Classical Electromagnetic Theory

Second Edition

Fundamental Theories of Physics

An International Book Series on The Fundamental Theories of Physics: Their Clarification, Development and Application

Volume 145

Classical Electromagnetic Theory

Second Edition

by

Jack Vanderlinde

University of New Brunswick,
Fredericton, NB, Canada

KLUWER ACADEMIC PUBLISHERS
NEW YORK, BOSTON, DORDRECHT, LONDON, MOSCOW

eBook ISBN: 1-4020-2700-1
Print ISBN: 1-4020-2699-4

Created in the United States of America

Visit Springer's eBookstore at: http://ebooks.springerlink.com
and the Springer Global Website Online at: http://www.springeronline.com

Preface

In questions of science, the authority of a thousand is not worth the humble reasoning of a single individual.
Galileo Galilei, physicist and astronomer (1564-1642)

This book is a second edition of "Classical Electromagnetic Theory" which derived from a set of lecture notes compiled over a number of years of teaching electromagnetic theory to fourth year physics and electrical engineering students. These students had a previous exposure to electricity and magnetism, and the material from the first four and a half chapters was presented as a review. I believe that the book makes a reasonable transition between the many excellent elementary books such as Griffith's *Introduction to Electrodynamics* and the obviously graduate level books such as Jackson's *Classical Electrodynamics* or Landau and Lifshitz' *Electrodynamics of Continuous Media*. If the students have had a previous exposure to Electromagnetic theory, all the material can be reasonably covered in two semesters. Neophytes should probable spend a semester on the first four or five chapters as well as, depending on their mathematical background, the Appendices B to F. For a shorter or more elementary course, the material on spherical waves, waveguides, and waves in anisotropic media may be omitted without loss of continuity.

In this edition I have added a segment on Schwarz-Christoffel transformations to more fully explore conformal mappings. There is also a short heuristic segment on Cherenkov radiation and Bremstrahlung. In Appendix D there is a brief discussion of orthogonal function expansions. For greater completeness, Appendices E and F have been expanded to include the solution of the Bessel equation and Legendre's equation as well as obtaining the generating function of each of the solutions. This material is not intended to supplant a course in mathematical methods but to provide a ready reference provide a backstop for those topics missed elsewhere. Frequently used vector identities and other useful formulas are found on the inside of the back cover and referred to inside the text by simple number (1) to (42).

Addressing the complaint "*I don't know where to start, although I understand all the theory*", from students faced with a non-transparent problem, I have included a large number of examples of varying difficulty, worked out in detail. This edition has been enriched with a number of new examples. These examples illustrate both the theory and the techniques used in solving problems. Working through these examples should equip the student with both the confidence and the knowledge to solve realistic problems. In response to suggestions by my colleagues I have numbered all equations for ease of referencing and more clearly delineated examples from the main text.

Because students appear generally much less at ease with magnetic effects than with electrical phenomena, the theories of electricity and magnetism are developed in parallel. From the demonstration of the underlying interconvertability of the fields in Chapter One to the evenhanded treatment of electrostatic and magnetostatic problems to the covariant formulation, the treatment emphasizes the relation between the electric and magnetic fields. No attempt has been made to follow the historical development of the theory.

An extensive chapter on the solution of Laplace's equation explores most of the techniques used in electro- and magnetostatics, including conformal mappings and separation of variable in Cartesian, cylindrical polar, spherical polar and oblate ellipsoidal coordinates. The magnetic scalar potential is exploited in many examples to demonstrate the equivalence of methods used for the electric and magnetic potentials. The next chapter explores the use of image charges in solving Poisson's equation and then introduces Green's functions, first heuristically, then more formally. As always, concepts introduced are put to use in examples and exercises. A fairly extensive treatment of radiation is given in the later portions of this book. The implications of radiation reaction on causality and other limitations of the theory are discussed in the final chapter.

I have chosen to sidestep much of the tedious vector algebra and vector calculus by using the much more efficient tensor methods, although, on the advice of colleagues, delaying their first use to chapter 4 in this edition. Although it almost universally assumed that students have some appreciation of the concept of a tensor, in my experience this is rarely the case. Appendix B addresses this frequent gap with an exposition of the rudiments of tensor analysis. Although this appendix cannot replace a course in differential geometry, I strongly recommend it for self-study or formal teaching if students are not at ease with tensors. The latter segments of this appendix are particularly recommended as an introduction to the tensor formulation of Special Relativity.

The exercises at the end of each chapter are of varying difficulty but all should be within the ability of strong senior students. In some problems, concepts not elaborated in the text are explored. A number of new problems have been added to the text both as exercises and as examples. As every teacher knows, it is essential that students consolidate their learning by solving problems on a regular basis. A typical regimen would consist of three to five problems weekly.

I have attempted to present clearly and concisely the reasoning leading to inferences and conclusions without excessive rigor that would make this a book in Mathematics rather than Physics. Pathological cases are generally dismissed. In an attempt to have the material transfer more easily to notes or board, I have labelled vectors by overhead arrows rather than the more usual bold face. As the material draws fairly heavily on mathematics I have strived to make the book fairly self sufficient by including much of the relevant material in appendices.

Rationalized SI units are employed throughout this book, having the advantage of yielding the familiar electrical units used in everyday life. This connection to reality tends to lessen the abstractness many students impute to electromagnetic theory. It is an added advantage of SI units that it becomes easier to maintain a clear distinction between B and H, a distinction frequently lost to users of gaussian units.

I am indebted to my students and colleagues who provided motivation for this book, and to Dr. Matti Stenroos and Dr. E.G. Jones who class tested a number of chapters and provided valuable feedback. Lastly, recognizing the unfortunate number of errata that escaped me and my proofreaders in the first edition, I have made a significantly greater effort to assure the accuracy of this edition.

Jack Vanderlinde
email: jvdl@unb.ca

Table of Contents

Chapter 1

Static Electric and Magnetic Fields in Vacuum

1.1 Static Charges

Static electricity, produced by rubbing different materials against one another, was known to the early Greeks who gave it its name (derived from ἤλεκτρον, pronounced ēlectron, meaning amber). Experiments by Du Fay in the early 18th century established that there are two kinds of electricity, one produced by rubbing substances such as hard rubber and amber, called resinous, and another produced by rubbing glassy substances such as quartz, dubbed vitreous. Objects with like charge were found to repel one another, while objects with unlike charge were found to attract. Benjamin Franklin attempted to explain electricity in terms of an excess or deficiency of the vitreous electric fluid, leading to the designations *positive* and *negative*.

