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Vanishing and Finiteness Results in Geometric Analysis

A Generalization of the Bochner Technique

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Introduction

This book originated from a graduate course given during the Spring of 2005 at the University of Milan. Our goal was to present an extension of the original Bochner technique describing a selection of results recently obtained by the authors, in non-compact settings where in addition one didn't assume that the relevant curvature operators satisfied signum conditions. To make the course accessible to a wider audience it was decided to introduce many of the more advanced analytical and geometrical tools along the way.

The initial project has grown past the original plan, and we now aim at treating in a unified and detailed way a variety of problems whose common thread is the validity of Weitzenböck formulae.

As is well illustrated in the elegant work by H.H. Wu, [165], typically, one is given a Riemannian (Hermitian) vector bundle E with compatible fiber metric and considers a geometric Laplacian L on E which is related to the connection (Bochner) Laplacian $-tr(D^*D)$ via a fiber bundle endomorphism $\mathfrak{R}$ which is in turn related to the curvature of the base manifold M. Because of this relationship, the space of L-harmonic sections of E reflects the geometric properties of M.

To illustrate the method, let us consider the original Bochner argument to estimate the first real Betti number $b^1(M)$ of a closed oriented Riemannian manifold $(M, \langle\,,\rangle)$.

By the Hodge–de Rham theory, $b^1(M)$ equals the dimension of the space of harmonic 1-forms $\mathcal{H}^1(M)$. A formula of Weitzenböck, independently rediscovered by Bochner, states that for every harmonic 1-form ω,

$$\frac{1}{2}\Delta|\omega|^2 = |D\omega|^2 + \operatorname{Ric}\left(\omega^\#, \omega^\#\right), \tag{0.1}$$

where Δ and Ric are the Laplace–Beltrami operator (with the sign convention $+d^2/dx^2$) and the Ricci curvature of M, respectively, D denotes the extension to 1-forms of the Levi–Civita connection, and $\omega^\#$ is the vector field dual to ω, defined by $\langle\omega^\#, X\rangle = \omega(X)$ for all vector fields X. In particular $|\omega|^2$ satisfies the differential inequality

$$\Delta|\omega|^2 - q(x)|\omega|^2 \geq 0,$$

where $q(x)/2$ is the lowest eigenvalue of the Ricci tensor at x. Thus, if Ric ≥ 0, then $|\omega|$ is subharmonic. Since M is closed, we easily conclude that $|\omega|$ =const. This can be done using two different viewpoints, (i) the L^∞ and (ii) the $L^{p<+\infty}$ one. As for (i), note that the smooth function $|\omega|$ attains its maximum at some point and, therefore, by the Hopf maximum principle we conclude that $|\omega|$ =const. In case (ii) we use the divergence theorem to deduce

$$0 = \int_M \operatorname{div}\left(|\omega|^2\nabla|\omega|^2\right) = \int_M \left|\nabla|\omega|^2\right|^2 + \int_M |\omega|^2\Delta|\omega|^2 \geq \int_M \left|\nabla|\omega|^2\right|^2 \geq 0.$$

This again implies $|\omega| =$const.

Now, since $\mathrm{Ric} \geq 0$, using this information in formula (0.1) shows that ω is parallel, i.e., $D\omega = 0$. As a consequence, ω is completely determined by its value at a given point, say $p \in M$. The evaluation map $\varepsilon_p : \mathcal{H}^1(M) \to \Lambda^1\left(T_p^* M\right)$ defined by

$$\varepsilon_p(\omega) = \omega_p$$

is an injective homomorphism, proving that, in general,

$$b^1(M) = \dim \mathcal{H}^1(M) \leq m.$$

Note that (0.1) yields

$$0 = \mathrm{Ric}\left(\omega_p^\#, \omega_p^\#\right) \text{ at } p.$$

Therefore, if $\mathrm{Ric}(p) > 0$, we get $\omega_p = 0$ which, in turn, implies $\omega = 0$. This shows that, when Ric is positive somewhere,

$$b^1(M) = \dim \mathcal{H}^1(M) = 0.$$

The example suggests that one can generalize the investigation in several directions. One can relax the assumption on the signum of the coefficient $q(x)$, consider complete non-compact manifolds, or both.

