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Selected Works of Oded Schramm

Volume 1

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Editors

Selected Works of Oded Schramm

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Preface to the Series

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The subjects of the volumes have been selected by an editorial board consisting of Anirban DasGupta, Peter Hall, Jim Pitman, Michael Sörensen, and Jon Wellner.

Preface

Oded Schramm was born on December 10, 1961, in Jerusalem, and died at the age of 46 in a climbing accident on Guye Peak, WA, on September 1, 2008. In between, he made profound and beautiful contributions to mathematics that will have a lasting influence.

In these two volumes, we have collected some of his papers, supplemented with three survey papers by Steffen Rohde, Olle Häggström and Cristophe Garban that further elucidate his work. Despite the seemingly generous size of the collection, spatial considerations neverthelss forced us to omit most of Oded's papers, and the mere fact that all of them are inspiring pieces of work led to some difficult issues on what to include and what to omit. The reader should not view our choices as an attempt to separate his best works from those that are merely great. Rather, we have tried to put together a representative collection that shows the breadth, depth, enthusiasm and clarity of his work. Others may have diverging opinions on what should or should not have been included, but we do hope that Oded himself would not have been too displeased by our choices. The papers we have included speak for themselves; let us just say a few words about how we have organized them into five sections.

Oded began his mathematical career as a geometer, making GEOMETRY the natural topic for Section 1. This section opens with Oded's first two papers from his Master's thesis in 1987 under Gil Kalai at the Hebrew University. We then move on to circle packing and conformal geometry, including examples of his extraordinarily fruitful collaboration with Zheng-Xu Hu, and end the section with the joint paper with Mario Bonk on embeddings of hyperbolic spaces. Of course geometric aspects permeate also all of the following sections. In fact, Oded once mentioned to one of us that in order to be able to think about a problem he always liked it to have a geometric component.

In the mid 1990's, Oded became interested in the topic of probability theory, which dominates Sections 2-5 of this collection. Section 2 deals in particular with the study of NOISE SENSITIVITY, pioneered in a joint paper with Itai Benjamini and Gil Kalai that opens the section. Noise sensitivity turned out to be a rich topic with applications ranging from voting systems to percolation. In the two other papers of this section, the first coauthored with Jeff Steif and the second with Christophe Garban and Gabor Pete, Oded went progressively deeper into noise sensitivity in a percolation setting, arriving at surprisingly detailed insights.

In Section 3 we have collected some of Oded's papers on RANDOM WALKS AND GRAPH LIMITS. This includes (i) a paper with Itai Benjamini establishing recurrence of random walks on suitably defined limits of finite planar graphs, (ii) the joint paper with his student Omer Angel on distributional limits of triangulations, (iii) a singly-authored paper on compositions of random transpositions, (iv) a collaboration with Yuval Peres, Scott Sheffield and David Wilson on the remarkably fruitful interplay between the infinity Laplacian and certain board games, and (v) the concise 2008 paper on so called hyperfinite graph limits.

One of the probabilistic objects that caught the strongest grip on Oded's imagination was PERCOLATION, which appeared prominently in Section 2 as a major testing ground for noise sensitivity. Papers on other aspects of percolation are collected in Section 4. Here we will see how Oded joined forces with Itai Benjamini, Russ Lyons and Yuval Peres in order to uncover many of the new and interesting phenomena that happen when we move beyond the usual setting of percolation in a Euclidean geometry, and instead study what happens on hyperbolic lattices and other nonamenable graph structures. Most of the papers in this section are joint work with (subsets of) this team of coauthors, plus Harry Kesten who joined them in establishing a beautiful result on uniform spanning trees. We end the section on a different note, namely Oded's joint paper with Itai Benjamini and Gil Kalai making progress on the important open problem of determining the order of magnitude of fluctuations of first passage percolation on the Euclidean lattice.

Despite strong competition from Oded's other works, there seems to be a consensus view that the most important of all his contributions to mathematics is his discovery and subsequent study of SCHRAMM-LOEWNER EVOLUTION (or stochastic Loewner avolution as Oded himself preferred to call it; conveniently, the abbreviation SLE works either way), which is the topic of Section 5. SLE is a family of conformally invariant random processes in the plane that turn out to appear as the scaling limit of percolation and a variety of other critical models. We begin this section with the famous paper, published in 2000 in the Israel Journal of Mathematics, where Schramm singlehandedly discovered SLE and obtained the first preliminary results on scaling limits. Then followed a series of papers with Greg Lawler and Wendelin Werner in which SLE was exploited to deduce deep results on intersection properties of random walks; here we include only some of the highlights. We furthermore include important joint papers with Steffen Rohde, Scott Sheffield, David Wilson and Stanislav Smirnov, plus Oded's contribution to the International Congress of Mathematicians in Madrid, 2006, in which he gives a survey of the field with an emphasis on open problems.

