Progress in Mathematics

Volume 285

Series Editors

H. Bass
J. Oesterlé
A. Weinstein

Alexander Soifer
Editor

Ramsey Theory

Yesterday, Today, and Tomorrow

Birkhäuser

Editor
Alexander Soifer
University of Colorado at Colorado Springs
College of Letters, Arts, and Sciences
1420 Austin Bluffs Parkway
Colorado Springs, CO 80918
USA
asoifer@uccs.edu

ISBN 978-0-8176-8091-6 e-ISBN 978-0-8176-8092-3
DOI 10.1007/978-0-8176-8092-3
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010938367

© Springer Science+Business Media, LLC 2011
All rights reserved. This work may not be translated or copied in whole or in part without the written permission of the publisher (Springer Science+Business Media, LLC, 233 Spring Street, New York, NY 10013, USA), except for brief excerpts in connection with reviews or scholarly analysis. Use in connection with any form of information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed is forbidden.
The use in this publication of trade names, trademarks, service marks, and similar terms, even if they are not identified as such, is not to be taken as an expression of opinion as to whether or not they are subject to proprietary rights.

Front cover utilizes the photograph of Frank Plumpton Ramsey (1903–1930), aged 18. Reproduced by kind permission of the Provost and the Scholars of King's College, Cambridge.

www.birkhauser-science.com

Dedicated to Paul Erdős, who envisioned Ramsey Theory,
shaped much of it by his problems, conjectures and results,
and inspired us all to dive into its deep and beautiful waters.

What This Book Is About and How It Came into Being

Ramsey theory is a fascinating, approximately 100-year-old field of mathematics that has a non-empty intersection with combinatorics, number theory, geometry, ergodic theory, topology, combinatorial geometry, set theory, measure theory, and so on. Ramsey theory possesses its own unifying ideas, and some of its results are among the most beautiful theorems of mathematics. The main mathematical idea of Ramsey theory is this: no matter how large and elaborate a system S is, and how large a positive integer n is, we can choose a large enough super system Q containing S, so that no matter how Q is colored in n colors, Q contains a monochromatic copy of S. Thus one can say that Ramsey theory studies *mathematics of coloring*.

In 2008 the director, Fred Roberts, and the executive committee of DIMACS[1] invited me to organize a three-day workshop on Ramsey theory. In response to Dr. Roberts' desire to host a nongeneric original view of the field, I proposed *Ramsey Theory: Yesterday, Today, and Tomorrow.* This was approved, and the workshop took place on May 27–29, 2009 at the Busch Campus of Rutgers University in Piscataway, New Jersey. The workshop looked at the emergence and history of Ramsey theory (*Yesterday*), its results (*Today*), and its future (*Tomorrow*) through its open problems, conjectures, and aspirations. In addition to mathematical and historical research, we also looked at how Ramsey theory can harness the power of computing in discovering mathematical results.

The workshop turned out to be an international event. It attracted researchers from the United States, England, Czech Republic, Hungary, and Germany. The speakers included world-renowned leaders of the field, such as Ronald L. Graham, Joel H. Spencer, and Jaroslav Nešetřil. It also included some of the most promising young researchers such as Jacob Fox of Princeton University, Andrzej Dudek of Carnegie Mellon University, Lynn Scow of the University of California Berkeley, and Dmytro Karabash of the Courant Institute of Mathematical Sciences.

[1] The Center for Discrete Mathematics and Theoretical Computer Science, a collaborative project of Rutgers and Princeton Universities, AT&T Labs – Research, Alcatel–Lucent Bell Labs, Cancer Institute of New Jersey (CINJ), NEC Laboratories America, and Telcordia Technologies.

The workshop inspired Ann Kostant, the executive editor of mathematics at Birkhäuser, to propose that I organize and edit this volume of surveys authored or coauthored by workshop participants under the title of the workshop *Ramsey Theory: Yesterday, Today, and Tomorrow* for its "Progress in Mathematics" Birkhäuser series.

