Matthew Robinson

Symmetry and the Standard Model

Mathematics and Particle Physics

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To our teachers

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Robert Manweiler
Stan Zygmunt

for showing us that anything that
can be explained can be explained clearly

Preface

Motivation for This Series

First of all, we want to point out that this book is by no means meant to compete with or take the place of any of the standard quantum field theory, particle physics, or mathematics texts currently available. There are too many outstanding choices to try to add yet another to the list. Our goal is, simply put, to teach physicists the math that is used in particle physics.

The origin of this goal is the plight of upper-level undergraduate and first/second year graduate students in physics, especially those in theory. Generally, after four years of standard undergraduate coursework and two years of standard graduate coursework, the road to understanding a modern research paper in particle theory is a long, hard hike. And as the physics becomes more and more advanced, the necessary math becomes more sophisticated at an overwhelming rate. At least, it is overwhelming for those of us who don't understanding everything immediately.

To make matters worse, the way physicists and mathematicians think about nearly everything in math and physics can be (and usually is) vastly different. The way a mathematician approaches differential equations, Lie groups, or fiber bundles is typically unlike the way a physicist approaches them. The language used is often very different, and the things that are important are almost always different. When physics students realize that they need a better understanding of how one does analysis on manifolds (e.g. at a graduate level), reading a graduate-level book about analysis on manifolds (by a mathematician) is very often frustrating and unhelpful. This shouldn't be taken to reflect poorly on our friends in the math department or on their pedagogical abilities. It is simply indicative of the wide gulf between two different disciplines.

Nevertheless, despite being different disciplines, the language of physics is mathematics. If you want to understand the inner workings of nature, you have to understand analysis on manifolds, as well as countless other topics, at a graduate level at least. But because physicists are not used to thinking in the way mathematicians think,[1] they will make much more progress when things are explained in a way that is “friendly to a physicist”, at least at first. For example, if you show a physicist the formal definition of an ideal or of cohomology (when they’ve never encountered those ideas before) they will usually find it very difficult to intuit what they actually are. However, if you say something like “an ideal is essentially all the multiples of something, like all the multiples of 7 on the real line” or “cohomology is essentially a way of measuring how many holes are in something”, and slowly build up the formal definition from there, progress will be much faster. The downside to this approach is that any mathematicians standing nearby will likely get very annoyed because “friendly to a physicist” usually translates to “lacking rigor”.

While mathematicians are correct in pointing out that we often (usually) lack rigor in how we think about their ideas, it is still good for us to do it. For example, after understanding that cohomology is essentially a way of measuring how many holes there are in something, the physicist will have a much easier time parsing through the formal definition. If there are a few non-trivial (but still simple) examples scattered along the way, there is a good chance that the physicist will develop a very good understanding of the “real” mathematical details.

However, with only a few exceptions, there are math books and there are physics books. When physicists write physics books they generally try to go as far as they can with as little advanced math as they can, or they assume that the reader is already familiar with the underlying math. And when mathematicians write math books, they either don’t care about the physical applications or they mention them only briefly and maintain abstract mathematical formalism with some physics vocabulary. While neither of these situations is the fault of the authors, it can often be to the detriment of helping eager physics students get any real intuition for what lines drawn around a hole have to do with magnetic fields.

So that is the context of this series of books. We’re trying to teach math in a way that is ~~lacking rigor~~ friendly to physicists. The goal is that after reading this, one of the many excellent introductory texts on relativistic quantum mechanics, quantum field theory, or particle physics will be much more accessible.

Outline of This Series

This is the first in a series of books intended to teach math to physicists. The current plans are for at least four volumes. Each of the first four volumes will discuss a variety of mathematical topics, but each will have a particular emphasis. Furthermore, a substantial portion of each will discuss, in detail, how specific mathematical ideas are used in particle physics.