A report by Benjamin Franklin that a cork ball inside an electrically charged metal cup is not attracted to the inside surface of the cup led Joseph Priestly to infer that, like gravity, electrical forces obey an inverse square law. This hypothesis was almost immediately confirmed (to limited accuracy) by John Robison, but the results were not published for almost 50 years. Cavendish, in an elegant experiment, showed that if a power law holds,[1] the exponent of r in the force law could not differ from minus two by more than 1 part in 50, but he failed to publish his results. Charles Augustin de Coulomb, who, in the late 18th century, measured both the attractive and repulsive force between charges with a delicate torsion balance, is credited with the discovery of the force law bearing his name – he found that the force is proportional to the product of the charges, acts along the line joining the charges, and decreases inversely as the square of the distance between them. Charges of opposite sign attract one another, whereas charges of the same sign repel. It has been verified experimentally that the exponent of r varies from minus two by no more than 1 part in 10^{16} over distances of order one meter.

[1] A modern interpretation suggests that a test of the exponent is not appropriate because a power law is not the anticipated form. In line with considerations by Proca and Yukawa, the potential should take the form $e^{-\beta r}/r$ ($\beta = m_\gamma c/\hbar$) where m_γ is the rest mass (if any) of the photon. Astronomical measurements of Jupiter's magnetic field place an upper limit of 4×10^{-51} kg on the mass of the photon

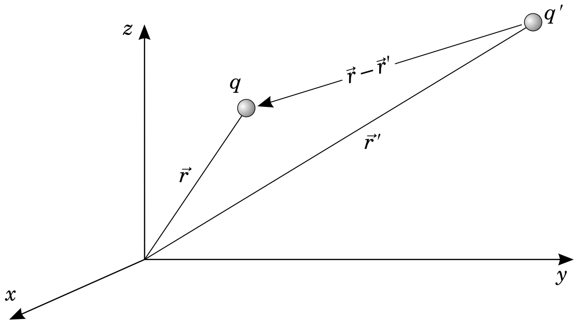

Figure 1.1: When q and q' are situated at r and r' respectively, the vector pointing from q' to q is $(\vec{r}-\vec{r}\,')$.

1.1.1 The Electrostatic Force

The inverse square electric force on a particle with charge q located at $\vec{r}$ due to a second charged particle with charge q' located at $\vec{r}\,'$ is (Figure 1.1) economically expressed by Coulomb's law:

$$\vec{F}_q = k_e\,\frac{qq'(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \tag{1–1}$$

Various system of units assign different values to k_e. *Gaussian Units* (Appendix A), used in many advanced texts, set $k_e \equiv 1$ thereby defining the unit of charge, the *esu.* In Gaussian units, the force is measured in *dynes* and r in cm. In this book we will uniformly use SI units, which have the advantage of dealing with ordinary electrical units such as volt and amperes at the cost of requiring k_e to take on a value of roughly 9×10^9 N-m^2/C^2. Anticipating later developments, we write $k = 1/4\pi\varepsilon_0$ to obtain:

$$\vec{F}_q = \frac{1}{4\pi\varepsilon_0}\,\frac{qq'(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \tag{1–2}$$

where ε_0, the *permittivity of free space*, is experimentally determined to be 8.84519×10^{-12} C^2 /N-m^2. More properly, as we will see, ε_0 can be derived from the (defined) speed of light in vacuum, $c \equiv 299{,}792{,}458$ m/s and the (defined) *permeability of free space*, $\mu_0 \equiv 4\pi\times10^{-7}$ kg-m/C^2, to give $\varepsilon_0 \equiv 1/\mu_0 c^2 = 8.85418781\cdots\times10^{-12}$ C^2 /N-m^2.

The force of several charges q_i' on q is simply the vector sum of the force q_1' exerts on q plus the force of q_2' on q and so on until the last charge q_n'. This statement may be summarized as

$$\vec{F}_q = \frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i'(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} \tag{1–3}$$

or, to translate it to the language of calculus, with the small element of source charge denoted by dq'

$$\vec{F}_q = \frac{q}{4\pi\varepsilon_0}\int\frac{(\vec{r}-\vec{r}\,')dq'}{|\vec{r}-\vec{r}\,'|^3} \tag{1–4}$$

Although we know that electric charges occur only in discrete quanta $\pm e = \pm 1.6021917 \times 10^{-19}$ coulomb (or $\pm\frac{1}{3}e$ and$\pm\frac{2}{3}e$ if quarks are considered), the elementary charge is so small that we normally deal with many thousands at a time and we replace the individual charges by a smeared-out *charge density.* Thus the charge distribution is described by a charge density $\rho(\vec{r}\,') = n(\vec{r}\,')e$, with n the *net* number (positive minus negative) of positive charges per unit volume centered on r'. For a distributed charge, we may generally write

$$\vec{F}_q = \frac{q}{4\pi\varepsilon_0}\int_\tau \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r' \tag{1–5}$$

where $dq\,'$ has been replaced by$\rho(\vec{r}\,')d^3r'$. The differential d^3r' represents the three-dimensional differential volume in arbitrary coordinates. For example, in Cartesians $d^3r' \equiv dx'dy'dz'$, whereas in spherical polars, $d^3r' \equiv r'^2\sin\theta' dr' d\theta' d\varphi'$. For charges distributed over a surface, it suffices to replace $dq\,'$ by $\sigma(\vec{r}\,')dA'$ and for line charges we write $dq' = \lambda(\vec{r}\,')d\ell'$.

If required, the lumpiness of a point charge q' can be accommodated in (1–5) by letting the charge density have the form of a three-dimensional Dirac δ function.[2] Line charges and surface charges can similarly be accommodated.

The original form (1–2) is easily recovered by setting $\rho(\vec{r}\,') = q'\delta(\vec{r}\,' - \vec{r}_q)$ with $\delta(\vec{r}\,' - \vec{r}_q) = \delta(x' - x_q)\delta(y' - y_q)\delta(z' - z_q)$ [3] and carrying out the integration called for in (1–5).

Example 1.1: Find the force on a charge q lying on the z axis above the center of a circular hole of radius a in an infinite uniformly charged flat plate occupying the x-y plane, carrying surface charge density σ (Figure 1.2).

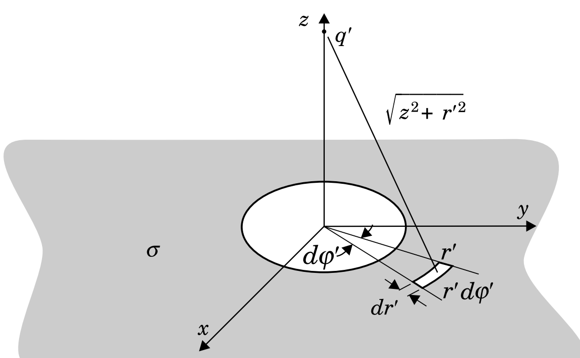

Figure 1.2: Example 1.1 – A uniformly distributed charge lies on the x-y plane surrounding the central hole in the plate.