This volume opens with "Yesterday", surveys of the prehistory and early history of Ramsey theory. They are followed by surveys of progress that has been made in Ramsey theory and in areas that arose from Ramsey theory, the descendants of Ramsey theory; these surveys point out directions in which Ramsey theory and its descendants may move in the future. The last three surveys address Euclidean Ramsey theory and related coloring problems. The survey on open problems is coauthored by Ronald L. Graham, one of the authors of Euclidean Ramsey theorems I, II, and III, 1973–1975, which constitute a major portion of the foundation of the subject. This survey is followed by a history of the mysterious problem of the chromatic number of the plane, and the final survey is on similar problems for the rational points in real Euclidean spaces.

In addition to invited and contributed talks, the workshop featured a "Problem Posing Session." Accordingly, this volume includes a section of open problems proposed at the workshop.

On behalf of all contributors to this volume, I thank Fred Roberts and the DIMACS Executive Committee for inviting and supporting the workshop, and the National Science Foundation for financial support. I thank the entire DIMACS' staff for their wonderful help, especially Nicole Clark, Linda Casals, and Mel Janowitz. I thank Ann Kostant of Birkhäuser for offering us such a fine vehicle for spreading the Ramseyan word, and the editors of the "Progress in Mathematics" series Hyman Bass, Joseph Oesterlé, and Alan Weinstein for accepting this volume in their enlightened series.

May 2010 Alexander Soifer

Some of the plenary speakers of the workshop, from the right: Ronald L. Graham, Jaroslav Nešetřil, Joel H. Spencer, and Alexander Soifer

Contents

What This Book Is About and How It Came into Being vii

Ramsey Theory Before Ramsey, Prehistory and Early History: An Essay in 13 Parts 1
Alexander Soifer

1 Overture 1
2 David Hilbert's 1892 Cube Lemma 2
3 The Issai Schur 1916 Theorem 3
4 The Baudet–Schur–Van der Waerden 1927 Theorem 7
5 The Generalized 1928 Schur Theorem 8
6 The Frank Plumpton Ramsey Principle 9
7 The Paul, György, and Esther Happy End Problem 11
8 Richard Rado's Regularity 14
9 Density and Arithmetic Progressions 17
10 The Tibor Gallai Theorem 22
11 De Bruijn–Erdős's 1951 Compactness Theorem 23
12 Khinchin's Small Book of Big Impact 23
13 Long Live the Young Theory! 24
References 25

Eighty Years of Ramsey $R(3, k)$ . . . and Counting! 27
Joel Spencer

1 Basics 27
2 George, Esther, Paul 28
3 Erdős Magic 29
4 An Erdős Gem 30
5 Upper Bounds 31
6 The Lovász Local Lemma 33
7 Random Greedy Triangle-Free 34
8 *R(3, k)* Resolved! 35
9 Random Greedy Triangle-Free Redux 36

10 Epilogue . . . 38
References . . . 38

Ramsey Numbers Involving Cycles . . . 41
Stanisław P. Radziszowski
1 Scope and Notation . . . 41
2 Two-Color Numbers Involving Cycles . . . 42
2.1 Cycles . . . 43
2.2 Cycles Versus Complete Graphs . . . 44
2.3 Cycles Versus Wheels . . . 46
2.4 Cycles Versus Books . . . 47
2.5 Cycles Versus Other Graphs . . . 49
3 Multicolor Numbers for Cycles . . . 50
3.1 Three Colors . . . 50
3.2 More Colors . . . 53
3.3 Cycles Versus Other Graphs . . . 54
4 Hypergraph Numbers for Cycles . . . 55
References . . . 56

On the Function of Erdős and Rogers . . . 63
Andrzej Dudek and Vojtěch Rödl
1 Introduction . . . 63
2 The Most Restrictive Case . . . 64
2.1 Proof of $f_{s,s+1}(n) \leq O\left(n^{1-1/O(s^4 log\, s)}\right)$ [1] . . . 65
2.2 Proof of $\Omega\left(n^{\frac{1}{2}}\right) \leq f_{s,s+1}(n)$ for $s \geq 2$ [2] . . . 67
2.3 Proof of $f_{s,s+1}(n) \leq O\left(n^{2/3}\right)$ for $s \geq 2$ [4] . . . 67
3 General Bounds . . . 69
3.1 Proof of $\Omega\left(n^{a_k(s)}\right) \leq f_{s,s+k}(n)$ [14, 15] . . . 71
3.2 Sketch of the Proof of $f_{s,s+k}(n) \leq O\left(n^{((k+1)/(2k+1))+\varepsilon}\right)$ for $s \geq s_0 = s_0(\varepsilon, k)$ [4] . . . 73
4 Concluding Remarks . . . 75
References . . . 75