Maintaining compacteness, one can sometimes allow negative values of $q(x)$ using versions of the generalized maximum principle, according to which if $\psi \geq 0$ satisfies

$$\Delta\psi - q(x)\psi \geq 0, \tag{0.2}$$

and M supports a solution $\varphi > 0$ of

$$\Delta\varphi - q(x)\varphi \leq 0, \tag{0.3}$$

then the ratio $u = \psi/\varphi$ is constant. Combining (0.2) and (0.3) shows that ψ satisfies (0.2) with equality sign. In particular, according to (0.1), $\psi = |\omega|^2$ satisfies (0.2), and therefore, if M supports a function φ satisfying (0.3), we conclude, once again, that ω is parallel, thus extending the original Bochner vanishing result to this situation.

It is worth noting that the existence of a function φ satisfying (0.3) is related to spectral properties of the operator $-\Delta + q(x)$, and that the conclusion of the generalized maximum principle is obtained by combining (0.2) and (0.3) to show that the quotient u satisfies a differential inequality without zero-order terms; see Section 2.5 in [133].

In the non-compact setting the relevant function may fail to be bounded, and even if it is bounded, it may not attain its supremum. In the latter case, one may use a version of the maximum principle at infinity introduced by H. Omori, [124] and generalized by S.T. Yau, [167], and S.Y Cheng and Yau, [34], elaborating ideas of L.V. Ahlfors. An account and further generalizations of this technique, which however works under the assumption that $q(x)$ is non-negative, may be found in [131].

Here we consider the case where the manifold is not compact and the function encoding the geometric problem is not necessarily bounded, but is assumed to satisfy suitable L^p integrability conditions, and the coefficient $q(x)$ in the differential (in)equality which describes the geometric problem is not assumed to be non-negative.

Referring to the previous example, the space of harmonic 1-forms in L^2 describes the L^2 co-homology of a complete manifold, and under suitable assumptions it has a topological content sensitive to the structure at infinity of the manifold. It turns out to be a bi-Lipschitz invariant, and, for co-compact coverings, it is in fact a rough isometry invariant.

As in the compact case described above, one replaces the condition that the coefficient $q(x)$ is pointwise positive, with the assumption that there exists a function φ satisfying (0.3) on M or at least outside a compact set. Again, one uses a Weitzenböck-type formula to show that the geometric function $\psi = |\omega|$ satisfies a differential inequality of the form (0.2).

Combining (0.2) and (0.3) and using the integrability assumption, one concludes that either ψ vanishes and therefore the space $L^2\mathcal{H}^1(M)$ of L^2-harmonic 1-forms is trivial or that $L^2\mathcal{H}^1(M)$ is finite-dimensional.

The method extends to the case of L^p-harmonic k-forms, even with values in a fibre bundle, and in particular to harmonic maps with L^p energy density, provided we consider an appropriate multiple of $q(x)$ in (0.3), and restrict the integrability coefficient p to a suitable range. Harmonic maps in turn yield information, as in the compact case, on the topological structure of the underlying domain manifold.

This relationship becomes even more stringent in the case where the domain manifold carries a Kählerian structure. Indeed, for complex manifolds, the splitting in types allows to consider, besides harmonic maps, also pluriharmonic and holomorphic maps. If, in addition, the manifold is Kähler, the relevant Weitzenböck identity for pluriharmonic functions (which in the L^2 energy case coincides with a harmonic function with L^2 energy) takes on a form which reflects the stronger rigidity of the geometry and allows us to obtain stronger conclusions. Thus, on the one hand one can enlarge the allowed range of the integrability coefficient p, and on the other hand one may deduce structure theorems which have no analogue in the purely Riemannian case.

The extension to the non-compact case introduces several additional technical difficulties, which require specific methods and tools. The description of these is in fact a substantial part of the book, and while most, but not all, of the results are well known, in many instances our approach is somewhat original. Further, in some cases, one needs results in a form which is not easily found, if at all, in the literature.

When we feel that these ancillary parts are important enough, or the approach sufficiently different from the mainstream treatment, a fairly detailed description is given. Thus we provide, for instance, a rather comprehensive treatment of comparison methods in Riemannian geometry or of the spectral theory of Schrödinger operators on manifolds. In other situations, the relevant tools are introduced when needed. For instance this is the case of the Poincaré inequalities or of the Moser iteration procedure.