There were no signs of a decrease in creativity or productivity on Oded's part until the untimely and tragic end, and we can only guess what further discoveries we miss because of it. Substantial parts of his work, joint with others, is still not completely written up and will appear in the coming years.

However, Oded will be missed not just because of his mathematics, but even more because of the gentle, warm-hearted and generous person that he was. Of course, the loss of him is felt most strongly by his wife Avivit, his daughter Tselil and his son Pele. But he was very much loved by the mathematical community and by everyone who knew him, as amply witnessed on the memorial blog which was set up shortly after his death: http://odedschramm.wordpress.com/

We hope that this collection will contribute, however modestly, to keeping the memory of Oded alive, and to nourishing the mathematical heritage he left for all of us. Yehi zichro baruch - may the memory of him be a blessing.

Itai Benjamini
Olle Häggström

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Part I
Geometry

Oded Schramm: From Circle Packing to SLE

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July 12, 2010

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1 Introduction

When I first met Oded Schramm in January 1991 at the University of California, San Diego, he introduced himself as a "Circle Packer". This modest description referred to his Ph.D. thesis around the Koebe-Andreev-Thurston theorem and a discrete version of the Riemann mapping theorem, explained below. In a series of highly original papers, some joint with Zhen-Xu He, he created powerful new tools out of thin air, and provided the field with elegant new ideas. At the time of his deadly accident on September 1st, 2008, he was widely considered as one of the most innovative and influential probabilists of his time. Undoubtedly, he is best known for his invention of what is now called the Schramm-Loewner Evolution (SLE), and for his subsequent collaboration with Greg Lawler and Wendelin Werner that led to such celebrated results as a determination of the intersection exponents of two-dimensional Brownian motion and a proof of Mandelbrot's conjecture about the Hausdorff dimension of the Brownian frontier. But already his previous work bears witness to the brilliance of his mind, and many of his early papers contain both deep and beautifully simple ideas that deserve better knowing.

In this note, I will describe some highlights of his work in circle packings and the Koebe conjecture, as well as on SLE. As Oded has co-authored close to 20 papers related to circle packings and more than 20 papers involving SLE, only a fraction can be discussed in detail here. The transition from circle packing to SLE was through a long sequence of influential papers concerning probability on graphs, many of them written jointly with Itai Benjamini. I will present almost no work from that period (some of these results are described elsewhere in this volume, for instance in Christophe Garban's article on Noise Sensitivity). In that respect, the title of this note is perhaps misleading.

In order to avoid getting lost in technicalities, arguments will be sketched at best, and often ideas of proofs will be illustrated by analogies only. In an attempt to present the evolution of Oded's mathematics, I will describe his work in essentially chronological order.

Oded was a truly exceptional person: not only was his clear and innovative way of thinking an inspiration to everyone who knew him, but also his caring, modest and relaxed attitude generated a comfortable atmosphere. As inappropriate as it might be, I have included some personal anecdotes as well as a few quotes from email exchanges with Oded, in order to at least hint at these sides of Oded that are not visible in the published literature.

This note is not meant to be an overview article about circle packings or SLE. My prime concern is to give a somewhat self-contained account of Oded's contributions. Since SLE has been featured in several excellent articles and even a book, but most of Oded's work on circle packing is accessible only through his original papers, the first part is a bit more expository and contains more background. The expert in either field will find nothing new, and will find a very incomplete list of references. My apology to everyone whose contribution is either unmentioned or, perhaps even worse, mentioned without proper reference.

Acknowledgement: I would like to thank Mario Bonk, Jose Fernández, Jim Gill, Joan Lind, Don Marshall, Wendelin Werner and Michel Zinsmeister for helpful comments on a first draft. I would also like to thank Andrey Mishchenko for generating Figure 3, and Don Marshall for Figure 6.

2 Circle Packing and the Koebe Conjecture

Oded Schramm was able to create, seemingly without effort, ingenious new ideas and methods. Indeed, he would be more likely to invent a new approach than to search the literature for an existing one. In this way, in addition to proving wonderful new theorems, he rediscovered many known results, often with completely new proofs. We will see many examples throughout this note.

Oded received his Ph.D. in 1990 under William Thurston's direction at Princeton. His thesis, and the majority of his work until the mid 90's, was concerned with the fascinating topic of circle packings. Let us begin with some background and a very brief overview of some highlights of this field prior to Oded's thesis. Other surveys are [Sa] and [Ste2].