Large Monochromatic Components in Edge Colorings of Graphs: A Survey . . . 77
András Gyárfás
1 Introduction . . . 77
1.1 A Remark of Erdős and Rado and Its Extension . . . 77
1.2 Colorings from Affine Planes . . . 78
1.3 Extending Colorings by Substitutions . . . 79
2 2-Colorings . . . 79
2.1 Type of Spanning Trees, Connectivity, Diameter . . . 79
2.2 Gallai-Colorings: Substitutions to 2-Colorings . . . 82
3 Multicolorings: Basic Results and Proof Methods . . . 83
3.1 Complete Bipartite Graphs: Counting Double Stars . . . 83

3.2 Fractional Transversals: Füredi's Method . . . 85
3.3 Fine Tuning . . . 85
3.4 When Both Methods Work: Local Colorings . . . 86
3.5 Hypergraphs . . . 88
4 Multicolorings: Type of Components . . . 89
4.1 Components with Large Matching . . . 89
4.2 Double Stars . . . 90
5 Variations . . . 90
5.1 Vertex-Coverings by Components . . . 91
5.2 Coloring by Group Elements . . . 91
5.3 Coloring Geometric Graphs . . . 91
5.4 Coloring Noncomplete Graphs . . . 92
References . . . 94

Szlam's Lemma: Mutant Offspring of a Euclidean Ramsey Problem from 1973, with Numerous Applications . . . 97
Jeffrey Burkert and Peter Johnson, Jr.
1 1973: A Volcano Erupts . . . 97
2 Some Definitions and More Background . . . 98
3 What Happened to the Rather Red Coloring Problem from 1973? . 101
4 Distance Graphs . . . 102
5 Szlam's Lemma, a Connection Between Rather Red Colorings and Chromatic Numbers . . . 104
6 van der Waerden Numbers, Cyclic van der Waerden Numbers, and a Lower Bound on Them Both . . . 107
References . . . 112

Open Problems in Euclidean Ramsey Theory . . . 115
Ron Graham and Eric Tressler
1 Introduction . . . 115
2 Ramsey Sets . . . 117
3 Unit Distance Graphs . . . 117
4 More General Distance Graphs . . . 119
References . . . 119

Chromatic Number of the Plane & Its Relatives, History, Problems and Results: An Essay in 11 Parts . . . 121
Alexander Soifer
1 The Problem . . . 121
2 The History . . . 124
3 Polychromatic Number of the Plane & Results Near the Lower Bound . . . 135
4 De Bruijn–Erdős Reduction to Finite Sets and Results Near the Lower Bound . . . 138
5 Polychromatic Number of the Plane & Results Near the Upper Bound . . . 141

6 Continuum of 6-Colorings of the Plane 144
7 Chromatic Number of the Plane in Special Circumstances 148
8 Colored Space 149
9 Rational Coloring 151
10 Axioms of Set Theory and the Chromatic Number of Graphs 153
11 Predicting the Future 156
References 157

Euclidean Distance Graphs on the Rational Points 163
Peter Johnson, Jr.
1 Definitions 163
2 The Search for $\chi(\mathbb{R}^n, 1)$ Leads to the Search for $\chi(\mathbb{Q}^n, 1)$ 164
3 Distances Other Than 1? 167
4 Problems 172
References 174

Open Problems Session 177
1 Problems Submitted by William Gasarch 179
2 Problems Submitted by Peter Johnson, Jr. 180
3 Problems on Topological Stability of Chromatic Numbers Submitted by Dmytro Karabash 182
4 Problem on the Gallai–Ramsey Structure, Submitted by Colton Magnant 183
5 Problems Involving Triangles, Submitted by Stanisław P. Radziszowski 185
6 Problems on Chromatic Number of the Plane and Its Relatives, Submitted by Alexander Soifer 189

