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Continuum Mechanics

Advanced Topics and Research Trends

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Preface

In the companion book (*Continuum Mechanics Using Mathematica*®) to this volume, we explained the foundations of continuum mechanics and described some basic applications of fluid dynamics and linear elasticity. However, deciding on the approach and content of this book, *Continuum Mechanics: Advanced Topics and Research Trends*, proved to be a more difficult task. After a long period of reflection, we made the decision to direct our efforts into drafting a book that demonstrates the flexibility and great potential of continuum physics to describe the wide range of macroscopic phenomena that we can observe. It is the opinion of the authors that this is the most stimulating way to learn continuum mechanics. However, it is also quite evident that this aim cannot be fully realized in a single book. Consequently, in this book we chose to present only the basics of interesting continuum mechanics models, along with some important applications of them.

We assume that the reader is familiar with all of the basic principles of continuum mechanics: the general balance laws, constitutive equations, isotropy groups for materials, the laws of thermodynamics, ordinary waves, etc. All of these concepts can be found in *Continuum Mechanics Using Mathematica* and many other books.

We believe that this book gives the reader a sufficiently wide view of the "boundless forest" of continuum mechanics, before focusing his or her attention on the beauty and complex structure of single trees within it (indeed, we could say that *Continuum Mechanics Using Mathematica* provides only the fertile humus on which the trees of this forest take root!).

The topics that we have selected for this book in order to show the power of continuum mechanics to characterize the experimental behavior of real bodies, and the order in which these topics are discussed here, are described below.

In Chap. 1, we discuss some interesting aspects of nonlinear elasticity. We start with the equilibrium equations and their variational formulation and discuss some peculiarities of the boundary value problems of nonlinear elasticity. We then analyze the homogeneous equilibrium solutions of isotropic materials together with the universal equilibrium solutions of Ericksen for compressible elastic materials. Moreover, some experimental results for constitutive equations in nonlinear elasticity are briefly explored. The existence and uniqueness theorems of Van Buren and Stoppelli, as well as Signorini's method, are presented with some recent extensions to live loads. Finally, the chapter concludes with a survey of the propagation of acceleration waves in an elastic body, and a new perturbation method for the analysis of these waves is presented.

In Chap. 2, we discuss the theory of continua with directors, which was proposed at the beginning of the twentieth century by the Cosserat brothers and was subsequently developed by many other authors. In this model, a continuous system S is no longer considered a collection of simple points defined by their coordinates in a frame of reference; instead, S is regarded as a set of complex particles that also possess a certain number of vectors that move independently of the particles with which they are associated. Such a model provides a better description of aggregates of microcrystals, polarized dielectrics, ferromagnetic substances, and one-dimensional and two-dimensional bodies. It can also be applied whenever the system contains a length that: (i) is less than the limit considered in continuum mechanics; (ii) characterizes the dimensions of microscopic regions that influence the macroscopic behavior of the body through their internal evolutions.

In Chap. 3, we consider a simplified model of a continuum with a *nonmaterial* moving surface across which the bulk fields can exhibit discontinuities. The general balance equations of this model are formulated together with the associated local field equations and jump conditions. In Chap. 4, this model is used to describe the phase equilibrium of two different phases. In particular, Maxwell's rule and Clapeyron's equation are derived.

The same model is applied in Chap. 5 to describe dynamical phase changes like melting and evaporation. The related difficult free-boundary problems are stated together with some numerical results.

Chapter 6 introduces the principles of mixture theory. This model, which allows us to describe the evolution of each constituent of a mixture as well as the whole mixture, is very useful in chemistry, biology, and mineralogy (alloys). This chapter contains a proof for the Gibbs rule, together with an analysis of phase equilibrium in a binary mixture.

Chapters 7 and 8 describe the interactions of electric and magnetic fields with matter using a continuum model with a nonmaterial interface. After a general discussion of the different properties resulting from a change of reference frame for the mechanical and electromagnetic equations, the approximations of quasi-electrostatics and quasi-magnetostatics are discussed. In particular, by adopting a continuum mechanics approach, we show that various physical models that have been proposed to explain the behavior of dielectrics and magnetic bodies are actually equivalent from a macroscopic perspective. In other words, different microscopic models can lead to the same macroscopic behavior.

In Chap. 9, we present the macroscopic approach to micromagnetism together with the very difficult mathematical problems associated with this model. Among other things, it is shown that the model of a continuum with a nonmaterial interface can be used to determine the form of Weiss' domains for some crystals and geometries.

Chapter 10 provides an introduction to continua in special relativity. After a brief analysis of the historical motivations of this theory, Minkowski's geometrical model of spacetime is presented. The relativistic balance equations are then formulated in terms of the symmetric momentum–energy four-tensor. After an accurate description of Fermi transport, the intrinsic deformation gradient is introduced, in order to define elastic materials by extending the objectivity principle to special relativity. We then justify the different transformation formulae adopted in the literature for the total work, the total energy and the total heat of an homogeneous system through a wide-ranging discussion of the absolute and relative viewpoints. At the end of this chapter, the fundamental problem of the interaction between matter and electromagnetic fields is analyzed, together with the different models that have been adopted to describe it. Finally, we prove the equivalence of all of these proposals.

There are only a few notebooks written in ***Mathematica***® for this book (which can be downloaded from the publisher's website at http:// www.birkhauser.com/978-0-8176-4869-5), since the topics here discussed are more theoretical in nature than those treated in *Continuum Mechanics Using Mathematica*. However, many of the notebooks associated with that book can also be applied to the topics covered here.

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A. Marasco

Chapter 1

Nonlinear Elasticity

1.1 Preliminary Considerations

In this chapter we focus on the basics of nonlinear elasticity in order to show its interesting mathematical and physical aspects. Readers who are interested in delving deeper into this subject should refer to the many existing books on it (see, for instance, [1]–[15]). We start by listing the main difficulties associated with this subject:

- The equations governing the equilibrium and the motion of an elastic body are nonlinear.
- Instead of being expressed by given functions assigned to the boundary of the region occupied by the elastic body, the boundary conditions are generally functions of the *unknown* deformation.
- Finding the forms of the constitutive equations of an elastic isotropic material is a very complex experimental task. We must determine unknown functions instead of the two Lamé constants that characterize a linearly elastic material.

The nonlinearity of the basic equations of nonlinear elasticity make it difficult to determine explicit solutions for both the equilibrium equations and the motion equations, except in simple cases. For the same reason, it is also an arduous task to prove existence and uniqueness theorems for boundary value problems that can be applied to equilibrium or dynamical problems in nonlinear elasticity. In particular, wave propagation analysis is much more complex than in linear elasticity. In this chapter, we try to analyze all of the above problems. When the subject requires a deeper analysis, references will be suggested.