[2]The δ function $\delta(x - a)$ is a sharply spiked function that vanishes everywhere except at $x = a$, where it is infinite. It is defined by $\int f(x)\delta(x-a)dx = f(a)$ when a is included in the region of integration; it vanishes otherwise. For further discussion, see Appendix C.

[3]In non-Cartesian coordinate systems the δ function may not be so obvious. In spherical polar coordinates, for example, $\delta(\vec{r} - \vec{r}\,') = r'^{-2}\delta(r - r\,')\delta(\cos\theta - \cos\theta\,')\,\delta(\varphi - \varphi\,')$.

Solution: The boundary of the integration over the charge distribution is most easily accommodated by working in cylindrical polar coordinates. The field point is located at $z\hat{k}$ and the source points have coordinates r' and φ' giving $\vec{r}-\vec{r}\,' = z\hat{k} - r'\cos\varphi'\hat{\imath} - r'\sin\varphi'\hat{\jmath}$. An element of charge dq' takes the form $dq' = \sigma(r\,', \varphi')dA' = \sigma r'\,dr\,'d\varphi'$. The distance $|\vec{r}-\vec{r}\,'|$ of the source charge element from the field point is $\sqrt{z^2+r'^2}$ leading us to write:

$$\vec{F}_q = \frac{q}{4\pi\varepsilon_0}\int_a^\infty \int_0^{2\pi} \frac{\sigma(z\hat{k} - r'\cos\varphi\,\hat{\imath} - r'\sin\varphi\,\hat{\jmath})}{(z^2+r'^2)^{3/2}} r'dr'd\varphi' \qquad \text{(Ex 1.1.1)}$$

The integrations over $\sin\varphi$ and $\cos\varphi$ yield 0, reducing the integral to:

$$\vec{F}_q = \frac{q}{4\pi\varepsilon_0} 2\pi \int_a^\infty \frac{\sigma z\hat{k}}{(z^2+r'^2)^{3/2}} r'dr'$$

$$= \frac{q}{2\varepsilon_0} \left.\frac{-\sigma z\hat{k}}{\sqrt{z^2+r'^2}}\right|_a^\infty = \frac{q\sigma z\hat{k}}{2\varepsilon_0\sqrt{z^2+a^2}} \qquad \text{(Ex 1.1.2)}$$

The force points upward above the plane and downward below. At large distances it tends to a constant, $\frac{1}{2\varepsilon_0}q\sigma\hat{k}$, exactly what it would be in the absence of the hole. We further verify that as $z \to 0$, in the center of the hole, the force vanishes. It is probably worth mentioning that the charge would not distribute itself uniformly on a conducting plate so that we have not solved the problem of a charged conducting plate with a hole.

Example 1.2: Find the force exerted on a point charge Q located at $\vec{r}$ in the x-y plane by a long (assume infinite) line charge λ, uniformly distributed along a thin wire lying along the z axis (Figure 1.3).

Solution: An element of charge along the wire is given by $dq' = \lambda dz'$ so that using (1–4) we can write the force on the charge

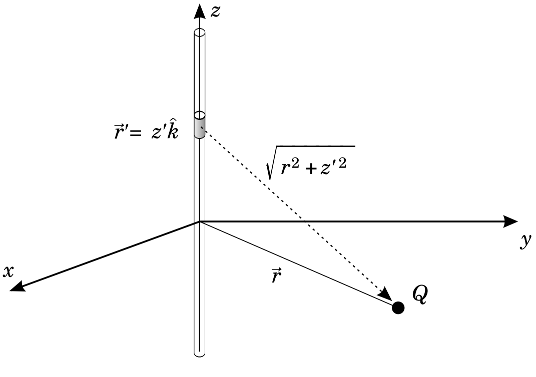

Figure 1.3: A line charge λ is distributed uniformly along the z axis.

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{(r\hat{r} - z'\hat{k})\,\lambda dz'}{(r^2 + z'^2)^{3/2}} \tag{Ex 1.2.1}$$

where $\hat{r}$ is a unit vector in the x-y plane pointing from the origin to the charge Q. The integral is best evaluated in two parts. The second part,

$$\int_{-\infty}^{\infty}\frac{-\lambda z'\hat{k}dz'}{(r^2 + z'^2)^{3/2}} = 0 \tag{Ex 1.2.2}$$

because the integrand is odd. The remaining integral is then,

$$\vec{F}_Q = \frac{Q}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda r\hat{r}dz'}{(r^2 + z'^2)^{3/2}} \tag{Ex 1.2.3}$$

r and $\hat{r}$ are constants with respect to dz' so that we may use (28) to evaluate the integral:

$$\vec{F}_Q = \left.\frac{Q\lambda r\hat{r}z'}{4\pi\epsilon_0 r^2\sqrt{r^2 + z'^2}}\right|_{-\infty}^{\infty} = \frac{Q\lambda\hat{r}}{2\pi\epsilon_0 r} \tag{Ex 1.2.4}$$

This question will be revisited in example 1.5 where we will allow the charge carrying wire to have finite size.

1.1.2 The Electric Field

Although Coulomb's law does an adequate job of predicting the force one particle causes another to feel, there is something almost eerie about one particle pushing or pulling another without any physical contact. Somehow, it would be more satisfying if the charged particle felt a force due to some local influence, a *field*, created by all other charged particles. This field would presumably exist independent of the sensing particle q. (In quantum electrodynamics, even the notion of a field without a carrier [the photon] is held to be aphysical.)

The force on the sensing particle must be proportional to its charge; all the other properties of the force will be assigned to the *electric field*, $\vec{E}(\vec{r})$. Thus we define the electric field by

$$\vec{F} = q\vec{E}(\vec{r}) \tag{1–6}$$

where $\vec{F}$ is the force on the charge q situated at $\vec{r}$ and $\vec{E}(\vec{r})$ is the electric field at position $\vec{r}$ due to all other charges. (The source charge's coordinates will normally be distinguished from the *field point* by a prime [$'$].) One might well wonder why the sensing particle, q's field would not be a component of the field at its position. A simple answer in terms of the electric field's definition, (1–6) above, is that since a particle can exert no net force on itself, its own field cannot be part of the field it senses (in the same way that you cannot lift yourself by your bootstraps). Unfortunately, this appears to suggest that two point charges at the same position might well experience a different field. That argument, however, is somewhat academic, as two point charges at the same location would give rise to infinite interaction forces. One might also argue that, as a point particle's field must be spherically symmetric, it would, in fact make no difference whether we included the sensing particle's own field in computing the force on the particle. This particular point of view runs into trouble when we consider the no longer spherically symmetric fields of accelerated charges in Chapter 12. Whatever the best answer, this self field will continue to trouble us whenever we deal seriously with point particles.