Ramsey Theory Before Ramsey, Prehistory and Early History: An Essay in 13 Parts[1]

Alexander Soifer

What amazes us today is, of course, that no one in Hamburg (including Schreier and Artin) had known about Schur's work [1916]. In that connection we must realize that the kind of mathematics involved in the [Baudet–Schur] conjecture was not mainstream, and that combinatorics was not a recognized field of mathematics at all.

– Nicolaas G. de Bruijn[2]

It takes a long time to become young.

– Pablo Picasso

1 Overture

How does a new theory emerge? It usually manifests itself in the older and established areas of mathematics. Gradually a critical mass of results appears, prompting a realization that what we have is a new identifiable field of mathematical thought, with its own set of problems and methods. As a fetus in a womb, the new theory eventually does not fit in the existing classification of mathematical thought. That is when the child is born. Ramsey theory has not been an exception.

In retrospect we all put history in a neat chronological order. In reality, older results may not have been known in time to provide the influence that our chronological order would suggest. We plug historical holes in the emergence of Ramsey theory not unlike filling holes in human evolution. History teaches that it is not enough to be right. In order to influence the evolution of one's field, one has to be lucky to be right at the right time. Aristarchus of Samos (Ἀριστταρχος, 310 BC–ca. 230 BC) was the first to explicitly conjecture a heliocentric model of the solar system. However, he did not influence the evolution of astronomy. Almost 1,800 years later, Nicolaus Copernicus (1473–1543), presented (more comprehensively) the conjecture of the heliocentric cosmology again, and did influence the evolution of astronomy in a major way.

[1] Much of this material is contained in the author's monograph [Soi], however, this text contains new facts and observations that were not known to the author in 2008 when [Soi] was published. Also, the emphasis here is quite different from [Soi].

[2] E-mail to A. Soifer, January 5, 2004.

A. Soifer
University of Colorado at Colorado Springs, Coloredo Springs, CO 80918, USA
e-mail: asoifer@uccs.edu

A. Soifer (ed.), *Ramsey Theory: Yesterday, Today, and Tomorrow*,
Progress in Mathematics 285, DOI 10.1007/978-0-8176-8092-3_1,
© Springer Science+Business Media, LLC 2011

Let me summarize in one paragraph the fetal development preceding Ramsey theory's birth. As we believe now, David Hilbert's cube lemma was the first Ramseyan result, but it did not influence anyone at the time and thus did not give birth to Ramsey theory. Issai Schur's 1916 theorem could have remained unnoticed too, but Schur was first to realize that he had run into something new and striking. And so Schur continued by conjecturing the result on monochromatic arithmetic progressions. However, Schur's conjecture did not reach Bartel Leendert van der Waerden, and the Ramseyan train of thought risked running out of fuel. The unborn Ramseyan mathematics was very lucky that another person, Pierre Joseph Henry Baudet, independently of Schur posed the same conjecture. Baudet passed away at the tender age of 30, but his conjecture impressed his friend and mentor Frederik Schuh. Schuh or Schuh's circle at the University of Amsterdam was the source of the conjecture for the 23-year old Van der Waerden. Having proved the conjecture, Van der Waerden walked away from Ramseyan prehistory. Issai Schur, on the other hand, continued to produce Ramseyan mathematics, and moreover directed and inspired his PhD students Richard Rado, Hildegard Ille and Alfred Brauer to do the same. Then came Frank Plumpton Ramsey who delivered the Two Commandments, the principles of the theory later named in his honor. Ramsey died at the age of 26, before his publication announced to the world the birth of the new theory.

In what follows, we take a more detailed look at the emergence of Ramsey theory, and trace how the Ramseyan baton was passed. A much more detailed exposition of Ramsey theory's prehistory and early history required a whole book, the book [Soi] that now exists. I have continued historical investigations, and this survey contains some new facts.