We assume that the reader is familiar with the foundations of continuum mechanics. Therefore, all of the basic concepts (such as the balance equations) are provided without explanations. If necessary, the reader can

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consult other books on this subject (see, for instance, [1]–[15]); in particular, [16] utilizes the same notation as we have adopted here.

1.2 The Equilibrium Problem

Let S be a homogeneous elastic system in the reference configuration C_*. From now on, S is assumed to be at a constant and uniform temperature. The system S adopts an equilibrium configuration C in the presence of body forces acting on the region C and surface tensions across the whole boundary ∂C or to a part Σ of ∂C. The task of *elastostatics* is to determine the *finite deformation* $\mathbf{x} = \mathbf{x}(\mathbf{X})$, where $\mathbf{X} \in C_*$, $\mathbf{x} \in C$, or, equivalently, the ***displacement*** $\mathbf{u}(\mathbf{X}) = \mathbf{x}(\mathbf{X}) - \mathbf{X}$ that S undergoes when moving from C_* to C under the influence of the applied forces mentioned above. We denote the *deformation gradient* by $\mathbf{F} = (\partial x_i/\partial X_L)$, the *displacement gradient* by $\mathbf{H} = (\partial u_i/\partial X_L) = \mathbf{F} - \mathbf{I}$, and the *right Cauchy–Green tensor* by $\mathbf{C} = \mathbf{F}\mathbf{F}^T$.

The equilibrium equations, the jump conditions across a surface Σ_1 separating two different materials, and the boundary conditions are, respectively:

$$
\begin{aligned}
\nabla_{\mathbf{x}} \cdot \mathbf{T} + \rho \mathbf{b} &= \mathbf{0}, \qquad \text{in } C - \Sigma_1, && (1.1)\\
[[\mathbf{T} \cdot \mathbf{n}]] &= \mathbf{0}, \qquad \text{on } \Sigma_1, && (1.2)\\
\mathbf{T} \cdot \mathbf{N} &= \mathbf{t}, \qquad \text{on } \Sigma, && (1.3)
\end{aligned}
$$

where ρ is the mass density in C, $\mathbf{T}$ is the *Cauchy stress tensor*, $\mathbf{n}$ is the unit vector normal to Σ_1, and $\mathbf{N}$ is the unit vector normal to the part (denoted Σ) of ∂C where surface forces act with a density of $\mathbf{t}$.

It is convenient to use the *Lagrangian equilibrium conditions*, since the unknown function $\mathbf{x} = \mathbf{x}(\mathbf{X})$ depends on the point $\mathbf{X}$ in C_*. Another reason to use these equations is that the forces acting on the part Σ of the boundary ∂C cannot be assigned because ∂C is unknown. The Lagrangian formulation corresponding to (1.1)–(1.3) is expressed by the following equations:[1]

$$
\begin{aligned}
\nabla_{\mathbf{X}} \cdot \mathbf{T}_* + \rho_* \mathbf{b} &= \mathbf{0}, \qquad \text{in } C_* - \Sigma_{*1}, && (1.4)\\
[[\mathbf{T}_* \cdot \mathbf{n}_*]] &= \mathbf{0}, \qquad \text{on } \Sigma_{*1}, && (1.5)\\
\mathbf{T}_* \cdot \mathbf{N}_* &= \mathbf{t}_*, \qquad \text{on } \Sigma_*, && (1.6)
\end{aligned}
$$

where $\mathbf{T}_*$ is the *Piola–Kirchhoff tensor* and ρ_*, $\mathbf{n}_*$, σ_*, $\mathbf{N}_*$, Σ_*, Σ_{*1}, and $\mathbf{t}_*$ are the Lagrangian quantities corresponding to $\mathbf{T}$, ρ, $\mathbf{n}$, σ, $\mathbf{N}$, Σ, Σ_1, and $\mathbf{t}$,

[1] See [16], p. 148.

respectively. In this chapter we will frequently use the following relations:[2]

$$\mathbf{T}_* = J\mathbf{T}(\mathbf{F}^{-1})^T, \tag{1.7}$$

$$d\sigma = J\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*}\, d\sigma_*, \tag{1.8}$$

$$\mathbf{N} = \frac{(\mathbf{F}^{-1})^T}{\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*}}\mathbf{N}_*, \tag{1.9}$$

$$\mathbf{t} = \frac{1}{J}\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1}\mathbf{N}_*}\,\mathbf{t}_*. \tag{1.10}$$

In a hyperelastic material, the Piola–Kirchhoff stress tensor $\mathbf{T}_*$ is expressed in terms of the specific *elastic potential* ψ by the relation (see [16], p. 161)

$$\mathbf{T}_* = \rho_* \frac{\partial \psi(\mathbf{F})}{\partial \mathbf{F}} = \rho_* \frac{\partial \widehat{\psi}(\mathbf{H})}{\partial \mathbf{H}}, \tag{1.11}$$

where we have introduced the notation $\psi(\mathbf{F}) = \psi(\mathbf{I} + \mathbf{H}) \equiv \widehat{\psi}(\mathbf{H})$.

Substituting (1.11) into (1.4) and introducing the fourth-order *elasticity tensor*

$$A_{ijLM}(\mathbf{H}) = \rho_* \frac{\partial T_{*iL}}{\partial H_{jM}} = \rho_* \frac{\partial^2 \widehat{\psi}}{\partial H_{iL}\partial H_{jM}}, \tag{1.12}$$

we obtain the following second-order quasi-linear partial differential system

$$A_{ijLM}(\mathbf{H}) \frac{\partial^2 u_j}{\partial X_L \partial X_M} + \rho_* b_i = 0, \tag{1.13}$$

whose unknowns are the components u_i, $i = 1, 2, 3$, of the displacement.

One of the main aims of elasticity is to verify that the system (1.13) and the boundary conditions (1.2)–(1.3) allow us to determine (at least in principle) the finite deformation $\mathbf{x} = \mathbf{x}(\mathbf{X})$; i.e., the equilibrium configuration of the body S to which a given load is applied. In other words, we need to establish the conditions for the unknown displacement field that make it possible to prove existence and uniqueness theorems for the boundary value problem obtained by associating the boundary conditions (1.2)–(1.3) with the equilibrium equations (1.13).

In the next section, some specific difficulties of this boundary value problem will be highlighted.

[2] See [16], p. 82, p. 148.

1.3 Remarks About Equilibrium Boundary Problems

We assume that the fields that appear in the equilibrium equations and the boundary conditions are smooth enough to allow us to perform all of the differentiation operations required. Moreover, the boundary part $\partial C_* - \Sigma_*$ is assumed to be fixed or deformed in a known manner. Formally, we write

$$\mathbf{x}(\mathbf{X}) = \mathbf{x}_0(\mathbf{X}) \quad \text{on } \partial C_* - \Sigma_*. \tag{1.14}$$

If $\Sigma_* = \emptyset$, the corresponding boundary value problem (BVP) is said to be one of *place*; if $\partial C_* = \Sigma_*$, then the BVP is one of *traction*. Finally, the BVP is said to be *mixed* when $\Sigma_* \subset \partial C_*$.