The material is organized as follows.

In Chapter 1, after a quick review of harmonic maps between Riemannian manifolds, where in particular we describe the Weitzenböck formula and derive a sharp version of Kato's inequality, we introduce the basic facts on the geometry of complex manifolds, and Hermitian bundles, concentrating on the Kähler case. Our approach is inspired by work of S.S. Chern, and is based on analyzing the Riemannian counterpart of the Kähler structure.

The same line of arguments allows us to extend a result of J.H. Sampson, [143], concerning the pluriharmonicity of a harmonic map from a compact Kähler manifold into a Riemannian target with negative Hermitian curvature to the case of a non-compact domain. This in turn yields a sharp version of a result of P. Li, [96], for pluriharmonic real-valued functions. The chapter ends with a derivation of Weitzenböck-type formulas for pluriharmonic and holomorphic maps.

Chapter 2 is devoted to a detailed description of comparison theorems in Riemannian Geometry under curvature conditions, both pointwise and integral, which will be extensively used throughout the book. We begin with general comparison results for the Laplacian and the Hessian of the distance function. The approach, which is indebted to P. Petersen's treatment, [128], is analytic in that it only uses comparison results for ODEs avoiding the use of Jacobi fields, and it is not limited to the case where the bound on the relevant curvature is a constant, but is given in terms of a suitable function G of the distance from a reference point. Some effort is also made to describe explicit bounds in a number of geometrically significant situations, namely when $G(r) = -B(1+r^2)^{\alpha/2}$, or when $G(t)$ satisfies the integrability condition $tG(t) \in L^1([0,+\infty))$ considered, among others, by U. Abresch, [1], and by S.H. Zhu, [171].

These estimates are then applied to obtain volume comparisons. Even though the method works both for upper and lower estimates, we concentrate on upper bounds, which hold under less stringent assumptions on the manifold, and in particular depend on lower bounds for the Ricci curvature alone, and do not require topological restrictions. We also describe volume estimates under integral Ricci curvature conditions which extend previous work of S. Gallot, [57], and, more recently, by Petersen and G. Wei, [129]. We then describe remarkable lower estimates for the volume of large balls on manifolds with almost non-negative Ricci curvature obtained by P. Li and R. Schoen, [95] and Li and M. Ramachandran, [98], elaborating on ideas of J. Cheeger M. Gromov and M. Taylor, [33]. These estimates in particular imply that such manifolds have infinite volume. We conclude the chapter with a version of the monotonicity formula for minimal submanifolds valid for the volume of intrinsic (as opposed to extrinsic) balls in bi-lipschitz harmonic immersions.

Chapter 3 begins with a quick review of spectral theory of self-adjoint operators on Hilbert spaces modelled after E.B. Davies' monograph, [41]. In particular, we define the essential spectrum and index of a (semibounded) operator, and apply the minimax principle to describe some of their properties and their mutual relationships. We then concentrate on the spectral theory of Schrödinger operators on manifolds, in terms of which many of the crucial assumptions of our geometrical results are formulated.

After having defined Schrödinger operators on domains and on the whole manifold, we describe variants of classical results by D. Fisher-Colbrie, [53], and Fisher-Colbrie and Schoen, [54], which relate the non-negativity of the bottom of the spectrum of a Schrödinger operator L on a domain Ω to the existence of a positive solution of the differential inequality $L\varphi \leq 0$ on Ω.

Since, as already mentioned above, the existence of such a solution is the assumption on which the analytic results depend, this relationship allows us to interpret such hypothesis as a spectral condition on the relevant Schrödinger operator. This is indeed a classical and natural feature in minimal surfaces theory where the stability, and the finiteness of the index of a minimal surface, amount to the fact that the stability operator $-\Delta - |II|^2$ has non-negative spectrum, respectively finite Morse index.

In describing these relationships we give an account of the links between essential spectrum, bottom of the spectrum, and index of a Schrödinger operator L on a manifold, and that of its restriction to (internal or external) domains. With a somewhat different approach and arguments, our presentation follows the lines of a paper by P. Berard, M.P. do Carmo and W. Santos, [13].