Factoring the charge q from Coulomb's law (1–5), we find that the electric field produced by a charge distribution $\rho(\vec{r}\,')$ must be

$$\vec{E}(\vec{r}\,) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3r' \tag{1–7}$$

where the integration is carried out over all space (ρ must of course vanish at sufficiently large r, making the volume of integration less than infinite). We reiterate that coordinates of the source of the field will be primed, while the *field* points are denoted by unprimed coordinates.

Example 1.3: Find the electric field above the center of a flat, circular plate of radius R, bearing a charge Q uniformly distributed over the top surface (Figure 1.3).

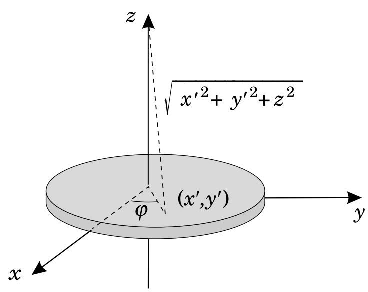

Figure 1.4: The field at height z above a uniformly charged disk.

Solution: The charge density on the plate takes the form $\rho(\vec{r}\,) = \frac{Q}{\pi R^2}\delta(z')$ for $x'^2 + y'^2 \le R^2$ and 0 elsewhere. Using

$$\vec{E}(\vec{r}\,) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,')\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3r' \tag{Ex 1.3.1}$$

we obtain the explicit expression

$$\vec{E}(0,0,z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\int_{-R}^{R}\int_{-\sqrt{R^2-y'^2}}^{\sqrt{R^2-y'^2}} \frac{z\hat{k} - x'\hat{\imath} - y'\hat{\jmath}}{(x'^2+y'^2+z^2)^{3/2}}\,dx'dy' \tag{Ex 1.3.2}$$

In cylindrical polar coordinates, $x' = r'\cos\varphi'$, $y' = r'\sin\varphi'$, and $dx'dy' = r'dr'd\varphi'$, giving

$$\vec{E}(0,0,z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\pi R^2}\int_0^R\int_0^{2\pi} \frac{(z\hat{k} - r'\cos\varphi'\hat{\imath} - r'\sin\varphi'\hat{\jmath}\,)\,d\varphi'\,r'\,dr'}{(r'^2+z^2)^{3/2}} \tag{Ex 1.3.3}$$

The integration over φ' eliminates the $\sin\varphi'$ and the $\cos\varphi'$ terms, leaving only

$$\vec{E}(0,0,z) = \frac{Q}{2\pi\varepsilon_0 R^2}\int_0^R \frac{z\hat{k}r'dr'}{(r'^2+z^2)^{3/2}} = -\frac{Q}{2\pi\varepsilon_0 R^2}\left.\frac{z\hat{k}}{(r'^2+z^2)^{1/2}}\right|_0^R$$

$$= \frac{Q}{2\pi\varepsilon_0 R^2}\left(\frac{z\hat{k}}{\sqrt{z^2}} - \frac{z\hat{k}}{\sqrt{R^2+z^2}}\right) = \frac{Q\hat{k}}{2\pi\varepsilon_0 R^2}\left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) \quad \text{(Ex 1.3.4)}$$

When z is small compared to R the field reduces to $(\sigma/2\varepsilon_0)\hat{k}$, the value it would have above an infinite sheet, whereas at large distances it tends to $Q/(4\pi\varepsilon_0 z^2)$. It is worth noting that adding to Ex 1.3.4 the field of the plate with the hole deduced from Ex 1.1.2 gives precisely the field of the infinite plate with the hole filled in.

The invention of the electric field appears at this point no more than a response to a vague uneasiness about the action at a distance implicit in Coulomb's law. As we progress we will endow the field, $\vec{E}$, with properties such as energy and momentum, and the field will gain considerable reality. Whether $\vec{E}$ is merely a mathematical construct or has some independent objective reality cannot be settled until we discuss radiation in Chapter 10.

1.1.3 Gauss' Law

It is evident that the evaluation of $\vec{E}$, even for relatively simple source charge distributions, is fairly cumbersome. When problems present some symmetry, they can often be solved much more easily using the integral form of Gauss' law, which states

$$\oint_S \vec{E}(\vec{r})\cdot d\vec{S}(\vec{r}) = \frac{q'}{\varepsilon_0} \tag{1–8}$$

where S is any closed surface, q' is the charge enclosed within that surface, $d\vec{S}$ is surface element of S pointing in the direction of an outward-pointing normal, and $\vec{r}$ is the location of the element $d\vec{S}$ on the surface. Note that S need not be a physical surface.

To prove this result, we expand $\vec{E}$ in (1–8) using (1–7) to obtain

$$\oint_S \vec{E}(\vec{r})\cdot d\vec{S}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\oint\left[\int \frac{\rho(\vec{r}\,')(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3r'\right]\cdot d\vec{S}(\vec{r})$$

$$= \frac{1}{4\pi\varepsilon_0}\int\left[\oint \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\cdot d\vec{S}(\vec{r})\right]\rho(\vec{r}\,')\,d^3r' \tag{1–9}$$

We must now evaluate the surface integral $\oint \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\cdot d\vec{S}$.

We divide the source points into those lying inside the surface S and those lying outside. The divergence theorem (20) generally allows us to write

$$\oint \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\cdot d\vec{S} = \int \vec{\nabla}\cdot\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}d^3r \tag{1–10}$$

For any fixed source point with coordinate $\vec{r}\,'$ outside the closed surface, S, the field point $\vec{r}$ (located inside S for the volume integration resulting from the application of the divergence theorem) never coincides with $\vec{r}\,'$, and it is easily verified by direct differentiation that the integrand on the right hand side of (1–10), the divergence of $(\vec{r}-\vec{r}\,')/|\vec{r}-\vec{r}\,'|^3$, vanishes identically. We conclude, therefore, that charges lying outside the surface make no contribution to the surface integral of the electric field.