The first volume will emphasize algebra, primarily group theory. In the first part we will discuss at length the nature of group theory and the major related

[1]The converse is typically very true as well.

ideas, with a special emphasis on Lie groups. The second part will then use these ideas to build a modern formulation of quantum field theory and the tools that are used in particle physics. In keeping with the theme, the formulations and tools will be approached from a heavily algebraic perspective. Finally, the first volume will discuss the structure of the standard model (again, focusing on the algebraic structure) and the attempts to extend and generalize it. As a comment, this does not mean that this volume is *solely* about algebra. We will talk about and use a variety of mathematics (i.e. we'll use analysis, geometry, statistics, etc.) – we'll just be primarily using algebra.

The second volume will emphasize geometry and topology in a fairly classical way. The first part will discuss differential geometry and algebraic topology, and the second part will combine these ideas to discuss more fundamental formulations of classical field theory and electrodynamics, and gravitation.

The third volume will once again emphasize geometry and topology, but in a more modern context – namely through fibre bundles. The major mathematical goal will be to build up a primer on global analysis (mathematical relationships between locally defined geometric data and globally defined topological data). The physical application in the second part of this volume will be a fairly comprehensive and robust overview of gauge theory, and a reformulation of the standard model in modern terms.

Finally, the fourth volume will emphasize real and complex analysis. The physical application will then be the study of particle interactions (a topic that is glaringly, but deliberately, absent from the first volume). This will include detailed discussions of renormalization, scattering amplitudes, decay rates, and all of the other topics that generally make up the major bulk of introductory quantum field theory and particle physics courses.

So, over the first four volumes, we will cover algebra, geometry, topology, and real and complex analysis – four of the major areas in mathematics. Furthermore, we will have discussed how all of this math ties in to classical field theory, quantum field theory, general relativity, gauge theory, non-perturbative quantum field theory, particle interactions, and renormalization. In other words, the first four volumes are intended to be a fairly comprehensive introduction to modern physics.

However, we do wish to reiterate that while we are hoping to be as comprehensive as possible, we only mean in scope, not in depth. Once again, our goal is *not* to replace any of the standard physics or math texts currently available. Rather, our goal is that these volumes will act as either a primer for those texts (so that after reading these books, you will find those to be highly approachable), or as supplemental references (to assist you when an idea is not clear).

As a final comment before moving on, we do have long term plans for more volumes. We would likely break the theme of having one mathematical emphasis and simply build on the math from the first four volumes as necessary. The tentative plan for the topics in the later volumes is:

- Volume V – Supersymmetry and Supergravity
- Volume VI – Conformal Field Theory and Introductory String Theory

- Volume VII – D-Branes and M-Theory
- Volume VIII – Algebraic Geometry and Advanced Topics in String Theory
- Volume IX – Cosmology and Astrophysics
- Volume X – String Cosmology

As a warning, this series is a "spare-time project" for all of us and the timeline for these volumes will, unfortunately, be quite drawn out.

Outline of This Volume

As we said, the emphasis of this book is algebra, and the physical application is the (algebraic) structure of the standard model. However, because this book is the first in the series, there is quite a bit of additional material that is not entirely vital to the logical flow of this volume or the series as a whole.

Chapter 1 (which is vital to the overall flow) is a primer in the classical prerequisites. In short, it is undergraduate physics using graduate notation. Consequently it is much more cursory than the subsequent chapters. The major idea is to review:

- The variational calculus formulation of classical mechanics
- Special relativity
- Classical field theory (primarily electromagnetism)

Chapter 2 is meant to serve as a reminder that despite all of the math, we are still trying to describe something physical at the end of the day. This chapter is therefore an overview of how experimentalists think about particle physics. We talk about the history of particle physics and how particles and interactions are organized. The content of this chapter is not vital to the overall flow of the book, but is vital to any self-respecting theorist and should be read carefully.

Chapter 3 (which is vital) is meant to serve as an introduction to group theory. There are three major sections:

- Basic group theory. This section focuses on finite, discrete groups because many group theoretic ideas are easier to intuit in this setting. We won't use finite discrete groups much later in the book (or series), but the ideas illustrated by them will be constantly used.
- Basic Lie group theory.
- A specific Lie group: the Lorentz group of special relativity.