Chapter 4 and Chapter 5 are the analytic heart of the book. In Chapter 4 we prove a Liouville-type theorem for L^p solutions u of divergence-type differential inequalities of the form

$$u\mathrm{d}iv(\varphi\nabla u) \geq 0,$$

where φ is a suitable positive function. An effort is made to state and prove the result under the minimal regularity assumptions that will be needed for geometric applications. As a consequence we deduce the main result of the chapter, namely a vanishing theorem for non-negative solutions of the Bochner-type differential inequality

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geq 0. \tag{0.4}$$

Assuming the existence of a positive solution of the inequality

$$\Delta\varphi + Ha(x)\varphi \leq 0, \tag{0.5}$$

for a suitable constant H, one proceeds similarly to what we described above, and shows that an appropriate combination u of the function ψ and φ satisfies the hypotheses of the Liouville-type theorem.

In Chapter 6 the analytic setting is similar, one considers vector spaces of L^p-sections whose lengths satisfy the differential inequality (0.4) and proves that such spaces are finite-dimensional under the assumption that a solution φ to the differential inequality (0.5) exists in the complement of a compact set K in M. The idea of the proof is to show that there exists a constant C depending only on the geometry of the manifold in a neighborhood of K such that the dimension of every finite-dimensional subspace is bounded by C. The proof is based on a version of a lemma by Li, and uses a technique of Li and J. Wang, [104] and [105], combined with the technique of the coupling of the solutions ψ and φ which allows us to deal with L^p sections with p not necessarily equal to 2. The proof requires a number of technical results which are described in detailed, in some cases new, direct proofs.

Chapter 6 to 9 are devoted to applications in different geometric contexts. In Chapter 6 we specialize the vanishing results to the case of harmonic maps with finite L^p energy, and derive results on the constancy of convergent harmonic maps, and a Schwarz-type lemma for harmonic maps of bounded dilation. We then describe topological results by Schoen and Yau, [146], concerning the fundamental group of manifolds of non-negative Ricci curvature and of stable minimal hypersurfaces immersed in non-positively curved ambient spaces. While the main argument is the same as Schoen and Yau's, the use of our vanishing theorem allows us to relax their assumption that the Ricci curvature of the manifold is non-negative. The chapter ends by generalizing to non-compact settings the finiteness theorems of L. Lemaire, [93], for harmonic maps of bounded dilation into a negatively curved manifold, on the assumption that the domain manifold has a finitely generated fundamental group.

In Chapter 7 we use the techniques developed above to describe the topology at infinity of a Riemannian manifold M, and more specifically the number of unbounded connected components of the complement of a compact domain D in M, namely the ends of M with respect to D.

The number of ends of a manifold will in turn play a crucial role in the structure results for Kähler manifolds, and in the derivation of metric rigidity in the Riemannian setting (see Chapters 8 and 9, respectively).

The chapter begins with an account of the theory relating the topology at infinity and suitable classes of harmonic functions on the manifold as developed by Li and L.F. Tam and collaborators. At the basis of this theory is the fact that, via the maximum principle, the parabolicity/non-parabolicity of an end is intimately connected with the existence of a proper harmonic function on the end (the so-called Evans–Selberg potential of the end), or, in the non-parabolic case, of a bounded harmonic function on the end with finite Dirichlet integral. Combining these facts with the analytic results of the previous chapters in particular, we obtain that the manifold has only one, or at most finitely many non-parabolic ends, depending on spectral assumptions on the operator $L = -\Delta - a(x)$, where $-a(x)$ is the smallest eigenvalue of the Ricci tensor at x. To complete the picture, following H.-D. Cao, Y. Shen, S. Zhu, [25], and Li and Wang, [104], one shows that when the manifold supports an L^1-Sobolev inequality, then all ends are non-parabolic. This in particular applies to submanifolds of Cartan–Hadamard manifolds, provided that the second fundamental form is small in a suitable integral norm. In the chapter, using a gluing technique of T. Napier and Ramachandran, [117], we also provide the details of a construction sketched by Li and Ramachandran, [98] of harmonic functions with controlled L^2 energy growth that will be used in the structure theorems for Kähler manifolds. The last two sections of the chapter contain further applications of these techniques to problems concerning line bundles over Kähler manifolds, and to the reduction of codimension of harmonic immersions with less than quadratic p-energy growth.