We can make the following remarks about these BVPs.

Remark The boundary data of a BVP are *given functions* of the boundary of the domain in which the solution must be found. For instance, to solve the Laplace equation in a domain Ω, we can provide either the values of the unknown solution u on $\partial\Omega$ (Dirichlet's BVP) or the values of its normal derivative (Neumann's BVP). The following examples show that a different situation occurs in nonlinear elasticity.

- Let S be an elastic body at equilibrium, with a uniform pressure p_0 acting on the boundary ∂C of the equilibrium configuration. The Eulerian formulation of the corresponding BVP is expressed by the equations:

$$\nabla_{\mathbf{x}} \cdot \mathbf{T} = \mathbf{0} \qquad \text{in } C,$$
$$\mathbf{T} \cdot \mathbf{N} = -p_0 \mathbf{N} \qquad \text{on } \partial C.$$

  In this formulation, the pressure p_0 is assigned to the unknown boundary ∂C. Using (1.4), (1.6), (1.9), and (1.10), this BVP can be formulated in the following Lagrangian form:

$$\nabla_{\mathbf{X}} \cdot \mathbf{T}_* = \mathbf{0} \qquad \text{in } C_*,$$
$$\mathbf{T}_* \cdot \mathbf{N}_* = -p_0 J (\mathbf{F}^{-1})^T \mathbf{N}_* \equiv \mathbf{t}_*(\mathbf{X}) \qquad \text{on } \partial C_*.$$

  Consequently, $\mathbf{t}_*$ is not a known function of $\mathbf{X} \in \partial C_*$ since it depends on the gradient of the unknown deformation. In other words, the function $\mathbf{t}_*(\mathbf{X})$ cannot be assigned completely because we only know how it depends on the deformation.

- Analogously, consider the elastic system S in Fig. 1.1, and suppose that the specific force $\mathbf{t} = ks(\mathbf{x})\mathbf{i}$ acts on the part Σ of its boundary.

In the above expression, $s(\mathbf{x})$ is the lengthening of the spring at the point $\mathbf{x}$, k is its elastic constant, and $\mathbf{i}$ is the unit vector orthogonal to the wall l.

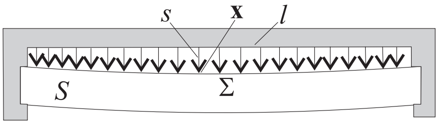

Fig. 1.1 A surface live load

In view of (1.10), the boundary condition to assign to the corresponding part Σ_* of ∂C_* is

$$\mathbf{t}_* = -J\sqrt{\mathbf{N}_* \cdot \mathbf{C}^{-1} \cdot \mathbf{N}_*}\, k\, s(\mathbf{x}(\mathbf{X}))\mathbf{i},$$

which again depends on the unknown deformation.

Any load which depends on the deformation in C_* is called a *live load*, whereas a load that is a known function of $\mathbf{X} \in \Sigma_*$ is said to be a *dead load*.

Dead loads have received a great deal of attention in the literature, but they are actually very difficult to realize. In fact, taking into account the condition that follows from (1.8) and (1.10)

$$\mathbf{t}_*(\mathbf{X}) = \frac{d\sigma}{d\sigma_*}\,\mathbf{t}(\mathbf{x}),$$

we see that the traction $\mathbf{t}$ at the boundary Σ must be given in such a way that $\mathbf{t}_*$ depends on $\mathbf{X}$ but not on the deformation. For instance, we could apply a specific force to a part of the boundary of the Eulerian equilibrium configuration given by

$$\mathbf{t} = \frac{d\mathbf{p}}{d\sigma},$$

where the force $d\mathbf{p}$ that acts on the elementary boundary area $d\sigma$ is constant. Clearly, it is not an easy task to achieve such a load experimentally. Even in the case of a uniform deformation ($\mathbf{F} = const$) under the action of a constant traction $\mathbf{t}$, the corresponding Lagrangian traction does not correspond to a dead load.

Remark A uniqueness theorem cannot hold for a BVP associated with nonlinear elasticity. Three classic examples illustrate this statement.

- There are deformations that coincide at the boundary but assume different values inside the body. For instance, John noted that if either the external or internal boundary of a spherical shell S is rotated by a multiple of 2π about an axis passing through its center without modifying the other boundary, the whole boundary of S will assume the same position but the internal state will be greatly modified.

- $\mathbf{u} = \mathbf{0}$ is an equilibrium solution of a thin hemispherical shell with zero surface traction. However, there is a second solution corresponding to the everted shell.

- Ericksen noted that in a pure traction problem with dead loads, a bar S that is subjected to equal and opposite axial forces at its ends should have at least two equilibrium configurations. In one of these, the forces are tractions; in the other, the bar is subjected to compressions after a rotation of π. Moreover, let S be at equilibrium in the Eulerian configuration C under the action of traction forces $\mathbf{t}$ acting at its ends σ_1 and σ_2 (see Fig. 1.2). If $\mathbf{t}_*$ is the traction per unit area in the reference configuration C_*, then it is easy to verify that S is still at equilibrium in the rotated configuration C' under the action of the compression $\mathbf{t}' = \mathbf{t}$. Let C'' denote the Eulerian equilibrium configuration corresponding to the Lagrangian equilibrium problem starting from the reference configuration C'.

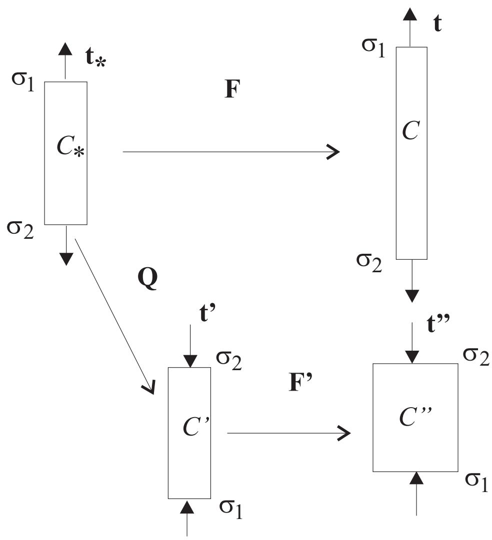

Fig. 1.2 Two possible equilibrium solutions of the same boundary problem

By applying the objectivity principle, and recalling that the loads are dead, we can easily prove that C'' is another possible equilibrium configuration corresponding to the Lagrangian boundary problem associated with C_*.