2 David Hilbert's 1892 Cube Lemma

As far as we know today, the first Ramseyan-type result appeared in 1892 as a little-noticed assertion in [Hil]. Its author was the great David Hilbert. In this work Hilbert proved the theorem of our interest merely as a tool for his study of irreducibility of rational functions with integral coefficients.

A set $Q_n\,(a, x_1, x_2, \ldots, x_n)$ of integers is called an *n-dimensional affine cube* if there exist $n + 1$ positive integers $a, x_1, \ldots, x_n$ such that

$$Q_n\,(a, x_1, x_2, \ldots, x_n) = \left\{a + \sum\nolimits_{i \in F} x_i : \varnothing \neq F \subseteq \{1, 2, \ldots, n\}\right\}.$$

It is convenient to use the symbol $[n]$ for the starting segment of positive integers:

$$[n] = \{1, 2, \ldots, n\}.$$

The theorem, which preceded the Schur and the Baudet–Schur–Van der Waerden theorems, reads as follows.

The Hilbert Cube Lemma 1. *For every pair of positive integers r, n, there exists a least positive integer $m = H(r, n)$ such that in every r-coloring of $[m]$ there exists a monochromatic n-dimensional affine cube.*

It seems that David Hilbert's monochromatic cube lemma was the first example of Ramseyan mathematics. Apparently nobody – including Hilbert – appreciated the lemma much. Hilbert did not continue research in the direction the lemma showed. The field did not learn about Hilbert's lemma until much later. The lemma was added as the first instance of Ramseyan thought, not due to its influence, which was nonexistent, but due to its respectable birth year of 1892.

3 The Issai Schur 1916 Theorem

Probably nobody remembered the 1892 Hilbert lemma by the time the second Ramseyan-type result appeared in 1916 as a little-noticed assertion in number theory. Its author was Issai Schur. Our interest here lies in the result he obtained during 1913–1916 when he worked at the University of Bonn as the successor to the famed topologist Felix Hausdorff. There Schur wrote his pioneering paper [Sch]: *Über die Kongruenz* $x^m + y^m \equiv z^m \pmod{p}$. In it Schur offered another proof of the 1908 theorem by Leonard Eugene Dickson [Dic], who was trying to prove Fermat's Last Theorem.

For use in his proof, Schur created, as he put it, "a very simple lemma, which belongs more to combinatorics than to number theory." Its setting is positive integers, colored in finitely many colors. The beautiful proof I present here utilizes coloring as well. Paul Erdős received this proof from Vera T. Sos, and included it in his talk at the 1970 International Congress of Mathematicians in Nice, France [E71.13]. We use the following natural lemma, which can be proven by a straightforward induction.

Issai Schur, courtesy of his daughter Hilde Abelin-Schur.

Lemma 2 (R. E. Greenwood and A. M. Gleason, 1955, [GG]). *For any positive integer n there is a positive integer $S(n)$ such that any n coloring of edges of the complete graph $K_{S(n)}$ contains a monochromatic triangle K_3.*

The Schur 1916 Theorem 3 ([Sch]). *For any positive integer n there is an integer $S(n)$ such that any n-coloring of the set $[S(n)]$ contains integers a, b, c of the same color such that $a + b = c$.*

In this case we call a, b, c a *monochromatic solution* of the equation $x + y = z$. In fact, Schur proved by induction that $S(n) = n!e$ would work.

Proof of Schur's Theorem. Let all positive integers be colored in n colors $c_1, c_2, \ldots, c_n$. Due to Lemma 2, there is $S(n)$ such that any n-coloring of edges of the complete graph $K_{S(n)}$ contains a monochromatic triangle K_3.

Construct a complete graph $K_{S(n)}$ with its vertices labeled with integers from the initial integer array $[S(n)] = \{1, 2, \ldots, S(n)\}$. Now color the edges of $K_{S(n)}$ in n colors as follows: let i and j, $i > j$, be two vertices of $K_{S(n)}$, color the edge ij in the color of the integer $i - j$ (remember, all positive integers were colored in n colors!). We get a complete graph $K_{S(n)}$ whose edges are colored in n colors. By Lemma 2, $K_{S(n)}$ contains a triangle ijk, $i > j > k$, whose edges ij, jk, and ik are colored in the same color (Fig. 2).