In Chapter 8 we concentrate on the Kähler setting. We begin by providing a detailed description of a result of Li and Yau, [107], on the constancy of holomorphic maps with values in a Hermitian manifold with suitably negative holomorphic bisectional curvature. We then describe two variations of the result, where the conclusion is obtained under different assumptions: in the first, using Poisson equation techniques, an integral growth condition on the Ricci tensor is replaced by a volume growth condition, while in the second one assumes a pointwise lower bound on the Ricci curvature which is not necessarily integrable, together with some spectral assumptions on a variant of the operator L. We then apply this in the proof of the existence of pluri-subharmonic exhaustions due to Li and Ramachandran, [98], which is crucial in obtaining the important structure theorem of Napier and Ramachandran, [117], and Li and Ramachandran, [98].

The unifying element of Chapter 9 is the validity of a Poincaré–Sobolev inequality. In the first section, we give a detailed proof of a warped product splitting theorem of Li and Wang, [104]. There are two main ingredients in the proof. The first is to prove that the metric splitting holds provided the manifold supports a non-constant harmonic function u for which the Bochner inequality with a sharp constant in the refined Kato's inequality is in fact an equality. The second ingredient consists of energy estimates for a suitable harmonic function u on M obtained by means of an exhaustion procedure. This is the point where the Poincaré–Sobolev inequality plays a crucial role. Finally, one uses the analytic techniques of Chapter 4 to show that u is the sought-for function which realizes equality in the Bochner inequality. In the second section we begin by showing that whenever M supports an L^2 Poincaré–Sobolev-type inequality, then a non-negative L^p solution ψ of the differential inequality (0.4),

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geq 0,$$

must vanish provided a suitable integral norm of the potential $a(x)$ is small compared to the Sobolev constant. This compares with the vanishing result of Chapter 4 which holds under the assumption that the bottom of the spectrum of $-\Delta + Ha(x)$ is non-negative. Actually, in view of the geometric applications that follow, we consider the case where M supports an inhomogeneous Sobolev inequality.

We then show how to recover the results on the topology at infinity for submanifolds of Cartan–Hadamard manifolds of Chapter 7. In fact, using directly the Sobolev inequality allows us to obtain quantitative improvements. Further applications are given to characterizations of space forms which extend in various directions a characterization of the sphere among conformally flat manifolds with constant scalar curvature of S. Goldberg, [61].

The book ends with two appendices. The first is devoted to the unique continuation property for solutions of elliptic partial differential systems on manifolds, which plays an essential role in the finite-dimensionality result of Chapter 5. Apart from some minor modifications, our presentation follows the line of J. Kazdan's paper [87].

In the second appendix we review some basic facts concerning the L^p cohomology of complete non-compact manifolds. We begin by describing the basic definitions of the L^p de Rham complex and discussing some simple, but significant examples. We then collect some classical results like the Hodge, de Rham, Kodaira decomposition, and briefly consider the role of L^p harmonic forms. Finally, we illustrate some of the relationships between L^p cohomology and the geometry and the topology of the underlying manifold both for $p = 2$ and $p \neq 2$. In particular we present (with no proofs) the Whitney-type approach developed by J. Dodziuk, [43] and V.M. Gol'dshtein, V.I. Kuz'minov, I.A. Shvedov, [63] and [64], where the topological content of the L^p de Rham cohomology is emphasized by relating it to a suitable, global simplicial theory on the underlying triangulated manifold.

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Chapter 1

Harmonic, pluriharmonic, holomorphic maps and basic Hermitian and Kählerian geometry

1.1 The general setting

The aim of the chapter is to review some basic facts of Riemannian and complex geometry, in order to compute, for instance, some Bochner-type formulas that we shall need in the sequel. In doing so, we do not aim at giving a detailed treatment of the subject, but only to set down notation and relevant results, illustrating some of the computational techniques involved in the proofs.