2.1 Background

According to the Riemann mapping theorem, every *simply connected* planar domain, except the plane itself, is conformally equivalent to a disc. The conformal map to the disc is unique, up to postcomposition with an automorphism of the disc (which is a Möbius transformation). The standard proof exhibits the map as a solution of an extremal problem (among all maps of the domain *into* the disc, maximize the derivative at a given point). The situation is quite different for *multiply connected* domains, partly due to the lack of a standard target domain. The standard proof can be modified to yield a conformal map onto a *parallel slit domain* (each complementary component is a horizontal line segment or a point). Koebe showed that every *finitely connected* domain is conformally equivalent to a *circle domain* (every boundary component is a circle or a point), in an essentially unique way. No proof similar to the standard proof of the Riemann mapping theorem is known.

Theorem 2.1 ([Ko1]). *For every domain $\Omega \subset \mathbb{C}$ with finitely many connected boundary components, there is a conformal map f onto a domain $\Omega' \subset \mathbb{C}$ all of whose boundary components are circles or points. Both f and Ω' are unique up to a Möbius transformation.*

Koebe conjectured (p. 358 of [Ko1]) that the same is true for infinitely connected domains. It later turned out that *uniqueness* of the circle domain can fail (for instance, it fails whenever the set of point-components of the boundary has positive area, as a simple application of the measurable Riemann mapping theorem shows). But *existence* of a conformally equivalent circle domain is still open, and is known as Koebe's conjecture or "Kreisnormierungsproblem". It motivated a lot of Oded's research.

There is a close connection between Koebe's theorem and circle packings. A *circle packing* P is a collection (finite or infinite) of closed discs D in the two dimensional plane $\mathbb{C}$, or in the two dimensional sphere S^2, with disjoint interiors. Associated with a circle packing is its *tangency graph* or *nerve* $G = (V, E)$, whose vertices correspond to the discs, and such that two vertices are joined by an edge if and only if the corresponding discs are tangent. We will only consider packings whose tangency graph is connected.

Conversely, the Koebe-Andreev-Thurston *Circle Packing Theorem* guarantees the existence of packings with prescribed combinatorics. Loosely speaking, a planar graph is a graph that can be drawn in the plane so that edges do not cross. Our graphs will not have double edges (two edges with the same endpoints) or loops (an edge whose endpoints coincide).

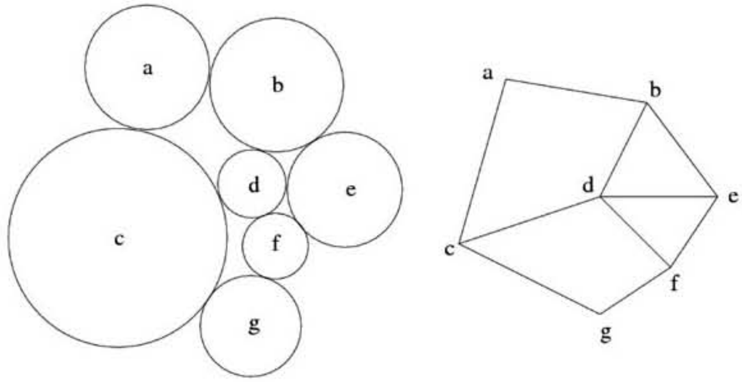

Figure 1: A circle packing and its tangency graph.

Theorem 2.2 ([Ko2], [T], [A1]). *For every finite planar graph G, there is a circle packing in the plane with nerve G. The packing is unique (up to Möbius transformations) if G is a triangulation of S^2.*

See the following sections for the history of this theorem, and sketches of proofs. In particular, in Section 2.3 we will indicate how the Circle Packing Theorem 2.2 can be obtained from the Koebe Theorem 2.1, and conversely that the Koebe theorem can be deduced from the Circle Packing Theorem. Every finite planar graph can be extended (by adding vertices and edges as in Figure 3(c)) to a triangulation, hence packability of triangulations implies packability of finite planar graphs (there are many ways to extend a graph to a triangulation, and uniqueness of the packing is no longer true). The situation is more complicated for infinite graphs. Oded wrote several papers dealing with this case. Thurston conjectured that circle packings approximate conformal maps, in the following sense: Consider the *hexagonal packing* H_ε of circles of radius ε (a portion is visible in Fig. 2 and Fig. 3(a)). Let $\Omega \subset \mathbb{C}$ be a domain (a connected open set). Approximate Ω from the inside by a circle packing P_ε of circles of $\Omega \cap H_\varepsilon$, as in Fig. 2 and Fig. 3(a) (more precisely, take the connected component containing p of the union of those circles whose six neighbors are still contained in Ω). Complete the nerve of this packing by adding one vertex for each connected component of the complement to obtain a triangulation of the sphere (there are three new vertices v_1, v_2, v_3 in Fig. 3(c); the three copies of v_3 are to be identified). By the Circle Packing Theorem, there is a circle packing P'_ε of the sphere with the same tangency graph (Figures 2 and 3(d) show these packings after stereographic projection from the sphere onto the plane; the circle corresponding to v_3 was chosen as the upper hemisphere and became the outside of the large circle after projection). Notice that each of the complementary components now corresponds to one ("large") circle of P'_ε, and the circles in the boundary of P_ε are tangent to these complementary circles. Now consider the map f_ε that sends the centers of the circles of P_ε to the corresponding centers in P'_ε, and extend it in a piecewise linear fashion. Rodin and Sullivan proved Thurston's conjecture that f_ε approximates