Denote $a = i - j$; $b = j - k$; $c = i - k$. Since all three edges of the triangle ijk are colored in the same color, the integers a, b, and c are colored in the same color in the original coloring of the integers (this is how we colored the edges of $K_{S(n)}$). In addition, we have the following equality:

$$a + b = (i - j) + (j - k) = i - k = c.$$

We are done! ■

The result of the Schur theorem 3 can be strengthened by an additional clever trick in the proof.

Strong Version of Schur's Theorem 4. *For any positive integer n there is an integer $S^*(n)$ such that any n-coloring of the initial positive integer array $[S^*(n)]$ contains distinct integers a, b, c of the same color such that $a + b = c$.*

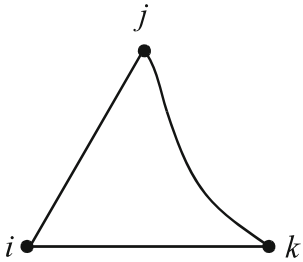


Fig. 2

Proof. Let all positive integers be colored in n colors $c_1, c_2, \ldots, c_n$. We add n more colors $c_1\prime, c_2\prime, \ldots, c_n\prime$ different from the original n colors and construct a complete graph $K_{S(2n)}$ with the set of positive integers $\{1, 2, \ldots, S(2n)\}$ labeling its vertices (see the definition of $S(2n)$ in the proof of Theorem 3). Now we are going to color the edges of $K_{S(2n)}$ in $2n$ colors.

Let i and j, $(i > j)$, be two vertices of $K_{S(2n)}$, and c_p be the color in which the integer $i - j$ is colored, $1 \le p \le n$ (remember, all positive integers are colored in n colors $c_1, c_2, \ldots, c_n$). Then we color the edge ij in color c_p if the number $\lfloor i/(i-j) \rfloor$ is even, and in color $c_p\prime$ if the number $\lfloor i/(i-j) \rfloor$ is odd (for a real number r, the symbol $\lfloor r \rfloor$, as usual, denotes the largest integer not exceeding r).

We get a complete graph $K_{S(2n)}$ whose edges are colored in $2n$ colors. By Lemma 2, $K_{S(2n)}$ contains a triangle ijk, $i > j > k$, whose edges ij, jk, and ik are colored in the same color (see Fig. 2).

Denote $a = i - j$; $b = j - k$; $c = i - k$. Since all three edges of the triangle ijk are colored in the same color, from the definition of coloring of edges of $K_{S(2n)}$ it follows that in the original coloring of positive integers, the integers a, b, and c were colored in the same color. In addition we have

$$a + b = (i - j) + (j - k) = i - k = c.$$

We are almost done. We only need to show (our additional pledge!) that the numbers a, b, c are all distinct. In fact, it suffices to show that $a \ne b$. Assume the opposite: $a = b$ and c_p is the color in which the number $a = b = i - j = j - k$ is colored. But then