Remark The *local* equilibrium of any elementary volume dc of S is described by conditions (1.1)–(1.3), which do not imply the *global* equilibrium of S. If we denote by $\boldsymbol{\Phi}$ and $\mathbf{M}_O$, respectively, the total force and torque of the reactions due to the constraints necessary to satisfy the displacement datum (1.14), then the following global equilibrium conditions hold:

$$\int_{C_*} \rho_* \mathbf{b}\, dc_* + \int_{\Sigma_*} \mathbf{t}_*\, d\sigma_* + \boldsymbol{\Phi} = \mathbf{0}, \tag{1.15}$$

$$\int_{C_*} \rho_* \mathbf{r} \times \mathbf{b}\, dc_* + \int_{\Sigma_*} \mathbf{r} \times \mathbf{t}_*\, d\sigma_* + \mathbf{M}_O = \mathbf{0}. \tag{1.16}$$

These conditions state that the resultant and the total torque (with respect to the pole O) of all of the forces acting on S vanish. It is clear that, if $\partial C_* - \Sigma_* \neq \emptyset$, the reaction fields $\boldsymbol{\Phi}$ and $\mathbf{M}_O$ satisfy (1.15)–(1.16). However, in a traction BVP, conditions (1.15) and (1.16) become

$$\int_{C_*} \rho_* \mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{t}_*\, d\sigma_* = \mathbf{0}, \tag{1.17}$$

$$\int_{C_*} \rho_* \mathbf{r} \times \mathbf{b}\, dc_* + \int_{\partial C_*} \mathbf{r} \times \mathbf{t}_*\, d\sigma_* = \mathbf{0}, \tag{1.18}$$

so that, due to the presence of $\mathbf{t}_*$ and $\mathbf{r} = \mathbf{x}(\mathbf{X}) - \mathbf{x}_0$, they depend on the deformation. Consequently, it is no longer possible to establish whether they are satisfied a priori. In other words, (1.17) and (1.18) represent *equilibrium compatibility conditions* for the data that can only be verified a posteriori.

1.4 Variational Formulation of Equilibrium

The equilibrium BVPs of an elastic system can also be formulated in variational terms. This means that the equilibrium solutions of the BVPs minimize suitable functionals. In this section, the deformation functions are assumed to be of class $C^2(C_*)$, since they must satisfy (1.4)–(1.6). However, if a weak solution is searched for,[3] then the deformation functions are assumed to belong to suitable Sobolev spaces.

[3] See Appendix A.

In order to apply this approach to the equilibrium problems, we introduce the Banach space

$$W = \{\mathbf{u}(\mathbf{X}) \in C^2(C_*) : \mathbf{u}(\mathbf{X}) = \mathbf{0}, \text{on}\, \partial C_* - \Sigma_*\} \tag{1.19}$$

with the norm

$$\|\mathbf{u}(\cdot)\| = \text{Max}_{\mathbf{X}\in C_*} \left\{ \left|u^i(\mathbf{X})\right|, \left|\frac{\partial u^i(\mathbf{X})}{\partial X_L}\right|, \left|\frac{\partial^2 u^i(\mathbf{X})}{\partial X_L \partial X_M}\right| \right\}. \tag{1.20}$$

If we denote the *elastic energy functional* defined on W by

$$\Psi[\mathbf{u}(\cdot)] = \int_{C_*} \rho_* \psi(\mathbf{H})\, dc_*, \tag{1.21}$$

then the following theorem holds.

Theorem 1.1
The displacement $\mathbf{u}_0(\mathbf{X})$ is an equilibrium displacement—i.e., it is a solution of the BVP (1.4)–(1.6)—if and only if it obeys the variational equality[4]

$$D\Psi\left[\mathbf{u}_0(\cdot)|\mathbf{h}(\cdot)\right] = \int_{C_*} \rho_* \mathbf{b} \cdot \mathbf{h}(\mathbf{X})\, dc_* + \int_{\Sigma_*} \mathbf{t}_* \cdot \mathbf{h}(\mathbf{X})\, d\sigma_*, \quad \forall \mathbf{h}(\cdot) \in W, \tag{1.22}$$

where $D\Psi$ is the Frechét differential of the functional (1.21)*.*

PROOF We have

$$\Psi[\mathbf{u}(\cdot) + \mathbf{h}(\cdot)] - \Psi[\mathbf{u}(\cdot)] = \int_{C_*} \rho_* \frac{\partial \psi}{\partial H_{iL}} \frac{\partial h_i}{\partial X_L} dc_* + O(\|\mathbf{h}(\cdot)\|),$$

so that, considering (1.11) and recalling that $\mathbf{h} = \mathbf{0}$ on $C_* - \Sigma_*$, we find that

$$D\Psi[\mathbf{u}_0(\cdot)|\mathbf{h}(\cdot)] = \int_{C_*} \rho_* \frac{\partial \psi}{\partial H_{iL}} \frac{\partial h_i}{\partial X_L}\, dc_*$$

[4]The operator $\mathcal{F} : F \longrightarrow F'$ between two Banach spaces F and F' is *Fréchet differentiable* at $\mathbf{u} \in F$ if

$$\mathcal{F}(\mathbf{u} + \mathbf{h}) = \mathcal{F}(\mathbf{u}) + D\mathcal{F}(\mathbf{u} \mid \mathbf{h}) + O(\| \mathbf{h} \|)$$

$\forall \mathbf{h} \in F$, where

$$D\mathcal{F}(\mathbf{u} \mid \cdot) : F \longrightarrow F'$$

is a linear continuous operator called the *Fréchet differential* of $\mathcal{F}$. The notation

$$D\mathcal{F}(\mathbf{u} \mid \cdot) = D_{\mathbf{u}}\mathcal{F} \cdot \mathbf{h},$$

defines the *Fréchet derivative* of $\mathcal{F}$ at $\mathbf{u}$.

$$= -\int_{C_*} \frac{\partial}{\partial X_L}\left(\rho_* \frac{\partial \psi}{\partial H_{iL}}\right) h_i \, dc_* + \int_{\Sigma_*} \rho_* \frac{\partial \psi}{\partial H_{iL}} h_i N_{*L} \, d\sigma_*$$
$$= -\int_{C_*} \mathbf{h} \cdot \nabla_{\mathbf{X}} \cdot \mathbf{T}_* \, dc_* + \int_{\Sigma_*} \mathbf{h} \cdot \mathbf{T}_* \mathbf{N}_* \, d\sigma_*.$$

It is now straightforward to show that (1.22) is equivalent to the equilibrium conditions (1.4)–(1.7). $\blacksquare$