Let $(M, \langle\, , \rangle)$ and $(N, (\, ,))$ be (real) smooth manifolds of (real) dimensions m and n respectively, endowed with the Riemannian metrics $\langle\, , \rangle$ and $(\, ,)$ and let $f : M \to N$ be a smooth map. The energy density $e(f) : M \to \mathbb{R}$ is the non-negative function defined on M as follows. Let $df \in \Gamma\left(T^*M \otimes f^{-1}TN\right)$ be the differential of f and set

$$e(f)(x) = \frac{1}{2}\left|d_x f\right|^2$$

where $|df|$ denotes the Hilbert-Schmidt norm of the differential map. In local coordinates $\left\{x^i\right\}$ and $\{y^\alpha\}$ respectively on M and N, $e(f)$ is expressed by

$$e(f) = \frac{1}{2}\langle\, , \rangle^{ij}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}(\, ,)_{\alpha\beta} = \frac{1}{2}\mathrm{tr}_{\langle\, , \rangle} f^*(\, ,).$$

Here $f^\alpha = y^\alpha \circ f$ and $\langle\, , \rangle^{ij}$ represents the inverse of the matrix coefficient $\langle\, , \rangle_{ij} = \left\langle \partial/\partial x^i, \partial/\partial x^j \right\rangle$.

If $\Omega \subset M$ is a compact domain we use the canonical measure

$$d\mathrm{Vol}_{\langle\, , \rangle} = \sqrt{\det \langle\, , \rangle_{ij}}\; dx^1 \wedge \cdots \wedge dx^m$$

associated to $\langle\, , \rangle$ to define the energy of $f|_\Omega : (\Omega, \langle\, , \rangle) \to (N, (\, ,))$ by

$$E_\Omega(f) = \int_\Omega e(f)\, d\mathrm{Vol}_{\langle\, , \rangle}.$$

Definition 1.1. A smooth map $f : (M, \langle\, , \rangle) \to (N, (\, ,))$ is said to be harmonic if, for each compact domain $\Omega \subset M$, it is a stationary point of the energy functional $E_\Omega : C^\infty(M, N) \to \mathbb{R}$ with respect to variations preserving f on $\partial\Omega$.

A vector field X along f, that is, a section of the bundle $f^{-1}TN \to M$ determines a variation f_t of f by setting

$$f_t(x) = \exp_{f(x)} tX_x.$$

If X has support in a compact domain $\Omega \subset M$, then

$$\left.\frac{d}{dt}\right|_{t=0} E_\Omega(f_t) = -\int_M \big(\tau(f)(x), X_x\big) d\mathrm{Vol}_{\langle\, ,\rangle}$$

where the Euler-Lagrange operator, called the *tension field* of f, is given by

$$\tau(f) = \mathrm{tr}_{\langle\, ,\rangle} Ddf,$$

$Ddf \in \Gamma\left(T^*M \otimes T^*M \otimes f^{-1}TN\right)$ being the (generalized) *second fundamental tensor* of the map f. As a consequence, $\tau(f) \in \Gamma\left(f^{-1}TN\right)$ and f is harmonic if and only if

$$\tau(f) = 0 \text{ on } M.$$

In local coordinates

$$\tau(f)^\gamma = \langle\, ,\rangle^{ij}\left(\frac{\partial^2 f^\gamma}{\partial x^i \partial x^j} - {}^M\Gamma^k_{ij}\frac{\partial f^\gamma}{\partial x^k} + {}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial f^\alpha}{\partial x^i}\frac{\partial f^\beta}{\partial x^j}\right)$$

where ${}^M\Gamma$ and ${}^N\Gamma$ are the Christoffel symbols of the Levi–Civita connections on M and N, respectively. Thus, the harmonicity condition is represented by a system of non-linear elliptic equations.

Observe that, when $f : (M, \langle\, ,\rangle) \to (N, (\, ,))$ is an isometric immersion, that is, $f^*(\, ,) = \langle\, ,\rangle$, then $\tau(f) = m\mathrm{H}$, with H the mean curvature vector field of the immersion. It is well known that the equation $\mathrm{H} \equiv 0$ is the Euler-Lagrange equation of the volume functional

$$V_\Omega(f) = \int_\Omega d\mathrm{Vol}_{\langle\, ,\rangle}$$

$\Omega \subset M$ a compact domain. Thus, an isometric immersion is *minimal* if and only if it is harmonic.