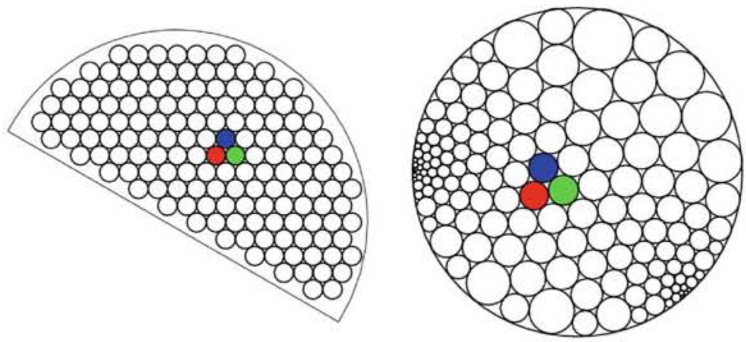

Figure 2: A circle packing approximation to a Riemann map.

the Riemann map, if Ω is simply connected (see Fig. 2):

Theorem 2.3. *[RSu] Let Ω be simply connected, $p, q \in \Omega$, and P'_ε normalized such that the complementary circle is the unit circle, and such that the circle closest to p (resp. q) corresponds to a circle containing* 0 *(resp. some positive real number). Then the above maps f_ε converge to the conformal map $f : \Omega \to \mathbb{D}$ that is normalized by $f(p) = 0$ and $f(q) > 0$, uniformly on compact subsets of Ω as $\varepsilon \to 0$.*

Their proof depends crucially on the non-trivial *uniqueness* of the hexagonal packing as the only packing in the plane with nerve the triangular lattice. Oded found remarkable improvements and generalizations of this theorem. See Section 2.6 for further discussion.

2.2 Why are Circle Packings interesting?

Despite their intrinsic beauty (see the book [Ste2] for stunning illustrations and an elementary introduction), circle packings are interesting because they provide a canonical and conformally natural way to embed a planar graph into a surface. Thus they have applications to combinatorics (for instance the proof of Miller and Thurston [MT] of the Lipton-Tarjan separator theorem, see e.g. the slides of Oded's circle packing talk on his memorial webpage), to differential geometry (for instance the construction of minimal surfaces by Bobenko, Hoffmann and Springborn [BHS] and their references), to geometric analysis (for instance, the Bonk-Kleiner [BK] quasisymmetric parametrization of Ahlfors 2-regular LLC topological spheres) to discrete probability theory (for instance, through the work of Benjamini and Schramm on harmonic functions on graphs and recurrence on random planar graphs [BS1],[BS2], [BS3]) and of course to complex analysis (discrete analytic functions, conformal mapping). However, Oded's work on circle packing did not follow any "main-stream" in conformal geometry or geometric function theory. I believe he continued to work on them just because he liked it. His interest never wavered, and many of his numerous late contributions to Wikipedia were about this topic.

Existence and uniqueness are intimately connected. Nevertheless, for better readability I will discuss them in two separate sections.