Theorem 1.2
If $\mathbf{b} = \mathbf{b}(\mathbf{X})$ and $\mathbf{t}_* = \mathbf{t}_*(\mathbf{X})$, then a displacement $\mathbf{u}_0(\mathbf{X}) \in W$ is an equilibrium displacement if and only if it is an extremal of the functional

$$F[\mathbf{u}(\cdot)] = \int_{C_*} \rho_* \psi[\mathbf{H}] \, dc_* - \int_{C_*} \rho_* \mathbf{b}(\mathbf{X}) \cdot \mathbf{u}(\mathbf{X}) \, dc_* - \int_{\Sigma_*} \mathbf{t}_*(\mathbf{X}) \cdot \mathbf{u}(\mathbf{X}) \, d\sigma_*; \tag{1.23}$$

i.e., if and only if the following condition holds:

$$DF[\mathbf{u}_0(\cdot)|\mathbf{h}(\cdot)] = 0, \quad \forall \mathbf{h}(\mathbf{X}) \in W. \tag{1.24}$$

PROOF If we note that

$$D\left(\int_{C_*} \rho_* \mathbf{b} \cdot \mathbf{u}(\mathbf{X}) \, dc_* + \int_{\Sigma_*} \mathbf{t}_* \cdot \mathbf{u}(\mathbf{X}) \, d\sigma_*\right)$$
$$= \int_{C_*} \rho_* \mathbf{b} \cdot \mathbf{h}(\mathbf{X}) \, dc_* + \int_{\Sigma_*} \mathbf{t}_* \cdot \mathbf{h}(\mathbf{X}) \, d\sigma_*,$$

then the proof follows from Theorem 1.1. $\blacksquare$

Theorem 1.3
If $\mathbf{b} = -\nabla_{\mathbf{x}} \varphi(\mathbf{x})$ and the body is subjected to a uniform pressure p_e, then $\mathbf{u}_0(\mathbf{X}) \in W$ is an equilibrium displacement if and only if it is an extremal of the functional

$$\bar{F}[\mathbf{u}(\cdot)] = \int_{C_*} (\rho_* \psi(\mathbf{H}) + \rho_* \varphi(\mathbf{u}) + p_e J) \, dc_*$$
$$= \int_{C} \rho \left(\psi(\mathbf{H}) + \varphi(\mathbf{u}) + \frac{p_e}{\rho}\right) dc; \tag{1.25}$$

i.e., if and only if

$$D\bar{F}[\mathbf{u}_0(\cdot)|\mathbf{h}(\cdot)] = 0, \quad \forall \mathbf{h}(\mathbf{X}) \in W. \tag{1.26}$$

PROOF First, taking into account the results of Theorem 1.1, we have

$$D\int_{C_*} (\rho_* \psi(\mathbf{H}) + \rho_* \varphi(\mathbf{u})) \, dc_*$$

$$
\begin{aligned}
&= -\int_{C_*} \mathbf{h}\cdot(\nabla_{\mathbf{X}}\mathbf{T}_* - \rho_*\nabla_{\mathbf{x}}\varphi)\,dc_* + \int_{\Sigma_*} \mathbf{h}\cdot\mathbf{T}_*\mathbf{N}d\sigma_* \\
&= -\int_{C_*} \mathbf{h}\cdot(\nabla_{\mathbf{X}}\mathbf{T}_* + \rho_*\mathbf{b})\,dc_* + \int_{\Sigma_*} \mathbf{h}\cdot\mathbf{T}_*\mathbf{N}_*\,d\sigma_*.
\end{aligned}
$$

If we prove that

$$
D\int_{C_*} p_e J\,dc_* = -\int_{\Sigma_*} p_e J(\mathbf{F})^{-1}\mathbf{N}_*\cdot\mathbf{h}\,d\sigma_*,
$$

then the condition $D\bar{F} = 0$ supplies the equilibrium equations and boundary conditions in the Lagrangian form. To this end, we note that (see (3.50) in [16])

$$
\begin{aligned}
D\int_{C_*} p_e J\,dc_* &= \int_{C_*} p_e dJ\,dc_* = \int_{C_*} p_e \frac{\partial J}{\partial F_{iL}}\frac{\partial h_i}{\partial X_L}\,dc_* \\
&= \int_{C_*} p_e J(F^{-1})_{iL}\frac{\partial h_i}{\partial X_L}\,dc_* \\
&= \int_{C_*} \frac{\partial}{\partial X_L}\left[p_e J(F^{-1})_{iL}h_i\right]\,dc_* \\
&\quad - \int_{C_*} \frac{\partial}{\partial X_L}\left[p_e J(F^{-1})_{iL}\right]h_i\,dc_* = -\int_{\Sigma_*} \mathbf{t}_*\cdot\mathbf{h}\,d\sigma_*,
\end{aligned}
$$

since

$$
\frac{\partial}{\partial X_L}\left(J(F^{-1})_{iL}\right) = 0. \tag{1.27}
$$

In fact, from the identity

$$
0 = \frac{\partial}{\partial X_M}\left[\frac{J}{J}F_{iL}(F^{-1})_{Mj}\right],
$$

when (3.49) of [16] is taken into account, we derive

$$
\begin{aligned}
0 &= J(F^{-1})_{Mi}\frac{\partial}{\partial X_M}\left[\frac{1}{J}F_{iL}\right] + \frac{F_{iL}}{J}\frac{\partial}{\partial X_M}[J(F^{-1})_{Mi}] \\
&= \frac{1}{J}F_{iL}\frac{\partial}{\partial X_M}[J(F^{-1})_{Mi}],
\end{aligned}
$$

so that

$$
\begin{aligned}
0 &= (F^{-1})_{Lj}F_{iL}\frac{\partial}{\partial X_M}[J(F^{-1})_{Mi}] \\
&= \delta_{ij}\frac{\partial}{\partial X_M}[J(F^{-1})_{Mi}] = \frac{\partial}{\partial X_M}[J(F^{-1})_{Mj}],
\end{aligned}
$$

and the theorem is proved. ∎

1.5 Isotropic Elastic Materials

Let S be an elastic body that is homogeneous and isotropic in the reference configuration C_*. In Sect. 7.2 of [16], it is shown that the elastic potential ψ of S is a function of the principal invariants I, II, and III of the left Cauchy–Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$

$$\psi = \psi(I, II, III), \tag{1.28}$$

and that the Cauchy stress tensor $\mathbf{T}$ can be written as follows:

$$\mathbf{T} = f_0\mathbf{I} + f_1\mathbf{B} + f_2\mathbf{B}^2, \tag{1.29}$$

where

$$f_0 = 2\rho\, III\frac{\partial\psi}{\partial III}, \tag{1.30}$$

$$f_1 = 2\rho\left(\frac{\partial\psi}{\partial I} + I\frac{\partial\psi}{\partial II}\right), \tag{1.31}$$

$$f_2 = -2\rho\frac{\partial\psi}{\partial II}. \tag{1.32}$$

On the other hand, from the Cayley–Hamilton theorem,[5]

$$\mathbf{B}^3 - I\mathbf{B}^2 + II\mathbf{B} - III\mathbf{I} = \mathbf{0}. \tag{1.33}$$