When $\vec{r}\,'$ is inside the bounding surface S, the singularity at $\vec{r}=\vec{r}\,'$ prevents a similar conclusion because the divergence of (1–10) becomes singular. To deal with this circumstance, we exclude a small spherical region of radius R centered on $\vec{r}\,'$ from the $\oint$ and integrate over this spherical surface separately. In the remaining volume, excluding the small sphere, $(\vec{r}-\vec{r}\,')/|\vec{r}-\vec{r}\,'|^3$ again has a vanishing divergence and presents no singularities, allowing us to conclude that it too, makes no contribution to the surface integral of $\vec{E}$. Setting $\vec{R}=(\vec{r}-\vec{r}\,')$ and $\hat{R}\cdot d\vec{S}=R^2 d\Omega$, with $d\Omega=\sin\theta\, d\theta\, d\varphi$, an element of solid angle, we may write the integral

$$\oint_{sphere}\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\cdot d\vec{S}(\vec{r}\,)=\oint_{sphere}\frac{(\vec{R}\cdot\hat{R})\,R^2\,d\Omega}{R^3}=4\pi \tag{1–11}$$

We substitute this result, (1–11) into (1–9) to get (1–8), the desired result

$$\oint_{sphere}\vec{E}(\vec{r}\,)\cdot d\vec{S}(\vec{r}\,)=\frac{1}{4\pi\varepsilon_0}\cdot 4\pi\int\rho(\vec{r}\,')d^3r'=\frac{q'}{\varepsilon_0}$$

A more geometric insight into the evaluation of (1–10) may be obtained by recognizing that the left hand side of (1–10) represents the solid angle covered by the surface as seen from $\vec{r}\,'$. When $\vec{r}\,'$ is inside the surface, S encloses the entire 4π solid angle, whereas when $\vec{r}\,'$ lies outside S, the contribution from near side of the surface makes the same solid angle as the back side, but the two bear opposite signs (because the normals point in opposite directions) and cancel one another.

To reiterate, only the charge inside the surface S enters into the integration. In words, Gauss' law states that the perpendicular component of the electric field integrated over a closed surface equals $1/\varepsilon_0$ times the charge enclosed within that surface, irrespective of the shape of the enclosing surface.

Example 1.4: Find the electric field at a distance r from the center of a uniformly charged sphere of radius R and total charge Q.

Solution: The charge $Q(r)$ enclosed within a sphere of radius r centered on the charge center is

$$Q(r)=\begin{cases}\left(\dfrac{r}{R}\right)^3 Q & \text{for } r\le R\\ Q & \text{for } r>R\end{cases} \tag{Ex 1.4.1}$$

On a spherical surface of radius r, symmetry requires $\vec{E}=E_r(r)\hat{r}$ with no φ dependence. Gauss' law then gives us

$$\oint\vec{E}\cdot d\vec{S}=\oint\vec{E}\cdot\hat{r}\;r^2 d\Omega=4\pi r^2\,E_r \tag{Ex 1.4.2}$$

Equating this to $Q(r)/\varepsilon_0$ yields

$$E_r = \begin{cases} \dfrac{rQ}{4\pi\varepsilon_0 R^3} & r \le R \\ \dfrac{Q}{4\pi\varepsilon_0 r^2} & r > R \end{cases} \tag{Ex 1.4.3}$$

The electric field inside the sphere grows linearly with radius and falls off quadratically outside the sphere. As Gauss' law applies equally to gravity and electrostatics (it depends only on the r^{-2} nature of the force), the same field dependence pertains to gravitational fields within gravitating bodies.

EXAMPLE 1.5: Find the electric field near a long, uniformly charged cylindrical rod of radius a.

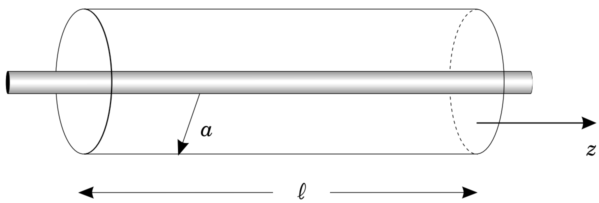

Figure 1.5: The cylinder about the rod forms a Gaussian surface perpendicular to the electric field. The end faces contribute nothing to the surface integral.

Solution: $E_z = 0$, as reversing the z axis, or translating the origin along the z axis does not change the problem. $E_\theta = 0$ as reversing θ or rotating the system about the z axis leaves the system invariant. Evidently $\vec{E}(r,\ \theta,\ z) = E_r(r)\,\hat{r}$. Drawing a cylinder about the rod as indicated in Figure 1.4 (the cylinder may be interior to the rod), we obtain from Gauss' law, (1–8),

$$\oint \vec{E}\cdot d\vec{S} = \int \frac{\rho}{\varepsilon_0}\, d^3r'$$

For $r > a$, this becomes

$$2\pi r\ell E_r = \frac{\rho\pi\, a^2\ell}{\varepsilon_0}, \qquad \text{or } E_r = \frac{\rho a^2}{2\varepsilon_0 r}, \tag{Ex 1.5.1}$$

while for $r < a$ we obtain

$$2\pi r\ell E_r = \frac{\rho\pi\, r^2\ell}{\varepsilon_0}, \qquad \text{or } E_r = \frac{\rho r}{2\varepsilon_0}. \tag{Ex 1.5.2}$$

We will make frequent use of Gaussian cylinders (and "pill-boxes" in the next example) throughout the remainder of this book.

Example 1.6: Find the electric field between two conducting infinite parallel plates bearing surface charge densities σ and $-\sigma$.

Solution: For a Gaussian enclosing surface we now choose a very flat cylinder (commonly referred to as a *pillbox*) that includes one of the two charged surfaces, say the top surface, of the bottom plate, as illustrated in Figure 1.6. The charge enclosed within the pillbox is σA where A is the flat area included in the box. Because of the symmetry we anticipate an electric field whose only non vanishing component is E_z. Gauss' law then becomes:

$$\oint \vec{E}\cdot d\vec{S} = A(E_{top} - E_{bott}) = \frac{\sigma A}{\varepsilon_0} \tag{Ex 1.6.1}$$

If the plates are conductors, then the electric field on the bottom surface of the pillbox lying inside the conductor must vanish (otherwise charges inside the metal would be subject to a Coulomb force and move until the field does vanish). We conclude, then, that the electric field at the top surface of the pillbox is $E_z = \sigma/\varepsilon_0$. Increasing the height of the pillbox straddling the bottom plate so that its top surface lies progressively closer to the top plate produces no variation of the enclosed charge; we infer that the field is uniform, (i.e., it does not vary with z.)

A similar argument could of course have been employed at the top plate, giving exactly the same result. This time below the plate, the surface of the pillbox along which $\vec{E}$ does not vanish points downward so that $-E_z = -\sigma\varepsilon_0$. As stated above, we shall make frequent use of the pillbox whenever we deal with the behavior of the field at surfaces, both conducting and nonconducting.