For later use, we show how to compute the tension field of $f : (M, \langle\, ,\rangle) \to (N, (,))$ with the moving frame formalism. Towards this aim, let $\{\theta^i\}$ and $\{e_i\}$, $i = 1, \ldots, m$, be local ortho-normal co-frame, and dual frame, on M with corresponding Levi–Civita connection forms $\{\theta^i_j\}$. Similarly, let $\{\omega^\alpha\}, \{\varepsilon_\alpha\}, \{\omega^\alpha_\beta\}$, $1 \le \alpha, \beta, \ldots \le n$ describe, locally, the Riemannian structure of $(N, (,))$. Then

$$f^*\omega^\alpha = f^\alpha_i \theta^i$$

so that

$$df = f^\alpha_i \theta^i \otimes \varepsilon_\alpha$$

and computing the covariant derivatives

$$\text{(i) } f^\alpha_{ij}\theta^j = df^\alpha_i - f^\alpha_j\theta^j_i + f^\beta_i\omega^\alpha_\beta, \qquad \text{(ii) } f^\alpha_{ij} = f^\alpha_{ji}$$

in such a way that

$$Ddf = f^\alpha_{ij}\theta^i \otimes \theta^j \otimes \varepsilon_\alpha$$

and

$$\tau(f) = \sum_i f^\alpha_{ii}\varepsilon_\alpha.$$

In what follows we shall also use the next *Bochner–Weitzenböck-type formula* for harmonic maps. Since we shall prove analogous formulas in Kählerian geometry we omit here its derivation. See, e.g., [47].

Theorem 1.2. *Let $f : (M, \langle\,,\rangle) \to (N, (\,,))$ be a smooth map. Then*

$$\frac{1}{2}\Delta|df|^2 = |Ddf|^2 - \mathrm{tr}_{\langle\,,\rangle}(D\tau(f), df) + \sum_i \left(df\left({}^M\mathrm{Ric}(e_i,\cdot)^\#\right), df(e_i)\right)$$
$$- \sum_{i,j}\left({}^N\mathrm{Riem}(df(e_i), df(e_j))\,df(e_j), df(e_i)\right)$$

with $\{e_i\}$ as above and ${}^M\mathrm{Ric}$, ${}^N\mathrm{Riem}$ respectively the Ricci tensor of M and the Riemannian curvature tensor of N. In particular, if f is harmonic,

$$\frac{1}{2}\Delta|df|^2 = |Ddf|^2 + \sum_i \left(df\left({}^M\mathrm{Ric}(e_i,\cdot)^\#\right), df(e_i)\right)$$
$$- \sum_{i,j}\left({}^N\mathrm{Riem}(df(e_i), df(e_j))\,df(e_j), df(e_i)\right).$$

Futher, assuming that f is a harmonic function the formula specializes to Bochner's formula

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}\, f|^2 + \mathrm{Ric}(\nabla f, \nabla f). \tag{1.1}$$

Weitzenböck formulae will be repeatedly used in the sequel. Here we give a sharp estimate from below of the term $|Ddf|^2$. This type of estimate goes under the name of *refined Kato inequalities.* Their relevance will be clarified by their analytic consequences. For a more general and abstract treatment, we refer to work by T. Branson, [21], and by D.M.J. Calderbank, P. Gauduchon, and M. Herzlich, [24].

Proposition 1.3. *Let $f : M \to N$ be a harmonic map between Riemannian manifolds of dimensions $\dim M = m$ and $\dim N = n$. Then*

$$|Ddf|^2 - |\nabla|df||^2 \geq \frac{1}{(m-1)}|\nabla|df||^2$$

pointwise on the open, dense subset $\Omega = \{x \in M : |df|(x) \neq 0\}$ and weakly on all of M.

Remark 1.4. The dimension n of the target manifold plays no role.