Multiplying by $\mathbf{B}^{-1}$ yields

$$\mathbf{B}^2 = I\mathbf{B} - II\mathbf{I} + III\mathbf{B}^{-1}.$$

For this relation, we can write (1.29) in the equivalent form

$$\mathbf{T} = \varphi_0\mathbf{I} + \varphi_1\mathbf{B} + \varphi_2\mathbf{B}^{-1}, \tag{1.34}$$

where

$$\varphi_0 = f_0 - II\, f_2, \tag{1.35}$$

$$\varphi_1 = f_1 + I\, f_2, \tag{1.36}$$

$$\varphi_2 = III\, f_2, \tag{1.37}$$

and

$$\varphi_0 = 2\rho\left(II\frac{\partial\psi}{\partial II} + III\frac{\partial\psi}{\partial III}\right), \tag{1.38}$$

$$\varphi_1 = 2\rho\frac{\partial\psi}{\partial I}, \tag{1.39}$$

$$\varphi_2 = -2\rho III\frac{\partial\psi}{\partial II}. \tag{1.40}$$

[5]See p. 92 of [16].

For an incompressible elastic body, we have $III = \det \mathbf{B} = 1$, and the above formulae become (see p. 165 of [16])

$$\psi = \psi(I, II), \tag{1.41}$$

$$\mathbf{T} = -p\mathbf{I} + 2\rho\frac{\partial\psi}{\partial I}\mathbf{B} - 2\rho\frac{\partial\psi}{\partial II}\mathbf{B}^{-1}, \tag{1.42}$$

where p is an undetermined pressure that depends on the point $\mathbf{x}$.

In view of some of the problems that we consider later, it is useful to introduce the *elastic energy per unit volume* of the reference configuration

$$\Psi = \rho_*\psi, \tag{1.43}$$

through which the relation (1.34) for an elastic compressible material becomes

$$\mathbf{T} = \frac{2}{J}\left(II\frac{\partial\Psi}{\partial II} + III\frac{\partial\Psi}{\partial III}\right)\mathbf{I} + \frac{2}{J}\frac{\partial\Psi}{\partial I}\mathbf{B} - 2\frac{\partial\Psi}{\partial II}\mathbf{B}^{-1}, \tag{1.44}$$

since $\rho J = \rho\sqrt{III} = \rho_*$. For an incompressible elastic material ($III = 1$), (1.42) can be written as follows:

$$\mathbf{T} = -p\mathbf{I} + 2\frac{\partial\Psi}{\partial I}\mathbf{B} - 2J\frac{\partial\Psi}{\partial II}\mathbf{B}^{-1}. \tag{1.45}$$

1.6 Homogeneous Deformations

A deformation $C_* \to C$ of the elastic body S is said to be a *homogeneous deformation* if it has the form

$$\mathbf{x} = \mathbf{F}\,\mathbf{X} + \mathbf{c}, \tag{1.46}$$

where the deformation gradient $\mathbf{F}$ and the vector $\mathbf{c}$ are constant.

When the material is *compressible*, the stress tensor $\mathbf{T}$ is given by (1.29). Consequently, it is constant in any homogeneous deformation, and the equilibrium equation (1.4) is obeyed if and only if there is no body force. In other words, in the absence of body forces, *any* homogeneous deformation obeys the equilibrium equation (1.4) for *any* isotropic elastic material. However, the boundary condition (1.6) depends on both the material and the chosen homogeneous deformation.

When the material is *incompressible*, a homogeneous deformation obeys the equilibrium equation (1.4), even in the presence of body forces, due to the presence of the undetermined function $p(\mathbf{x})$. If, in particular, $\mathbf{b} = 0$, then the pressure p is constant. Again, the boundary condition depends on the material and the homogeneous deformation chosen.

We derive a very important conclusion from these remarks. At least in principle, *it is possible to determine the constitutive relations* (1.44) *and* (1.45) *using homogeneous deformations and surface forces.*

In the following sections we describe some important homogeneous deformations as well as some famous experiments to determine the forms of the constitutive equations of the stress tensor for particular isotropic elastic materials.

1.7 Homothetic Deformation

A *homothetic deformation* of an elastic system S is expressed by the equations

$$x_i = \lambda_i X_i, \quad i = 1, 2, 3, \tag{1.47}$$

where the constants λ_i are nonzero. If $\lambda_i > 1$, then the system S exhibits an extension along the axis X_i; if $0 < \lambda_i < 1$, then the system S exhibits a compression along the axis x_i. The deformation gradient of (1.47) is given by the matrix

$$\mathbf{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \tag{1.48}$$

so that the coordinate axes are the principal axes of deformation. From (1.48), we derive

$$J \equiv \det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3; \tag{1.49}$$

moreover, the left Cauchy–Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ and its inverse can, respectively, be written as

$$\mathbf{B} = \begin{pmatrix} \lambda_1^2 & 0 & \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{\lambda_1^2} & 0 & \\ 0 & \frac{1}{\lambda_2^2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3^2} \end{pmatrix}. \tag{1.50}$$

Finally, the principal invariants of $\mathbf{B}$ are

$$I = \mathrm{tr}\mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \tag{1.51}$$

$$II = \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2, \tag{1.52}$$

$$III = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{1.53}$$

Starting from (1.50)–(1.53) and (1.44), we derive the following expressions for the Cauchy stress tensor components in a compressible elastic material:

$$T_{11} = 2\lambda_1 \left\{ \frac{1}{\lambda_2\lambda_3} \left[\frac{\partial \Psi}{\partial I} + (\lambda_2^2 + \lambda_3^2) \frac{\partial \Psi}{\partial II} \right] + \lambda_2\lambda_3 \frac{\partial \Psi}{\partial III} \right\}, \quad (1.54)$$

$$T_{22} = 2\lambda_2 \left\{ \frac{1}{\lambda_1\lambda_3} \left[\frac{\partial \Psi}{\partial I} + (\lambda_1^2 + \lambda_3^2) \frac{\partial \Psi}{\partial II} \right] + \lambda_1\lambda_3 \frac{\partial \Psi}{\partial III} \right\}, \quad (1.55)$$

$$T_{33} = 2\lambda_3 \left\{ \frac{1}{\lambda_1\lambda_2} \left[\frac{\partial \Psi}{\partial I} + (\lambda_1^2 + \lambda_2^2) \frac{\partial \Psi}{\partial II} \right] + \lambda_1\lambda_2 \frac{\partial \Psi}{\partial III} \right\}, \quad (1.56)$$

$$T_{ij} = 0, \quad i \neq j. \quad (1.57)$$

These relations prove that the state of tension inside the body S is uniform, so the equilibrium equations are obeyed.