Chapter 4 (which is vital) then begins with the real physical content of this book. Using the algebraic machinery developed previously, we discuss the three major types of fields: spin-0 fields (also called scalar fields or Klein-Gordon fields), spin-$1/2$ fields (also called spinor fields or Dirac fields), and spin-1 fields (also called vector fields). Then we discuss gauge theory, the algebraic framework that seems to describe all of particle physics. Next is quantization, then symmetry breaking and non-Abelian gauge theory, and finally we look at the standard model itself.

Finally, Chap. 5 (which is not vital, but is highly recommended) is a survey of several of the extensions and generalizations of the standard model, including $SU(5)$ and $SO(10)$, supersymmetry, and approaches to quantum gravity. As a warning, this chapter is meant to be a vast, mountaintop overview of several ideas. It is not meant to be a thorough introduction to anything, and you will likely find that a lot of it leaves you wanting more. We will be coming back to almost every topic in Chap. 5 in later volumes in much greater detail. We encourage you to view it as a way of getting familiar with the generic ideas and vocabulary, not as a way of gaining deep understanding.

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Contents

Contributing Authors

Despite the single name on the cover, this book was by no means a solo effort. I've been fortunate enough to work with several other outstanding physicists/writers who are far more qualified than I am to write on many (or perhaps all) topics, and their contributions have made this book much better than I ever could have made it.

Chapter 2 was written by Dr. Jay Dittmann and Dr. Karen Bland of Baylor University. Dr. Dittmann is a professor at Baylor and he leads the Experimental High Energy Physics group's research on the CDF experiment at Fermilab, where he and Dr. Bland measure the properties of fundamental particles in proton-antiproton collisions.

Section 4.8 was written by Dr. Mario Serna of the United States Air Force Academy and myself.

Chapter 5 was written by Dr. Gerald Cleaver and Dr. Serna. Dr. Cleaver is a professor at Baylor University and the head of the Early Universe Cosmology and Strings group of Baylor's Center for Astrophysics, Space Physics, and Engineering Research (CASPER).

As for the rest of the book, I had tremendous help from all of the other authors. From editing to many (extremely) lengthy discussions on what needed to be added, changed, taken out, improved, and removed, the entire book is genuinely a collaborative effort. This is the reason for the use of first person plural throughout the book.

Chapter 1
Review of Classical Physics

As we said above, this chapter will be somewhat cursory. We assume that if you're reading this you are already at least somewhat familiar with these ideas, and we therefore won't spend much time detailing them. The following chapters will include much, much more explanation of the ideas contained in them.

The flip side of this is that you can think of this chapter as a good test of whether or not you have the prerequisites for this book. If you find that you can get through this chapter (even if you're limping a little at the end) then you'll probably be fine for the rest of the book. If, on the other hand, you can't follow what we're doing here at all, it may be best to look through some of the texts mentioned in the further reading section at the end of this chapter before diving in here.

1.1 Hamilton's Principle

Just about everything in physics begins with a **Lagrangian**, which is defined (the first time you see it) as the kinetic energy minus the potential energy,[1]

$$L = T - V, \tag{1.1}$$

where $T = T(q, \dot{q})$ and $V = V(q)$, and q is a general position coordinate (like x, y, r, θ, etc.). The dot in the term $\dot{q}$ represents a time derivative, so $\dot{q}$ represents velocity. Then the **Action** is defined as the integral of the Lagrangian from an initial time to a final time,

$$S = \int_{t_i}^{t_f} dt L(q, \dot{q}). \tag{1.2}$$

[1] We'll discuss a possible reason for this definition in a few pages.

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It is important to realize that S is a "functional" of the particle's world-line in $(q, \dot{q})$ space, not a function. This means that it depends on the entire path $(q, \dot{q})$ rather than a given point on the path (hence the integral). The only fixed points on the path are $q(t_i)$, $q(t_f)$, $\dot{q}(t_i)$, and $\dot{q}(t_f)$. The rest of the path is generally unconstrained, and the value of S depends on the entire path.