Gauss' law may be restated in terms of the local charge density by means of the divergence theorem. Writing the charge enclosed within the boundary S as the volume integral of the charge density enclosed, we find

$$\oint \vec{E}\cdot d\vec{S} = \int \frac{\rho}{\varepsilon_0}\, d^3r' \tag{1–12}$$

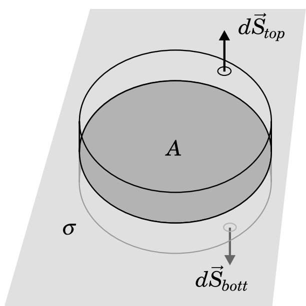

Figure 1.6: The "pillbox" encloses one of the charged surfaces. The electric field is parallel to the curved surface side so that the integral over the curved side makes no contribution to the surface integral.

With the aid of the divergence theorem, the surface integral of the electric field may be rewritten:

$$\int_{vol} \vec{\nabla} \cdot \vec{E}(\vec{r}) d^3r = \int_{vol} \frac{\rho(\vec{r}\,')}{\varepsilon_0}\, d^3r' \tag{1–13}$$

Since the boundary and hence the volume of integration was arbitrary but the same on both sides, we conclude that the integrands must be equal

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0} \tag{1–14}$$

1.1.4 The Electric Potential

The expression for the electric field arising from a charge distribution may be usefully expressed as a gradient of a scalar integral as follows.

$$\begin{aligned}\vec{E}(\vec{r}) &= \frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}\,') \frac{(\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3}\, d^3r' \\ &= -\frac{1}{4\pi\varepsilon_0} \int \rho(\vec{r}\,')\, \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}\,'|}\right) d^3r' \end{aligned} \tag{1–15}$$

As $\vec{\nabla}$ acts only on the unprimed coordinates, we may take it outside the integral (1–15) to obtain

$$\vec{E}(\vec{r}) = -\vec{\nabla} \left(\frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3r'\right) \equiv -\vec{\nabla} V \tag{1–16}$$

We identify the *electric potential*, V, with the integral

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}\, d^3r' \tag{1–17}$$

Since the curl of any gradient vanishes, we have immediately

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} V) = 0 \tag{1–18}$$

There are obvious advantages to working with the scalar V instead of the vector field $\vec{E}$. First, the integral for V requires computing only one component rather than the three required for $\vec{E}$. Second, the electric field obtained from several localized sources would require taking the vector sum of the fields resulting from each source. Since taking the gradient is a linear operation, the electric field could as well be found from

$$\begin{aligned}\vec{E} &= -\vec{\nabla} V_1 - \vec{\nabla} V_2 - \vec{\nabla} V_3 - \ldots \\ &= -\vec{\nabla}(V_1 + V_2 + V_3 + \ldots)\end{aligned}$$

where now we need only find a scalar sum of potentials. These simplifications make it well worthwhile to use the electric scalar potential whenever possible.

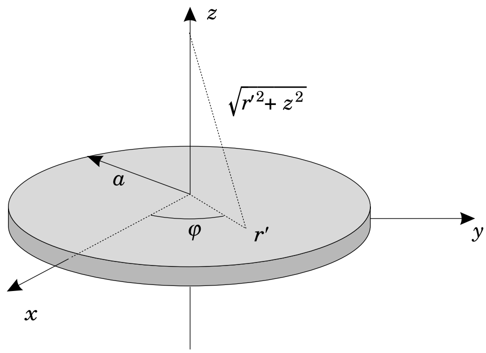

Figure 1.7: The circular plate is assumed to lie in the x-y plane.

EXAMPLE 1.7: Find the potential V at height z above the center of a disk of radius a carrying charge Q uniformly distributed over its top surface (Figure 1.7).

Solution: The charge density on the disk is

$$\rho = \begin{cases} \dfrac{Q}{\pi a^2}\,\delta(z') & \text{for } x'^2 + y'^2 \le a^2 \\ 0 & \text{elsewhere} \end{cases} \tag{Ex 1.7.1}$$

The potential above the center of the plate is then

$$V(0,0,z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec r\,')\,d^3r\,'}{|\vec r - \vec r\,'|} = \frac{1}{4\pi\varepsilon_0}\int_0^a\int_0^{2\pi} \frac{Q}{\pi a^2}\,\frac{r'dr'd\varphi'}{\sqrt{r'^2 + z^2}}$$

$$= \frac{Q}{2\pi\varepsilon_0 a^2}\,\sqrt{r'^2 + z^2}\,\Bigg|_0^a = \frac{Q}{2\pi\varepsilon_0 a^2}\left(\sqrt{a^2 + z^2} - z\right) \tag{Ex 1.7.2}$$

To find the field $\vec E$ as we have already done in the example 1.3, we merely find the gradient of V giving

$$E_z(0,0,z) = -\frac{\partial V}{\partial z} = \frac{Q}{2\pi\varepsilon_0 a^2}\left(1 - \frac{z}{\sqrt{a^2 + z^2}}\right) \tag{Ex 1.7.3}$$

At large distance the field reduces to that of a point charge. (Notice that because we have not calculated V as a function of x and y [or r and φ], no information about $E_x$or E_y can be obtained from V, however, symmetry dictates that they must vanish on the z axis.)

Like the electric field, the potential can also be expressed in terms of the local charge density by combining the differential form of Gauss' law with the definition of V, $-\vec\nabla V = \vec E$:

$$\vec\nabla \cdot \vec E = \vec\nabla \cdot\,(-\vec\nabla V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0} \tag{1–19}$$

The differential equation solved by V, $\nabla^2 V = -\rho/\varepsilon_0$, is known as Poisson's equation. Its homogeneous counterpart ($\rho = 0$) is the Laplace equation. The Laplace equation will be discussed in considerable detail in Chapter 5 and the solution of Poisson's equation is the subject of Chapter 6.

1.2 Moving Charges

In the remainder of this chapter we consider the forces and fields due to slowly moving charges. Although the charges are allowed to move, we do insist that a steady state exist so that the forces are static. This restriction will be lifted in later chapters.

1.2.1 The Continuity Equation

Among the most fundamental conservation laws of physics is conservation of charge. There is no known interaction that creates or destroys charge (unlike mass, which can be created or annihilated). This conservation law is expressed by the equation of continuity (1–24).