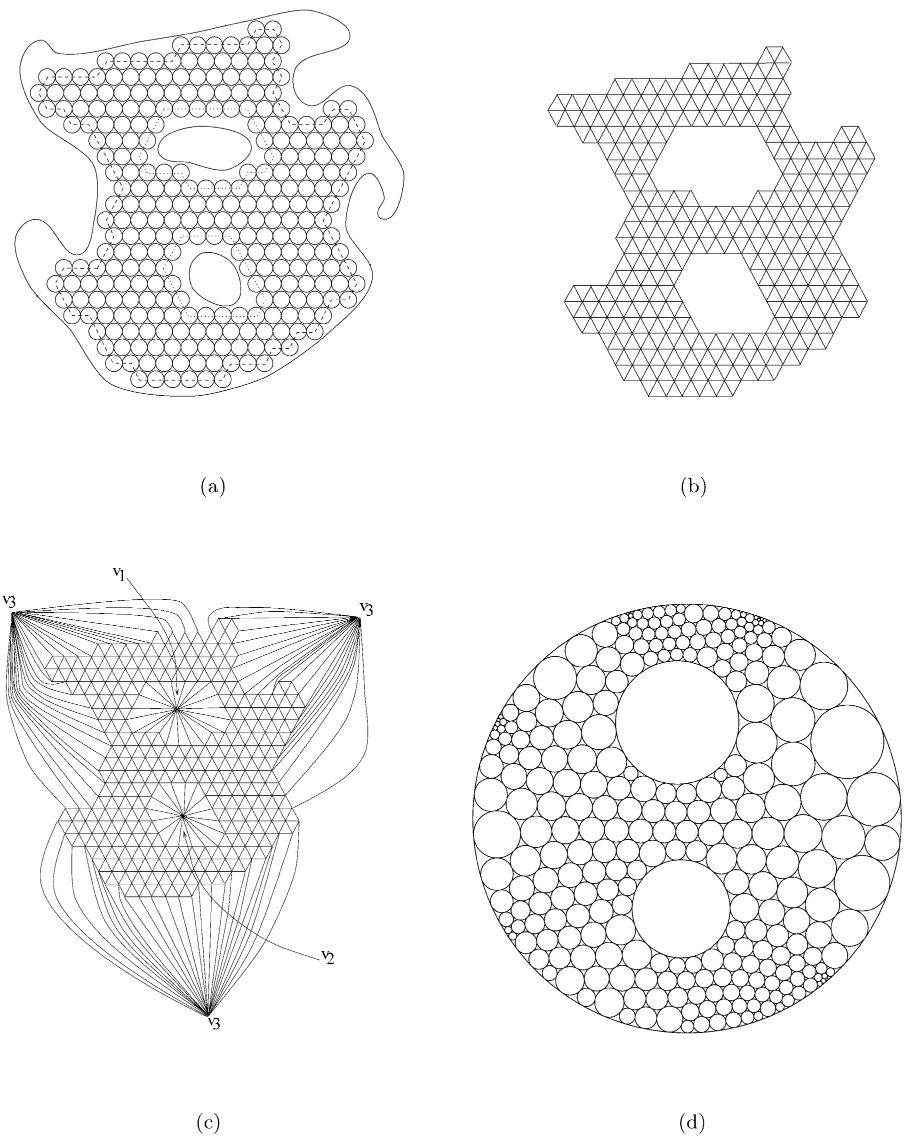

Figure 3: A circle packing approximation of a triply connected domain, its nerve, its completion to a triangulation of S^2, and a combinatorially equivalent circle packing; (a)-(c) are from Oded's thesis; thanks to Andrey Mishchenko for creating (d)

2.3 Existence of Packings

Oded applied the highest standards to his proofs and was not satisfied with "ugly" proofs. As we shall see, he found four (!) different new existence proofs for circle packings with prescribed combinatorics. Before discussing them, let us have a glance at previous proofs.

The Circle Packing Theorem was first proved by Koebe [Ko2] in 1936. Koebe's proof of existence was based on his earlier result that every planar domain Ω with *finitely many* boundary components, say m, can be mapped conformally onto a *circle domain*. A simple iterative algorithm, due to Koebe, provides an infinite sequence Ω_n of domains conformally equivalent to Ω and such that Ω_n converges to a circle domain. To obtain Ω_{n+1} from Ω_n, just apply the Riemann mapping theorem to the simply connected domain (in $\mathbb{C} \cup \{\infty\}$) containing Ω_n whose boundary corresponds to the $(n \mod m)-$th boundary component of Ω. With the conformal equivalence of finitely connected domains and circle domains established, a circle packing realizing a given tangency pattern can be obtained as a limit of circle domains: Just construct a sequence of m-connected domains so that the boundary components approach each other according to the given tangency pattern. For instance, if the graph $G = (V, E)$ is embedded in the plane by specifying simple curves $\gamma_e : [0, 1] \to S^2$, $e \in E$, then the complement Ω_ε of the set

$$\bigcup_{e \in E} \gamma_e[0, 1/2 - \varepsilon] \cup \bigcup_{e \in E} \gamma_e[1/2 + \varepsilon, 1]$$

is such an approximation. It is not hard to show that the (suitably normalized) conformally equivalent circle domains Ω'_ε converge to the desired circle packing when $\varepsilon \to 0$.

Koebe's theorem was nearly forgotten. In the late 1970's, Thurston rediscovered the circle packing theorem as an interpretation of a result of Andreev [A1], [A2] on convex polyhedra in hyperbolic space, and obtained uniqueness from Mostow's rigidity theorem. He suggested an algorithm to compute a circle packing (see [RSu]) and conjectured Theorem 2.3, which started the field of circle packing. Convergence of Thurston's algorithm was proved in [dV1]. Other existence proofs are based on a Perron family construction (see [Ste2]) and on a variational principle [dV2].