An analogy that may help is a room full of sand. Imagine that you have to walk from one corner of the room to another corner of the room in a certain amount of time, and at each step you have to pick up some amount of sand and drop some other amount of sand. And let's say that how much sand you have to pick up is a function of not only where you are in the room, but also how fast you're moving. And let's say that how much sand you drop is a function only of where you are in the room. If we then call the amount of sand you pick up $p = p(q, \dot{q})$, and the amount of sand you drop $d = d(q)$, then at each point you are gaining $l = p(q, \dot{q}) - d(q)$ units of sand.[2] So, after traveling some path across the room, you will have a total of

$$s = \int_{t_i}^{t_f} dt\, l(q, \dot{q}) \tag{1.3}$$

units of sand. This total amount of sand is a function of the path you took to cross the room as well as how fast you moved along that path. This poses an interesting mathematical problem. Let's say we want to move through the room collecting as little sand as possible. How could we go about finding the $(q, \dot{q})$ path that would do this?

This problem, it turns out, leads to one of the most fundamental ideas in physics: **Hamilton's Principle**. It says that as a particle moves through space, it "picks up" some kinetic energy $T(q, \dot{q})$ and "drops" some potential energy $V(q)$, and that nature will always choose a path that extremizes the amount of energy the particle picks up along the path.[3] In more formal language, Hamilton's principle says that nature will always choose a path in $(q, \dot{q})$ space that extremizes the functional S. Nature always chooses the most efficient way to get from one place to another.

But there is a practical problem with finding this "most efficient" path. Because S is a functional (and not a function), which depends on the entire path in $(q, \dot{q})$ space rather than a point, it cannot be extremized in the "Calculus I" sense of merely setting the derivative equal to 0. We'll need to do something slightly different.

To motivate the solution, let's look more closely at the "Calculus I" way of finding an extremum point. As we said above, we set the derivative equal to 0. In other words, x_0 is an extremal point of a function $f(x)$ if

$$\left.\frac{df(x)}{dx}\right|_{x_0} = f'(x_0) = 0 \tag{1.4}$$

[2]For the sake of the analogy let's say that you can have a "negative" amount of sand.

[3]Note that potential energy can be positive or negative.

(the term in the middle is a slight abuse of notation, but its meaning should be clear). Then, looking at the general Taylor expansion of $f(x)$ around some arbitrary point $x = x_0$, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \cdots . \tag{1.5}$$

Therefore, x_0 is an extremum of $f(x)$, the first-order term $(f'(x_0)(x - x_0))$ will be zero.

Put another way, equation (1.5) allows us to approximate the value of a function at one point in terms of information about the function at another point. Let's say we know everything about $f(x)$ at $x = x_1$ but we want to approximate $f(x)$ at $x = x_2$. If x_1 is reasonably close to x_2 we can keep only first-order terms and (1.5) gives

$$f(x_2) = f(x_1) + f'(x_1)(x_2 - x_1). \tag{1.6}$$

But as we just noted, if x_1 is an extremal point of $f(x)$, then we have

$$f(x_2) = f(x_1) \tag{1.7}$$

for x_2 sufficiently close to x_1. In this sense, we can think of the first-order term

$$\delta f = f'(x_1)(x_2 - x_1) \tag{1.8}$$

as a perturbation from $f(x_1)$ when moving a distance $(x_2 - x_1)$. So

$$f(x_2) = f(x_1) + \delta f. \tag{1.9}$$

And as we just saw, if x_1 is an extremal point, we have

$$\delta f = 0, \tag{1.10}$$

where δf is the first-order shift away from the original point x_1. In other words, a point is an extremum point if and only if a point very close to it has no first-order correction.