We define the current flowing into some volume as the rate that charge accumulates in that volume:

$$I = \frac{dQ}{dt} \tag{1–20}$$

More usefully, we express I as the net amount of charge crossing the boundary into the volume τ with boundary S per unit time,

$$I = -\oint_S \rho\vec{v}\cdot d\vec{S} = -\oint_S \vec{J}\cdot d\vec{S} \tag{1–21}$$

where $\vec{J} \equiv \rho\vec{v}$ is the *current density*. (Recall that $d\vec{S}$ points outward from the volume so that $\vec{J}\cdot d\vec{S}$ is an outflow of current, hence the negative sign for current flowing into the volume.) Combining (1–20) and (1–21) and replacing Q with the volume integral of the charge density, we obtain

$$\oint_S \vec{J}\cdot d\vec{S} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_\tau \rho\, d^3r \tag{1–22}$$

With the aid of the divergence theorem, the right and left hand side of this equation become

$$\int_\tau \vec{\nabla}\cdot\vec{J}\, d^3r = -\int_\tau \frac{\partial\rho}{\partial t}\, d^3r \tag{1–23}$$

Since the volume of integration τ was arbitrary, the integrands must be equal, giving the continuity equation

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{1–24}$$

(It is perhaps useful to maintain this intuitive view of divergence as the outflow of a vector field from a point.) The equation of continuity states simply that an increase in charge density can only be achieved by having more current arrive at the point than leaves it. The continuity equation expresses the conservation of charge, one of the cornerstones of physics and is fundamental to the study of electromagnetic theory. When dealing with electrostatics we have $\vec{\nabla} \cdot \vec{J} = 0$ (note that this does not preclude current flows, only that $\partial\rho/\partial t = 0$).

1.2.2 Magnetic Forces

Magnetic forces were familiar to Arab navigators before 1000 A.D. who used lodestones as primitive compasses. Magnetic poles of the earth were postulated in the thirteenth century but it was not until about 1820 that Biot, Savart and Ampère discovered the interaction between currents and magnets.

As Biot, Savart and Ampère discovered, when charges are in motion, a force additional to the electrical force appears. We could merely postulate a force law (Equation (1–38), but it would be more satisfying to demonstrate the intimate connection between electricity and magnetism by obtaining magnetism as a consequence of electricity and relativistic covariance.

As a simple demonstration of why we expect currents to interact, let us consider two long line charges, each of length L with linear charge density λ_1 (consider them tending to infinity, requiring only that λL be a finite constant) lying along the x-axis and λ_2 parallel to the λ_1, at distance r from the x-axis. As seen by a stationary observer, the force on wire *2* is (Ex 1.2.4)

$$F = F_e = \frac{1}{2\pi\varepsilon_0}\frac{\lambda_1\lambda_2 L}{r^2}\,\vec{r} \tag{1–25}$$

A second observer moving with velocity v along the x axis sees the line charges in motion with velocity $-v$. According to special relativity, the transverse (to the motion) components of forces in a stationary and a moving (indicated by a prime) reference frame are related by $F = \gamma F'$ $[\gamma \equiv (1-v^2/c^2)^{-1/2}]$. We deduce, therefore, that in the moving observer's frame, the total force of one wire on the other should be $F' = \gamma^{-1}F$. [4]

Alternatively, we calculate a new length L' for the moving line charges using the length contraction formula $L' = \gamma^{-1}L$, and we deduce, assuming conservation of charge, an appropriately compressed charge density $\lambda_1' = \gamma\lambda_1$ and $\lambda_2' = \gamma\lambda_2$ in the moving frame. Thus, if the same laws of physics are to operate, the moving observer calculates an electric force

$$\vec{F}_e' = \frac{1}{2\pi\varepsilon_0}\frac{\lambda_1'\lambda_2' L'}{r^2}\,\vec{r} = \frac{1}{2\pi\varepsilon_0}\frac{(\gamma\lambda_1)(\gamma\lambda_2)\gamma^{-1}L}{r^2}\,\vec{r} = \frac{\gamma}{2\pi\varepsilon_0}\frac{\lambda_1\lambda_2 L}{r^2}\,\vec{r} = \gamma\vec{F}_e \tag{1–26}$$

clearly not the result anticipated above.

In fact, the moving observer, who of course believes the line charges to be in motion, must invent a second force, say F_m, in order to reconcile the results of the alternative calculations.

Thus

[4] More properly, the transverse force F, on a particle moving with velocity v in system Σ, is related to F' in Σ' where the particle has velocity v' by $\gamma F = \gamma' F'$, and transverse means perpendicular to the velocity of frame Σ' with respect to Σ.

$$\vec{F}' = \frac{\vec{F}_e}{\gamma} = \vec{F}_e' + \vec{F}_m' = \gamma\vec{F}_e + \vec{F}_m' \tag{1–27}$$

which we may solve for $\vec{F}_m'$:

$$\begin{aligned}\vec{F}_m' &= \frac{\vec{F}_e}{\gamma} - \gamma\,\vec{F}_e = \gamma\,\vec{F}_e\left(\frac{1}{\gamma^2} - 1\right)\\
&= \gamma\vec{F}_e\left(1 - \frac{v^2}{c^2} - 1\right) = -\frac{v^2}{c^2}\,\gamma\vec{F}_e\\
&= -\,\vec{F}_e'\frac{v^2}{c^2} = \frac{(\lambda_1' v\lambda_2' v)L'}{2\pi\varepsilon_0 c^2 r^2}\,\vec{r}\\
&= -\,\frac{I_1' I_2' L'}{2\pi\varepsilon_0 c^2 r^2}\vec{r} \equiv -\frac{\mu_0\,I_1' I_2' L'}{2\pi r^2}\,\vec{r}\end{aligned} \tag{1–28}$$

In the frame where the line charges move, we find a force opposite to the electrical force, proportional to the product of the currents. Parallel currents attract one another; antiparallel currents repel. The term $1/\varepsilon_0 c^2$, conventionally abbreviated as μ_0, is called the permeability of free space. The constant μ_0 has a defined value $4\pi \times 10^{-7}$ kg-m/C^2. This choice fixes the unit of charge, the coulomb, which we have conveniently left undefined to this point.

Again shying away from action at a distance we invent a field $\vec{B}$, produced by I_2 at the location of the current I_1 with which the current I_1 interacts. Since $d\vec{F}_m$, the magnetic force on a short segment $Id\vec{\ell}$ of current *1*, is perpendicular to I_1, it must be of the form

$$d\vec{F}_m = I_1 d\vec{\ell} \times \vec{B} \tag{1–29}$$

where $\vec{B}$ is a yet undetermined vector field known as the *magnetic induction field*, or alternatively the *magnetic flux density.*

The magnetic force on a moving charged point particle is easily deduced by identifying $\vec{v}dq$ with $Id\vec{\ell}$ in (1–29) to obtain $d\vec{F}_m = dq(\vec{v} \times \vec{B})$. The total force on a charged particle in a static electromagnetic field is known as the *Lorentz* force