Oded's thesis [S1] was chiefly concerned with a generalization of the existence theorem to packings with prescribed convex shapes instead of discs, and to applications. A consequence ([S1], Proposition 8.1) of his "Monster packing theorem" is, roughly speaking, that the circle packing theorem still holds if discs are replaced by smooth convex sets.

Theorem 2.4. *([S1], Proposition 8.1) For every triangulation $G = (V, E)$ of the sphere, every $a \in V$, every choice of smooth strictly convex sets D_v for $v \in V \setminus \{a\}$, and every smooth simple closed curve C, there is a packing $P = \{P_v : v \in V\}$ with nerve G, such that P_a is the exterior of C and each $P_v, v \in V \setminus \{a\}$ is positively homothetic to D_v.*

Sets A and B are positively homothetic if there is $r > 0$ and $s \in \mathbb{C}$ with $A = rB + s$. Strict convexity (instead of just convexity) was only used to rule out that three of the prescribed sets could meet in one point (after dilation and translation), and thus his packing theorem applied in much more generality. Oded's approach was topological in nature: Based on a cleverly constructed

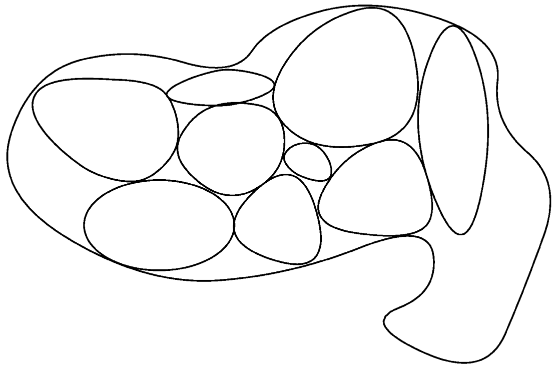

Figure 4: A packing of convex shapes in a Jordan domain, from Oded's thesis

spanning tree of G, he constructed what he called a "monster". This refers to a certain $|V|$-dimensional space of configurations of sets homothetic to the given convex shapes, with tangencies according to the tree, and certain non-intersection properties. Existence of a packing was then obtained as a consequence of Brower's fixed point theorem. Here is a poetic description, quoted from his thesis:

One can just see the terrible monster swinging its arms in sheer rage, the tentacles causing a frightful hiss, as they rub against each other.

Applying Theorem 2.4 to the situation of Figure 3, with D_v chosen as circles when $v \notin \{v_1, v_2, v_3\}$, and arbitrary convex sets D_{v_j}, Oded adopted the Rodin-Sullivan convergence proof to obtain a new proof of the following generalization of Koebe's mapping theorem. The original proof of Courant, Manel and Shiffman [CMS] employed a very different (variational) method.

Theorem 2.5. *([S1], Theorem 9.1; [CMS]) For every $n+1$-connected domain Ω, every simply connected domain $D \subset \mathbb{C}$ and every choice of n convex sets D_j, there are sets D'_j which are positively homothetic to D_j such that Ω is conformally equivalent to $D \setminus \cup_1^n D'_j$.*

Later [S7] he was able to dispose of the convexity assumption, and proved the packing theorem for smoothly bounded but otherwise arbitrary shapes. As a consequence, he was able to generalize Theorem 2.5 to arbitrary (not neccessarily convex) compact connected sets D_j, thus rediscovering a theorem due to Brandt [Br] and Harrington [Ha].

Oded then developed a differentiable approach to the circle packing theorem. In [S3] he shows

Theorem 2.6. *([S3],Theorem 1.1) Let P be a 3-dimensional convex polyhedron, and let $K \subset \mathbb{R}^3$ be a smooth strictly convex body. Then there exists a convex polyhedron $Q \subset \mathbb{R}^3$ combinatorially equivalent to P which midscribes K.*

Here "Q midscribes K" means that all edges of Q are tangent to ∂K. He also shows that the space of such Q is a six-dimensional smooth manifold, if the boundary of K is smooth and has positive Gaussian curvature. For $K = S^2$, Theorem 2.6 has been stated by Koebe [Ko2] and proved by Thurston [T] using Andreev's theorem [A1], [A2]. Oded notes that Thurston's midscribability proof based on the circle packing theorem can be reversed, so that Theorem 2.6 yields a new proof of the Circle Packing Theorem (given a triangulation, just take $K = S^2$, Q the midscribing convex polyhedron with the combinatorics of the packing, and for each vertex $v \in V$, let D_v be the set of points on S^2 that are visible from v).