So, looking back at our functional case, while this general idea will work, there is a complication because we can't do a Taylor expansion around a point – we have to expand around an entire $(q, \dot{q})$ path. Doing this (and the results that follow) is the underlying idea in **Variational Calculus**, which we will use in a great deal of this book. Our approach will be to start with the functional action S along some $(q, \dot{q})$ path, move slightly away from this path, to the path $(q + \delta q, \dot{q} + \delta \dot{q})$, expand this to get the first-order term δS, and then set this equal to 0 to see what constraints this places on $(q, \dot{q})$.

Consider some arbitrary path $(q_0, \dot{q}_0)$ with action

$$S_0 = \int_{t_i}^{t_f} dt\ L(q_0, \dot{q}_0). \tag{1.11}$$

We can shift slightly away from this path, and the resulting action will be

$$S_\delta = \int_{t_i}^{t_f} dt\ L(q_0 + \delta q, \dot{q}_0 + \delta \dot{q}). \tag{1.12}$$

Taylor expanding this to first order gives

$$\begin{aligned} S_\delta &= \int_{t_i}^{t_f} dt\ L(q_0, \dot{q}_0) + \int_{t_i}^{t_f} dt \left(\delta q \frac{\partial L}{\partial q} + \delta \dot{q} \frac{\partial L}{\partial \dot{q}} \right) \\ &= \int_{t_i}^{t_f} dt\ L(q_0, \dot{q}_0) + \int_{t_i}^{t_f} dt \left(\delta q \frac{\partial L}{\partial q} + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \delta q \right). \end{aligned} \tag{1.13}$$

Integrating the second term in the parentheses by parts (and taking the variation of δq and $\delta \dot{q}$ to be 0 at t_i and t_f – these are the fixed points of the path because we require it to start and stop at particular places), we have

$$\begin{aligned} S_\delta &= \int_{t_i}^{t_f} dt\ L(q_0, \dot{q}_0) + \int_{t_i}^{t_f} dt \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \\ &= S_0 + \delta S. \end{aligned} \tag{1.14}$$

So, our constraint on $(q, \dot{q})$ to be an extremal path (setting the first-order term equal to 0) is

$$\delta S = \int_{t_i}^{t_f} dt \delta q \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) = 0. \tag{1.15}$$

The only way to guarantee this for an arbitrary variation δq from the path $(q, \dot{q})$ is to require

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{1.16}$$

This equation is called the **Euler-Lagrange** equation, and it produces the equations of motion of the particle.

The generalization to multiple coordinates q_i $(i = 1, \ldots, n)$ is straightforward:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0. \tag{1.17}$$

If, for example, we have motion in two dimensions (x and y) with Lagrangian L, we would have two equations of motion

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0,\\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= 0.\end{aligned} \tag{1.18}$$

A simple mathematical example of this is to find the extremal path between two points in a plane. The Lagrangian in this case can be simply taken as the length travelled, ds:

$$S_{length} = \int_a^b ds, \tag{1.19}$$

where a and b are the initial and final x values for the path. The (infinitesimal form of the) Pythagorean formula allows us to rewrite ds using $ds^2 = dx^2 + dy^2$. So, we have

$$\begin{aligned}S_{length} &= \int_a^b ds\\ &= \int_a^b \sqrt{dx^2 + dy^2}\\ &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx\\ &= \int_a^b \sqrt{1 + (y')^2} dx.\end{aligned} \tag{1.20}$$

So the Lagrangian is $L = \sqrt{1 + (y')^2}$, where x takes the place of t and y takes the place of q. The Euler-Lagrange equation (1.17) here gives:

$$\begin{aligned}\frac{\partial L}{\partial y} &= 0\\ \frac{d}{dx}\frac{\partial L}{\partial y'} &= \frac{d}{dx}\frac{\partial}{\partial y'}\left(\sqrt{1 + (y')^2}\right)\\ &= \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + (y')^2}}\right).\end{aligned} \tag{1.21}$$