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{1–30}$$

Let us attempt to determine the magnetic induction field produced by the current I_2 assumed to run along the z axis. Equation (1–29) requires $\vec{B}$ to be perpendicular to $d\vec{F}_m$ (which is directed along $\vec{r}$, the cylindrical radial position vector of the current $I_1 d\ell$). If, in addition, we make the not unnatural assumption that $\vec{B}$ is also perpendicular to I_2, then $\vec{B}$ must be directed along $\vec{I}_2 \times \vec{r}$. Taking $\vec{B} = C(\vec{I}_2 \times \vec{r})$ and substituting this form into (1–29), we find

$$d\vec{F}_m = CI_1 d\vec{\ell} \times (\vec{I}_2 \times \vec{r}) = -CI_1 I_2 d\ell\,\vec{r}$$

which, when compared with the result from (1–28)

$$d\vec{F}_m = -\frac{\mu_0\,I_1 d\ell\,I_2}{2\pi r^2}\vec{r} \tag{1–31}$$

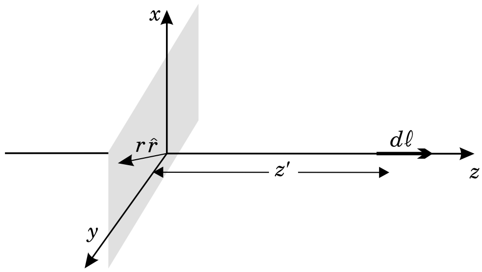

Figure 1.8: The current is assumed to run along the z axis, and we pick the observer in the x-y plane.

leads immediately to

$$\vec{B}(\vec{r}\,) = \frac{\mu_0}{2\pi}\frac{\vec{I}_2\ \times \vec{r}}{r^2} \tag{1–32}$$

1.2.3 The Law of Biot and Savart

The magnetic induction field, B, of the long straight wire (1–32), must in fact be the sum of contributions from all parts of wire *2* stretching from $-\infty$ to $+\infty$. Since the magnetic force is related to the electric force by a Lorentz transformation that does not involve r, $\vec{E}$ and $\vec{B}$ must have the same r dependence, $(1/r^2)$. The field, $d\vec{B}$, generated by a short segment of wire, $d\vec{\ell}$ carrying current I_2 at the origin, must therefore be given by

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I_2 d\vec{\ell}\,' \times \vec{r}}{r^3} \tag{1–33}$$

(The choice of numeric factor $[\mu_0/4\pi]$ will be confirmed below.) Equation (1–33) is easily generalized for current segments located at $\vec{r}\,'$, rather than at the origin, giving

$$d\vec{B}(\vec{r}\,) = \frac{\mu_0}{4\pi}\frac{I_2 d\vec{\ell}\,' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \tag{1–34}$$

Integrating over the length of the current source, we obtain the Biot-Savart law:

$$\vec{B}(\vec{r}\,) = \frac{\mu_0}{4\pi}\int \frac{I_2 d\vec{\ell}\,' \times (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \tag{1–35}$$

We might verify that this expression does indeed give the field (1–32) of the infinite straight thin wire. Without loss of generality we may pick our coordinate system as in Figure 1.8, with the wire lying along the z-axis and the field point in the (x-y plane. Then $\vec{r} - \vec{r}\,' = r\hat{r} - z'\hat{k}$, $|\vec{r} - \vec{r}\,'| = \sqrt{r^2 + z'^2}$, and $d\vec{\ell}\,' = \hat{k}\,dz'$. The flux density, $\vec{B}$, may now be calculated:

$$\vec{B}(\vec{r}\,) = \frac{\mu_0 I_2}{4\pi}\int_{-\infty}^{+\infty}\frac{\hat{k}\times(r\hat{r} - z'\hat{k}\,)}{(r^2 + z'^2)^{3/2}}dz' = \frac{\mu_0 I_2}{4\pi}\int_{-\infty}^{\infty}\frac{r\,(\hat{k}\times\hat{r})}{(r^2 + z'^2)^{3/2}}dz'$$

$$= \frac{\mu_0 I_2(\hat{k}\times\hat{r})r}{4\pi} \left. \frac{z'}{\sqrt{r^2+z'^2}} \right|_{-\infty}^{\infty} = \frac{\mu_0}{2\pi} \frac{\vec{I}_2\times\vec{r}}{r^2} \tag{1–36}$$

Noting that (1–36) reproduces (1–32), we consider the factor $\mu_0/4\pi$ confirmed.

As we will normally deal with currents that have finite spatial extent, the current element $I_2 d\vec{\ell}$ in (1–35) should in general be replaced by $\int_S \vec{J}\cdot d\vec{S}d\ell$, where S is the cross section I_2 occupies. The Biot-Savart law may then be written

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\, d^3r' \tag{1–37}$$

Equation (1–37) plays the same role for magnetic fields as Coulomb's law (1–7) does for electric fields.

Example 1.8: A circular loop of radius a carrying current I lies in the x-y plane with its center at the origin (Figure 1.9). Find the magnetic induction field at a point on the z-axis.

Solution: In cylindrical coordinates, $\vec{J}(\vec{r}) = I\delta(r'-a)\,\delta(z')\,\hat{\varphi}$ and $\vec{r}-\vec{r}\,' = z\hat{k} - a\hat{r}$. The numerator of the integrand of (1–37) then becomes

$$\vec{J}\times(\vec{r}-\vec{r}\,') = I\,\delta(r'-a)\,\delta(z')\,(z\hat{r}+a\hat{k}) \tag{Ex 1.8.1}$$

Thus (1–37) becomes:

$$\vec{B}(\,0,\,0,\,z) = \frac{\mu_0}{4\pi}\int \frac{I\,\delta(r'-a)\,\delta(z')\,(z\hat{r}+a\hat{k})}{(a^2+z^2)^{3/2}}\, r'\,dr'\,d\varphi'\,dz'$$

$$= \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{I(\,-z\hat{r}+a\hat{k})}{(a^2+z^2)^{3/2}}\, a\,d\varphi = \frac{\mu_0}{4\pi}\,\frac{2\pi a^2 I\hat{k}}{(a^2\,+\,z^2)^{3/2}} \tag{Ex 1.8.2}$$

where we have used the fact that $\oint \hat{r}d\varphi = 0$. In terms of the *magnetic moment*, $\vec{m} = I\pi a^2\hat{k}$, (defined in Chapter 2, usually just current times area of the loop) of the loop, we may approximate this result at large distances as

$$\vec{B}(0,\,0,\,z) = 2\,\frac{\mu_0}{4\pi}\,\frac{\vec{m}}{R^3} \tag{Ex 1.8.3}$$

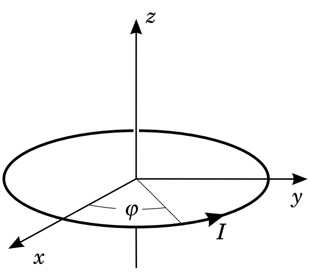

Figure 1.9: A circular loop carrying current I in the x-y plane.