Let us denote the parametric equations of the boundary of S in the reference configuration C_* by

$$\mathbf{X} = \mathbf{X}(u_1, u_2).$$

The parametric equations of the boundary ∂C in the deformed equilibrium configuration C are then

$$\mathbf{x} = \lambda_i \mathbf{X}(u_1, u_2). \quad (1.58)$$

It remains to evaluate the surface force

$$\mathbf{t} = \mathbf{T} \cdot \mathbf{N} \quad (1.59)$$

that must be applied to the unit surface of ∂C in order to make the deformation (1.47) possible. In the above equation, $\mathbf{N}$ denotes the unit vector normal to the known boundary surface (1.58).

We now apply the above considerations to the parallelepiped S shown in Fig. 1.3. We note that the faces of S remain parallel to each other under the deformation (1.47). Therefore, the unit vector $\mathbf{N}$ orthogonal to the face $ABCD$ after the deformation becomes

$$\mathbf{N} = (0, 1, 0),$$

whereas the unit vector ν tangent to the same face $ABCD$ has the components

$$\nu = (\alpha, 0, \beta),$$

where

$$\alpha^2 + \beta^2 = 1.$$

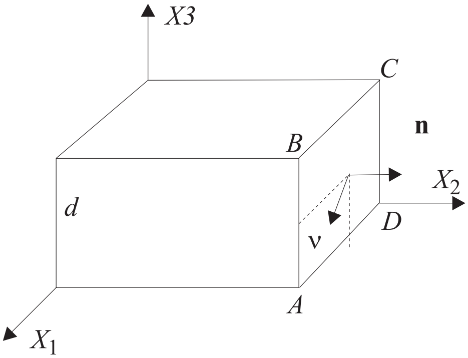

Fig. 1.3 Homothetic deformation of a parallelepiped

Consequently, the normal and tangential forces acting on this face are

$$\mathbf{t}_n = (\mathbf{n} \cdot \mathbf{T}\,\mathbf{n})\mathbf{n} = T_{22}\mathbf{n}, \quad \mathbf{t}_\nu = (\nu \cdot \mathbf{T}\,\mathbf{n})\nu = \mathbf{0}, \tag{1.60}$$

respectively. Applying the same considerations to the other faces, we conclude that the forces are orthogonal to the faces of the parallelepiped on which they act.

When S is incompressible $J = \lambda_1\lambda_2\lambda_3 = 1$, and, in view of (1.45) and (1.50), we can state that the stress tensor has the following components:

$$T_{ii} = -p + 2\frac{\partial\Psi}{\partial I}\lambda_i^2 - 2\frac{\partial\Psi}{\partial II}\frac{1}{\lambda_i^2}, \; i = 1, 2, 3, \tag{1.61}$$

$$T_{ij} = 0, \quad i \neq j. \tag{1.62}$$

In the absence of body force, equilibrium equation (1.1) is verified if the undetermined pressure p satisfies the equation

$$\frac{\partial p}{\partial x_i} = 0; \tag{1.63}$$

i.e., if it is equal to a constant p_0.

Finally, since $\lambda_1\lambda_2\lambda_3 = 1$, we have the following for an incompressible material:

$$T_{ii} = -p_0 + 2\frac{\partial\Psi}{\partial I}\lambda_i^2 - 2\frac{\partial\Psi}{\partial II}\frac{1}{\lambda_i^2}, \quad i = 1, 2, \tag{1.64}$$

$$T_{33} = -p_0 + 2\frac{\partial\Psi}{\partial I}\frac{1}{\lambda_1^2\lambda_2^2} - 2\frac{\partial\Psi}{\partial II}\lambda_1^2\lambda_2^2, \tag{1.65}$$

$$T_{ij} = 0, \quad i \neq j, \tag{1.66}$$

and the principal invariants of $\mathbf{B}$ become:

$$I = \lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2}, \tag{1.67}$$

$$II = \lambda_1^2\lambda_2^2 + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}, \tag{1.68}$$

$$III = 1. \tag{1.69}$$

We conclude this section by noting that, if there is no surface force on the face $X_3 = 0$ or on the face $X_3 = d$, $T_{33} = 0$ and (1.65) gives the following value for the pressure p_0:

$$p_0 = 2\frac{\partial\Psi}{\partial I}\frac{1}{\lambda_1^2\lambda_2^2} - 2\frac{\partial\Psi}{\partial II}\lambda_1^2\lambda_2^2. \tag{1.70}$$

Introducing this value of p_0 into (1.64), we obtain

$$T_{11} = 2\left(\lambda_1^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial\Psi}{\partial I} + \lambda_2^2\frac{\partial\Psi}{\partial II}\right), \tag{1.71}$$

$$T_{22} = 2\left(\lambda_2^2 - \frac{1}{\lambda_1^2\lambda_2^2}\right)\left(\frac{\partial\Psi}{\partial I} + \lambda_1^2\frac{\partial\Psi}{\partial II}\right). \tag{1.72}$$

1.8 Simple Extension of a Rectangular Block

The particular homothetic deformation

$$x_1 = \alpha X_1, \quad x_2 = \beta X_2, \quad x_3 = \beta X_3, \tag{1.73}$$

where α and β are positive real numbers, is termed a *simple extension*. The tensors $\mathbf{B}$ and $\mathbf{B}^{-1}$ that correspond to this deformation are

$$\mathbf{B} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \beta^2 & 0 \\ 0 & 0 & \beta^2 \end{pmatrix}, \qquad \mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{\alpha^2} & 0 & 0 \\ 0 & \frac{1}{\beta^2} & 0 \\ 0 & 0 & \frac{1}{\beta^2} \end{pmatrix}. \tag{1.74}$$

If S is compressible, the stress tensor is given by (1.54)–(1.57):

$$T_{11} = 2\alpha\left(\frac{1}{\beta^2}\frac{\partial\Psi}{\partial I} + 2\frac{\partial\Psi}{\partial II} + \beta^2\frac{\partial\Psi}{\partial III}\right), \tag{1.75}$$

$$T_{22} = T_{33} = 2\left[\frac{1}{\alpha}\left(\frac{\partial\Psi}{\partial I} + (\alpha^2+\beta^2)\frac{\partial\Psi}{\partial II}\right) + \alpha\beta^2\frac{\partial\Psi}{\partial III}\right], \tag{1.76}$$

$$T_{ij} = 0, \quad i \neq j. \tag{1.77}$$

Let S be a rectangular block with edges that are parallel to the coordinate axes. Also let $\mathbf{u}_i$, $i = 1, 2, 3$, be the unit vectors along these axes. If ∂S_i is the face with $\mathbf{u}_i$ as its unit normal vector, and $\partial S'_i$ is the face with $-\mathbf{u}_i$ as its unit normal vector, then the surface forces $\mathbf{t}_i$ and $\mathbf{t}'_i$ that must be applied to ∂S_i and $\partial S'_i$, respectively, in order to achieve the above deformation are

$$\mathbf{t}_i = T_{ii}\mathbf{u}_i, \quad \mathbf{t}'_i = -T_{ii}\mathbf{u}_i. \tag{1.78}$$