So

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0,$$
$$\Longrightarrow \left(\frac{y'}{\sqrt{1+(y')^2}}\right) = A, \tag{1.22}$$

where A is some constant. Rearranging this gives

$$(y')^2 = A^2\big(1+(y')^2\big) \Longrightarrow y' = \sqrt{\frac{A^2}{1-A^2}} = M, \tag{1.23}$$

where M is yet another constant. Integrating this is straightforward:

$$y' = \frac{dy}{dx} = M \Longrightarrow y(x) = Mx + B, \tag{1.24}$$

where B is a final constant. This is the equation of a straight line, and we have therefore proven that the shortest distance between two points in the plane is a line. While this isn't a terribly profound realization, the fact that our variational calculus approach has allowed us to prove this is certainly remarkable. And while we will look at many more physical examples later, suffice it now to say that different Lagrangians result in different $(q, \dot{q})$ paths just as a different choice for the Pythagorean theorem (say, the length formula on a sphere, which is very different than on the plane) result in different "shortest paths".[4]

1.2 Noether's Theorem

When taking introductory physics, students often notice that a lot of the tools they are using are "conservation" laws: conservation of energy, conservation of momentum, conservation of mass, conservation of charge, etc. This is a very important observation, and it turns out that these conservation laws are actually specific manifestations of very deep aspects of the mathematical structure of physics. The idea of all of the conservation laws is that no matter how the system changes, there is something (energy, momentum, charge, etc.) that stays the same.

[4] As a preview, this is in some sense the fundamental idea behind general relativity. The form the Pythagorean theorem (which we'll see in a few pages is also called the "metric") takes depends on the geometry of the space you're in – i.e. it is different on the flat plane than on the curved sphere. Mathematically the fundamental equation of general relativity, Einstein's field equation, is a relationship between energy and the metric. So just as the "most efficient path" on a sphere is different than in a plane, when energy (i.e. mass) changes the metric via Einstein's field equation, matter follows different paths. We call this effect 'gravity'.

It turns out that this idea flows very naturally from the Lagrangian structure we discussed in the previous section. Given a Lagrangian, we can find a special collection of mathematical transformations on the Lagrangian (see below) that correspond to the physical conservation laws mentioned above.

To see this, consider a Lagrangian $L = L(q, \dot{q})$, and then make an infinitesimal transformation away from the original path:

$$q \to q + \epsilon\delta q, \tag{1.25}$$

where ϵ is some infinitesimal constant ($\epsilon \ll 1$) included for later convenience. This transformation will give

$$L(q, \dot{q}) \to L(q + \epsilon\delta q, \dot{q} + \epsilon\delta\dot{q}) = L(q, \dot{q}) + \epsilon\delta q\frac{\partial L}{\partial q} + \epsilon\delta\dot{q}\frac{\partial L}{\partial \dot{q}}. \tag{1.26}$$

If the Euler-Lagrange equations of motion are satisfied, so that $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$, then under $q \to q + \epsilon\delta q$,

$$L \to L + \epsilon\delta q\frac{\partial L}{\partial q} + \epsilon\delta\dot{q}\frac{\partial L}{\partial \dot{q}} = L + \epsilon\delta q\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \epsilon\frac{\partial L}{\partial \dot{q}}\frac{d}{dt}\delta q = L + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\epsilon\delta q\right). \tag{1.27}$$

So under $q \to q + \epsilon\delta q$ we have the first-order change $\delta L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\epsilon\delta q\right)$. We define the **Noether Current**, j, as

$$j = \frac{\partial L}{\partial \dot{q}}\delta q. \tag{1.28}$$

Now, if we can find some transformation δq that leaves the action invariant, or in other words such that $\delta S = 0$, then

$$\delta L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\epsilon\delta q\right) = \frac{dj}{dt} = 0, \tag{1.29}$$

and so the current j is a constant in time. In other words, j is *conserved.*

As a familiar example, consider a projectile described by the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy. \tag{1.30}$$

This will be unchanged under the transformation

$$x \to x + \epsilon, \tag{1.31}$$

where ϵ is any constant (here, $\delta q = 1$ in the above notation), because

$$x \to x + \epsilon \Rightarrow \dot{x} \to \dot{x}. \tag{1.32}$$

So,

$$j = \frac{\partial L}{\partial \dot{q}} \delta q = m\dot{x} \tag{1.33}$$

is conserved. We recognize $m\dot{x}$ as the momentum in the x-direction, which we expect to be conserved by conservation of momentum from Physics I.