One defect of the continuity method in his thesis was that it did not provide a proof of uniqueness (see next section). In [S4] he presented a completely different approach to prove a far more general packing theorem, that had the added benefit of yielding uniqueness, too. A quote from [S4]:

It is just about the most general packing theorem of this kind that one could hope for (it is more general than I have ever hoped for).

A consequence of [S4] (Theorem 3.2 and Theorem 3.5) is

Theorem 2.7. *Let G be a planar graph, and for each vertex $v \in V$, let $\mathcal{F}_v$ be a proper 3-manifold of smooth topological disks in S^2, with the property that the pattern of intersection of any two sets in $\mathcal{F}_v$ is topologically the pattern of intersection of two circles. Then there is a packing P whose nerve is G and which satisfies $P_v \in \mathcal{F}_v$ for $v \in V$.*

The requirement that $\mathcal{F}_v$ is a 3-manifold requires specification of a topology on the space of subsets of S^2 : Say that subsets $A_n \subset S^2$ converge to A if $\limsup A_n = \liminf A_n = A$ and $A^c = \mathrm{int}(\limsup \mathrm{A_n^c})$. An example is obtained by taking a smooth strictly convex set K in $\mathbf{R}^3$ and letting $\mathcal{F}$ be the family of intersections $H \cap \partial K$, where H is any (affine) half-space intersecting the interior of K. Specializing to $K = S^2$, $\mathcal{F}$ is the familiy of circles and the choice $\mathcal{F}_v = \mathcal{F}$ for all v reduces to the circle packing theorem.

The proof of Theorem 2.7 is based on his *incompatibility theorem*, described in the next section. It provides uniqueness of the packing (given some normalization), which is key to proving existence, using continuity and topology (in particular invariance of domains).

2.4 Uniqueness of Packings

I was always impressed by the flexibility of Oded's mind, in particular his ability to let go of a promising idea. If an idea did not yield a desired result, it did not take long for him to come up with a completely different, and in many cases more beautiful, approach. He once told me that if he did not make progress within three days of thinking about a problem, he would move on to different problems.

Following Koebe and Schottky, **uniqueness** of finitely connected circle domains (up to Möbius images) is not hard to show, using the reflection principle: If two circle domains are conformally equivalent, the conformal map can be extended by reflection across each of the boundary circles, to obtain a conformal map between larger domains (that are still circle domains). Continuing in this fashion, one obtains a conformal map between complements of limit sets of reflection groups. As they are Cantor sets of area zero, the map extends to a conformal map of the whole sphere, hence is a Möbius transformation. Uniqueness of the (finite) circle packing can be proved in a similar fashion. To date, the strongest rigidity result whose proof is based on this method is the following theorem of He and Schramm. See [Bo] for the related rigidity of Sierpinski carpets.

Theorem 2.8 ([HS2], Theorem A). *If Ω is a circle domain whose boundary has $\sigma-$finite length, then Ω is rigid (any conformal map to another circle domain is Möbius).*

For *finite* packings, there are several technically simpler proofs. The shortest and most elementary of them is deferred to the end of this section, since I believe it has been discovered last. Rigidity of *infinite* packings lies deeper. The rigidity of the hexagonal packing, crucial in the proof of the Rodin-Sullivan theorem as elaborated in Section 2.6 below, was originally obtained from deep results of Sullivan's concerning hyperbolic geometry. He's thesis [He] gave a quantitative and simpler proof, still using the above reflection group arguments and the theory of quasiconformal maps. In one of his first papers [S2], Oded gave an elegant combinatorial proof that at the same time was more general:

Theorem 2.9 ([S2], Theorem 1.1). *Let G be an infinite, planar triangulation and P a circle packing on the sphere S^2 with nerve G. If $S^2 \setminus \text{carrier}(P)$ is at most countable, then P is rigid (any other circle packing with the same combinatorics is Möbius equivalent).*

The *carrier* of a packing $\{D_v : v \in V(G)\}$ is the union of the (closed) discs D_v and the "interstices" (bounded by three mutually touching circles) in the complement of the packing. The rigidity of the hexagonal packing follows immediately, since its carrier is the whole plane.

The ingenious new tool is his *Incompatibility Theorem*, a combinatorial analog to the conformal modulus of a quadrilateral. To fully appreciate it, lets first look at its classical continuous counterpart, and defer the statement of the Theorem to Section 2.4.2 below.

2.4.1 Extremal length and the conformal modulus of a quadrilateral

If you conformally map a 3x1-rectangle to a disc, such that the center maps to the center, what fraction of the circle does the image of one of the two short sides occupy? Despite having known the effect of "crowding" in numerical conformal mapping, I was surprised to learn of the numerical value of 0.0114... from Don Marshall (see [MS].) Of course, the precise value can be easily computed as an elliptic integral, but if asked for a rough guess, most answers are around 1/10 (the uniform measure with respect to length would give 1/8). Oded's answer, after a moments thought (during a tennis match in the early 90's), was 1/64, reasoning that this is the probability of a planar random walker to take each of his first three steps "to the right".