It is quite natural to wonder if a simple extension can be obtained by the action of normal forces on the faces ∂S_1 and $\partial S'_1$. In order to achieve this, we first apply the forces $\mathbf{t}_1$ and $\mathbf{t}'_1$ obtained from (1.78) for $i = 1$ to these faces; moreover, due to (1.77), we must satisfy the following condition if the forces acting on the other faces are to be eliminated:

$$T_{22} = \frac{1}{\alpha\beta}\frac{\partial \Psi}{\partial I} + \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)\frac{\partial \Psi}{\partial II} + \alpha\beta\frac{\partial \Psi}{\partial III} = 0. \tag{1.79}$$

For a given α (i.e., for an assigned extension or contraction along $\mathbf{u}_1$), the following three cases are possible:

1. Equation 1.79 allows a unique real positive solution β and the requested extension can be achieved.
2. Equation 1.79 does not permit real solutions, and so the assigned extension cannot be achieved in this material.
3. Equation 1.79 allows a number of real positive solutions $(\beta_1, \beta_2, \ldots)$. Then, by substituting the pairs (α, β_1), $(\alpha, \beta_2), \ldots$ into (1.75), we can derive the different forces that can be applied to ∂S_1 and $\partial S'_1$ to give the same extension.

The last case could not be verified for linear elasticity. In fact, in this approximation, when we denote Lamé's coefficients (see p. 176 of [16]) by λ and μ, (1.79) reduces to the condition

$$\lambda\alpha + 2\beta(\lambda + \mu) - 3\lambda - 2\mu = 0,$$

which is a first-degree equation. Consequently, for a given α, it allows one positive real solution β at most.

Again, we refer this deformation to an incompressible elastic parallelepiped S. For a simple extension that preserves the volume, we have $\beta^2 = 1/\alpha$, and the matrices $\mathbf{F}$, $\mathbf{B}$, and $\mathbf{B}^{-1}$ (see 1.74) become

$$\mathbf{F} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\alpha}} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \frac{1}{\alpha} & 0 \\ 0 & 0 & \frac{1}{\alpha} \end{pmatrix}, \quad \mathbf{B}^{-1} = \begin{pmatrix} \frac{1}{\alpha^2} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}. \tag{1.80}$$

From (1.45), which defines the Cauchy stress tensor for such a material, and from (1.80), we obtain

$$T_{11} = -p + 2\frac{\partial \Psi}{\partial I}\alpha^2 - 2\frac{\partial \Psi}{\partial II}\frac{1}{\alpha^2}, \tag{1.81}$$

$$T_{22} = T_{33} = -p + 2\frac{\partial \Psi}{\partial I}\frac{1}{\alpha} - 2\frac{\partial \Psi}{\partial II}\alpha, \tag{1.82}$$

$$T_{ij} = 0, \ i \neq j. \tag{1.83}$$

Now, in the absence of body forces, it is possible to achieve the simple extension without any surface forces on the faces parallel to the coordinate planes Ox_1x_2 and Ox_1x_3. According to these conditions we have $T_{22} = T_{33} = 0$, so the uniform pressure is given by the relation

$$p = \frac{2}{\alpha}\frac{\partial \Psi}{\partial I} - 2\alpha\frac{\partial \Psi}{\partial II}. \tag{1.84}$$

Substituting this expression into (1.81), we finally obtain

$$T_{11} = 2\left(\alpha^2 - \frac{1}{\alpha}\right)\frac{\partial \Psi}{\partial I} + 2\left(\alpha - \frac{1}{\alpha^2}\right)\frac{\partial \Psi}{\partial II}. \tag{1.85}$$

1.9 Simple Shear of a Rectangular Block

Let S be a rectangular block. The deformation

$$x_1 = X_1 + KX_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{1.86}$$

is called a *simple shear* of S. In this deformation, each plane $X_2 = const.$ slides on itself. Any plane $X_3 = const.$ undergoes a similar deformation. Finally, each plane $X_1 = const.$ rotates by the *shear angle* α, and $K = \arctan \alpha$ (see Fig. 1.4) is said to be the *amount of shear*.

The deformation gradient F and the left Cauchy–Green tensor are given by the matrices

$$\mathbf{F} = \begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.87}$$

Since

$$\det \mathbf{F} = 1, \tag{1.88}$$

the deformation preserves the volume.

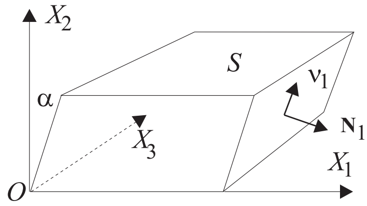

Fig. 1.4 Simple shear of a parallelepiped

The principal invariants of $\mathbf{B}$ are

$$I = 3 + K^2, \tag{1.89}$$
$$II = 3 + K^2, \tag{1.90}$$
$$III = 1, \tag{1.91}$$

and the matrix $\mathbf{B^{-1}}$ is

$$\mathbf{B}^{-1} = \begin{pmatrix} 1 & -K & 0 \\ -K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.92}$$

Using the equations of the two bent faces π_1 and π_2 of S,

$$x_1 + Kx_2 = 0, \quad x_1 - Kx_2 = a, \tag{1.93}$$

where a is the length of the edge between π_1 and π_2, we can derive the unit vectors normal to them:

$$\mathbf{N}_{1,2} = \left(\pm\frac{1}{\sqrt{1+K^2}}, \pm\frac{K}{\sqrt{1+K^2}}, 0 \right). \tag{1.94}$$

Consequently, the vectors tangent to π_1 and π_2 and parallel to the plane Ox_1x_2 are

$$\nu_{1,2} = \left(\pm\frac{K}{\sqrt{1+K^2}}, \pm\frac{1}{\sqrt{1+K^2}}, 0 \right). \tag{1.95}$$

Introducing (1.87)–(1.92) into (1.29)–(1.40), we derive

$$T_{11} = 2\left((1+K^2)\frac{\partial\Psi}{\partial I} + (2+K^2)\frac{\partial\Psi}{\partial II} + \frac{\partial\Psi}{\partial III} \right), \tag{1.96}$$

$$T_{22} = 2\left(\frac{\partial\Psi}{\partial I} + 2\frac{\partial\Psi}{\partial II} + \frac{\partial\Psi}{\partial III} \right), \tag{1.97}$$