For the sake of illustration, notice that we can find the equations of motion directly from this using the Euler-Lagrange equations (1.17):

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0, \\ \Longrightarrow \frac{d}{dt}(m\dot{x}) - 0 &= 0, \\ m\ddot{x} = F_x &= 0. \end{aligned} \tag{1.34}$$

and

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= 0, \\ \Longrightarrow \frac{d}{dt}(m\dot{y}) + mg &= 0, \\ m\ddot{y} = F_y &= -mg. \end{aligned} \tag{1.35}$$

So, using only the action, we have "derived" the conservation of momentum and we have derived Newton's Laws, from which we can derive the equations of motion for the particle:

$$\begin{aligned} m\ddot{x} = 0 \Longrightarrow \dot{x} &= v_{0,x} \\ \Longrightarrow x &= v_{0,x}t + x_0. \end{aligned} \tag{1.36}$$

and

$$\begin{aligned} m\ddot{y} = -mg \Longrightarrow \dot{y} &= v_{0,y} - gt \\ \Longrightarrow y &= y_0 + v_{0,y}t - \frac{1}{2}gt^2. \end{aligned} \tag{1.37}$$

You should recognize these as the standard kinematical equations for a projectile.

So to summarize the primary point of this section, **Noether's Theorem** says that whenever there is a continuous symmetry in the action for a physical object there is a corresponding conserved quantity. Conservation of linear momentum comes from an action that is invariant under continuous linear translations. Conservation of angular momentum comes from an action that is invariant under continuous rotations (write out, for example, the action for a planet in orbit and see if you can derive the conservation of angular momentum). Things like conservation of charge are a little more complicated, but we will see them later in this book.

While this is an enormously powerful idea in physics, it is actually fairly simple. Consider the projectile again. It should be clear that the Lagrangian should be invariant under translations in x. The path the ball moves in certainly shouldn't depend on where I'm standing across the surface of the earth (my x position). And if the motion is unaffected by where I am standing, then there's no reason its behavior "in the x direction" should ever change – whatever it starts off doing is what it should keep doing.[5] And therefore its x-motion, or momentum in the x direction, remains the same. However, this argument fails with the y component because of the gravitational field; the path the ball follows does depend on the y direction, and therefore the y momentum is not conserved.

The same thing holds for angular momentum. The behavior of a planet in orbit is unaffected by rotations around the planet, and therefore the Lagrangian is unaffected by these rotations. And, consequently, the momentum around the planet (or the angular momentum) is constant. All conservation laws are really saying is that if nothing changes in a certain direction, motion in that direction won't change either.

1.3 Conservation of Energy

In the last section we saw that the conservation of *momentum* comes from the invariance of the action under translations in *space*. In this sense momentum and space have a special relationship. Now, we'll see that energy and time share a similar relationship – namely conservation of energy comes from the invariance of an action under translations in *time*.

Consider the rate of change of the Lagrangian with respect to time. There is no reason to expect the Lagrangian to be constant with respect to time and therefore $\frac{dL}{dt} \neq 0$. However, there is a very important property we can get from $\frac{dL}{dt}$ if we assume the equations of motion of the particle are satisfied. Starting with the general total derivative of L with respect to time, we have

$$\frac{dL}{dt} = \frac{d}{dt}L(q,\dot{q}) = \frac{\partial L}{\partial q}\frac{dq}{dt} + \frac{\partial L}{\partial \dot{q}}\frac{d\dot{q}}{dt} + \frac{\partial L}{\partial t}. \tag{1.38}$$

[5] In terms of mechanics this is simply Newton's First Law. There is a very profound geometric generalization of this involving geodesics and curved spacetime manifolds that we'll look at in the next book in this series.