An important classical conformal invariant, masterfully employed by Oded in many of his papers, is the *modulus* of a quadrilateral. Let Ω be a simply connected domain in the plane that is bounded by a simple closed curve, and let p_1, p_2, p_3 and p_4 be four consecutive points on $\partial\Omega$. Then there is a unique $M > 0$ such that there is a conformal map $f : \Omega \to [0, M] \times [0, 1]$ and such that f takes the p_j to the four corners with $f(p_1) = 0$ (by a classical theorem of Caratheodory, f extends homeomorphically to the boundary of the domains). There are several quite different instructive proofs of uniqueness of M. Each of the following three techniques has a counterpart in the circle packing world that has been employed by Oded. Suppose we are given two rectangles and a conformal map f between them taking corners to corners.

One method to prove uniqueness is to repeatedly reflect f across the sides of the rectangles. The resulting extention is a conformal map of the plane, hence linear, and it follows that the aspect ratio is unchanged. This is similar to the aforementioned Schottky group argument.

A second method is to explicitly define a quantity λ depending on a configuration $(\Omega, p_1, ..., p_4)$ in such a way that it is conformally invariant and such that one can compute λ for the rectangle $[0, M] \times [0, 1]$. This is achieved by the *extremal length* of the family Γ of all rectifiable curves γ joining two opposite "sides" $[p1, p2]$ and $[p3, p4]$ of Ω. The extremal length of a curve family Γ is defined as

$$\lambda(\Gamma) = \sup_\rho \frac{(\inf_\gamma \int_\gamma \rho |dz|)^2}{\int_{\mathbb{C}} \rho^2 dxdy}, \tag{1}$$

where the supremum is over all "metrics" (measurable functions) $\rho : \mathbb{C} \to [0, \infty)$. For the family of curves joining the horizontal sides in the rectangle $[0, M] \times [0, 1]$, it is not hard to show $\lambda(\Gamma) = M$. This simple idea is actually one of the most powerful tools of geometric function theory. See e.g. [Po2] or [GM] for references, properties and applications.

Discrete versions of extremal length (or the "conformal modulus" $1/\lambda$) have been around since the work of Duffin [Duf]. In conformal geometry, they have been very succesfully employed beginning with the groundbreaking paper [Can]. Cannon's extremal length on a graph $G = (V, E)$ is obtained from (1) by viewing non-negative functions $\rho : V \to [0, \infty)$ as metrics on G, defining the length of a "curve" $\gamma \subset V$ as the sum $\sum_{v \in \gamma} \rho(v)$, and the "area" of the graph as $\sum \rho(v)^2$. See [CFP1] for an account of Cannon's discrete Riemann mapping theorem, and for instance the papers [HK] and [BK] concerning applications to quasiconformal geometry. Oded's applications to square packings and transboundary extremal length are briefly discussed in Section 2.7 below.

A third and very different method is topological in nature and is one of the key ideas in [HS1]. Suppose we are given two rectangles Ω, Ω' with different aspect ratio and overlapping as in Fig. 5, and a conformal map f between them mapping corners to corners. Then the difference $f(z) - z$ is $\neq 0$ on the boundary $\partial\Omega$. Traversing $\partial\Omega$ in the positive direction, inspection of Fig. 5 shows that the image curve under $f(z) - z$ winds around 0 in the negative direction. But a negative winding is impossible for analytic functions (by the argument principle, the winding number counts the number of preimages of 0).

2.4.2 The Incompatibility Theorem

Again consider the overlapping rectangles Ω, Ω' of Fig. 5, and two combinatorially equivalent packings P, P' whose nerves triangulate the rectangles, as in Fig. 6. Assume for simplicity that the sets D_v and D'_v of the packings are closed topological discs (except for the four sides $D_1, ...D_4$, $D'_1, ..., D'_4$ of the rectangles, which are considered to be sets of the packing). Intuitively, two topological discs D and D' are called *incompatible* if they intersect as in Fig. 5. More formally, say that D *cuts* D' if there are two points in $D' \setminus \text{interior}(D)$ that cannot be connected by a curve in $\text{interior}(D' \setminus D)$. Then Oded calls D and D' incompatible if D cuts D' or D' cuts D. As he notes, *the motivation for the definition comes from the simple but very important observation that the possible patterns of intersection of two circles are very special, topologically.* Indeed, any two circles are compatible.

Theorem 2.10 ([S2], Theorem 3.1)**.** *There is a vertex v for which D_v and D'_v are incompatible.*