}_N(t) - \overline{F}(t)\right) \to_D U \sim N\left(0, \left[\frac{\mathrm{d}}{\mathrm{dy}} p^{-1}(y)|_{y=\overline{F}_T(t)}\right]^2 F_T(t)\overline{F}_T(t)\right). \tag{1.26}$$

However, since p is a polynomial of (potentially high) degree n, one is not generally able to obtain the asymptotic variance of $\widehat{\overline{F}}_N(t)$ in (1.26) in closed form. It has been shown by Bhattacharya and Samaniego (2010) that the asymptotic result in (1.26) may be written in a somewhat more useful form as

$$\sqrt{N}\left(\widehat{\overline{F}}_N(t) - \overline{F}(t)\right) \to_D W \sim N\left(0, \left(\sum_{i=1}^{n-1} i a_i \left[p^{-1}(\overline{F}_T(t))\right]^{i-1}\right)^{-2} F_T(t)\overline{F}_T(t)\right). \tag{1.27}$$

Since $y = p(x)$ is a known function, its inverse $p^{-1}(y)$ may be obtained numerically at any given y (say, by interval halving). Suppose that the random sample of N system lifetimes gives rise to the set of ordered failure times $t_{1:N}, \ldots, t_{N:N}$. The reliability function $\overline{F}_T(t)$ is estimated by the empirical distribution $\widehat{\overline{F}}_{T,N}(t)$ of the observed system failure times, with

$$\widehat{\overline{F}}_{T,N}(t) = \begin{cases} 1 & \text{for} \quad t < t_{1:N}, \\ \frac{N-i}{N} & \text{for} \quad t_{i:N} \le t < t_{i+1:N}, \quad i = 1, \ldots, N-1, \\ 0 & \text{for} \quad t \ge t_{N:N}. \end{cases}$$

The estimator $\widehat{\overline{F}}_N(t)$ of $\overline{F}(t)$ is simply the step function with jumps at times $t_{1:N}, \ldots, t_{N:N}$ and values given by

$$\widehat{\overline{F}}_N(t) = \begin{cases} 1 & \text{for} \quad t < t_{1:N}, \\ p^{-1}\left(\frac{N-i}{N}\right) & \text{for} \quad t_{i:N} \le t < t_{i+1:N}, \quad i = 1, \ldots, N-1, \\ 0 & \text{for} \quad t \ge t_{N:N}. \end{cases}$$

The estimator $\widehat{\overline{F}}_N(t)$ is, asymptotically, the optimal estimator of $\overline{F}$ in the nonparametric setting explained above. Bhattacharya and Samaniego (2010) carried out Monte Carlo simulations, based on samples of size 50 and 100, to evaluate the performance of $\widehat{\overline{F}}_N$ for the well-known five-component bridge system. Five parametric models—exponential, Weibull, lognormal, gamma and Pareto—as component distributions F. These results provide support to the claim that $\widehat{\overline{F}}_N$ does indeed perform well, over a reasonably broad class of possible models for F, even for moderate sample sizes like $N = 50$ and certainly for sample sizes of 100 or more. Furthermore, by using (1.27), it is possible to obtain, numerically, approximate but quite reliable confidence intervals for $\overline{F}(t)$ when the sample size N is sufficiently large. Bhattacharya and Samaniego (2010) have also discussed some possible extensions of the above inversion method to non-i.i.d. settings. They have noted that, even in the case of independent but non-identically distributed (INID) component lifetimes, the estimation of component distributions is not generally possible, as the component distributions are not identifiable from data on system failure times. However, in the special case in which, for each i, the ith component of the system has reliability function $h_i(p)$, where h_i is strictly increasing for $p \in [0, 1]$, with $h_i(0) = 0$ and $h_i(1) = 1$, the inversion technique described above leads to the identification of the NPMLEs of the n component reliability functions and to the determination of their asymptotic distributions.

Another development on nonparametric inference for component lifetime distribution, different in nature compared to the one detailed above, is due to Balakrishnan et al. (2011b). This exact method utilizes the order statistics $T_{1:N} < \cdots < T_{N:N}$ obtained from the lifetimes $T_1, \ldots, T_N$ of N identical systems under test. Once again, by assuming that the components in each system are i.i.d. with a continuous distribution F, and that the signature of the system is specified, these authors have used order statistics $T_{i:N}$ to develop exact nonparametric