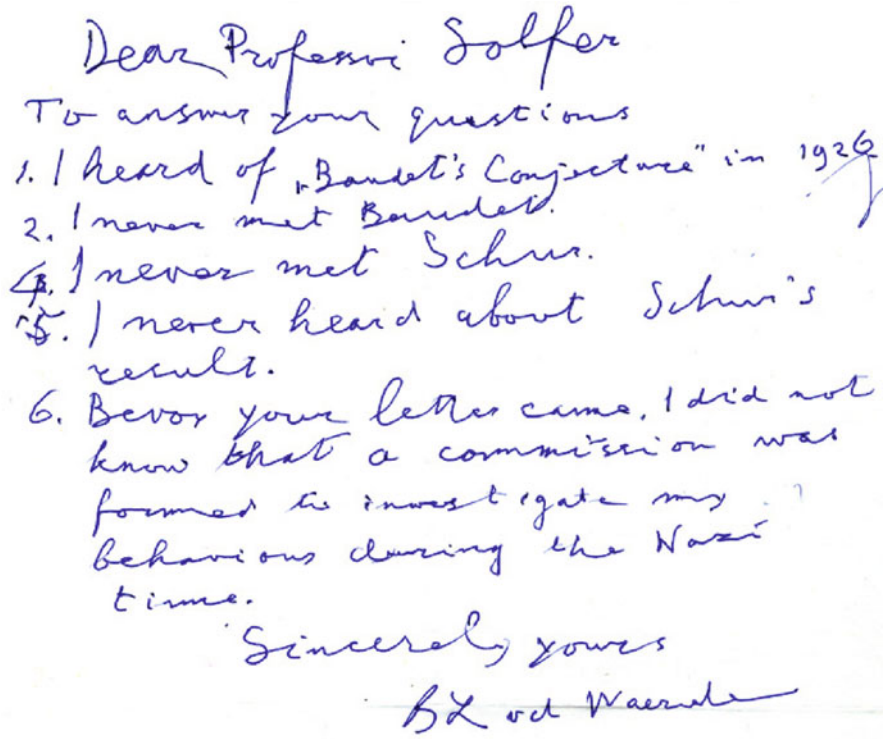
$$\left\lfloor \frac{i}{i-j} \right\rfloor = \left\lfloor 1 + \frac{j}{i-j} \right\rfloor = 1 + \left\lfloor \frac{j}{i-j} \right\rfloor = 1 + \left\lfloor \frac{j}{j-k} \right\rfloor,$$

i.e., the numbers $\lfloor i/(i-j) \rfloor$ and $\lfloor j/(j-k) \rfloor$ have different parity, thus the edges ij and jk of the triangle ijk must have been colored in different colors. This contradiction to the fact that all three edges of the triangle ijk have the same color proves that $a \ne b$. Theorem 4 is proven. ■

Nobody then asked questions of the kind Issai Schur posed and solved in this 1916 paper. Consequently, nobody appreciated this result much when it was published. Even Van der Waerden himself learned about the Schur theorem from me in 1995! See for yourselves Van der Waerden's April 24, 1995 letter to me:

Now Schur's theorem shines as one of the most beautiful, classic theorems of mathematics. Leon Mirsky loved this theorem, and wrote [Mir] on the occasion of the centenary of the birth of Schur:

> We have here a statement of the type: "if a system is partitioned arbitrarily into a finite number of subsystems, then at least one subsystem possesses a certain specified property." To the best of my knowledge, there is no earlier result which bears even a remote resemblance to Schur's theorem. It is this element of novelty that impresses itself so forcibly on the mind of the reader.


Dear Professor Soifer
To answer your questions
1. I heard of „Baudet's Conjecture" in 1926
2. I never met Baudet.
3. I never met Schur.
4. I never heard about Schur's result.
5. Bevor your letter came, I did not know that a commission was formed to investigate my behaviour during the Nazi time.
Sincerely yours
BL vd Waerden

Van der Waerden, April 24, 1995 letter to Alexander Soifer.

However, Mirsky continued [Mir]:

> After writing his paper, Schur never again touched on the problem discussed there; and this is in itself something of a mystery. For the strongest impression one receives on scanning his publications is the almost compulsive striving for comprehensiveness. There are few isolated investigations; in algebra, in analysis, in the theory of numbers, Schur reverts again and again to his original questions and pursues them to the point of where one feels that the last word has been spoken.... Why, then, did he not investigate any of the numerous questions to which his Theorem points so compellingly? There is no evidence to enable us to solve the riddle. (Footnote: As will emerge from the discussion below, Professor Rado, if anyone, should be able to throw light on the mystery – and he tells me that he cannot.)

Mirsky's statement, apparently backed by Richard Rado, was echoed in the authoritative book on Ramsey theory [GRS2, p. 70], thus becoming a universal view on this matter: "Schur never again touched on this problem." In fact, the new Ramseyan mathematics, discovered by Issai Schur in his 1916 paper, remained dear to his heart for years to come. Even though at the time, nobody was interested in the new direction the Schur 1916 theorem had shown, Issai Schur himself was: read on!