Proof. It suffices to consider the pointwise inequality on Ω. Let $\{f_i^\alpha\}$ and $\{f_{ij}^\alpha\}$ be the coefficients of the (local expressions of the) differential and of the Hessian of f, respectively. Then

$$|df| = \sqrt{\sum_{\alpha,i} \left(f_i^\alpha\right)^2}$$

so that

$$\nabla |df| = \frac{\sum\limits_i \left\{ \sum\limits_{\alpha,j} f_{ij}^\alpha f_j^\alpha \right\} e_i}{\sqrt{\sum\limits_{\alpha,i} \left(f_i^\alpha\right)^2}}$$

and we have

$$|Ddf|^2 - |\nabla |df||^2 = \sum_{\alpha,i,j} \left(f_{ij}^\alpha\right)^2 - \frac{\sum\limits_i \left\{ \sum\limits_{\alpha,j} f_{ij}^\alpha f_j^\alpha \right\}^2}{\sum\limits_{\alpha,i} \left(f_i^\alpha\right)^2}. \tag{1.2}$$

For $\alpha = 1, \ldots, n$, define

$$M^\alpha = \left(f_{ij}^\alpha\right) \in M_m(\mathbb{R}), \ y^\alpha = \left(f_i^\alpha\right)^t \in \mathbb{R}^m.$$

Note that each matrix M^α is traceless, by harmonicity of f, and symmetric. Then (1.2) reads

$$|Ddf|^2 - |\nabla |df||^2 = \sum_\alpha \|M^\alpha\|^2 - \frac{\left|\sum\limits_\alpha M^\alpha y^\alpha\right|^2}{\sum\limits_\alpha |y^\alpha|^2}$$

where $\|M\|^2 = tr\left(MM^t\right)$ and $|y|$ denotes the $\mathbb{R}^m$-norm of y. We have to show that

$$\sum_\alpha \|M^\alpha\|^2 - \frac{\left|\sum\limits_\alpha M^\alpha y^\alpha\right|^2}{\sum\limits_\alpha |y^\alpha|^2} \geq \frac{1}{(m-1)} \frac{\left|\sum\limits_\alpha M^\alpha y^\alpha\right|^2}{\sum\limits_\alpha |y^\alpha|^2}.$$

This inequality is an immediate consequence of the next simple algebraic lemma. □

Lemma 1.5. *For $\alpha = 1, \ldots, n$, let $M^\alpha \in M_m(\mathbb{R})$ be a symmetric matrix satisfying* $\operatorname{trace}(M^\alpha) = 0$*. Then, for every $y^1, \ldots, y^n \in \mathbb{R}^m$ with $\sum\limits_\alpha |y^\alpha|^2 \neq 0$,*

$$\sum_\alpha \|M^\alpha\|^2 - \frac{\left|\sum\limits_\alpha M^\alpha y^\alpha\right|^2}{\sum\limits_\alpha |y^\alpha|^2} \geq \frac{1}{(m-1)} \frac{\left|\sum\limits_\alpha M^\alpha y^\alpha\right|^2}{\sum\limits_\alpha |y^\alpha|^2}. \tag{1.3}$$

Moreover, suppose the equality holds. If $y^\alpha \neq 0$, then either $M^\alpha = 0$ or y^α is an eigenvector of M^α corresponding to an eigenvalue μ^α of multiplicity 1. Furthermore, the orthogonal complement $\langle y^\alpha \rangle^\perp$ is the eigenspace of M^α corresponding to the eigenvalue $-\mu^\alpha / (m-1)$ of multiplicity $(m-1)$.

Proof. First, we consider the case $\alpha = 1$. Let $\lambda_1 \leq \cdots \leq \lambda_s \leq 0 \leq \lambda_{s+1} \leq \cdots \leq \lambda_m$ be the eigenvalues of M. Without loss of generality we may assume that $\lambda_m \geq |\lambda_1|$. We are thus reduced to proving that

$$\sum_{i=1}^{m} \lambda_i^2 \geq \left(1 + \frac{1}{m-1}\right) \lambda_m^2.$$

To this end we note that, since M is traceless,

$$-\sum_{j=1}^{m-1} \lambda_j = \lambda_m \tag{1.4}$$

and therefore, from Schwarz's inequality,

$$\lambda_m^2 \leq (m-1) \sum_{j=1}^{m-1} \lambda_j^2. \tag{1.5}$$

This implies

$$\sum_{i=1}^{m} \lambda_i^2 = \lambda_m^2 + \sum_{j=1}^{m-1} \lambda_j^2 \geq \left(1 + \frac{1}{m-1}\right) \lambda_m^2,$$

as desired. Suppose now that $M \neq 0$, so that $\lambda_m > 0$, and assume that equality holds in (1.3) for some vector $y \neq 0$. Let $C \in O(m)$ be such that $CMC^t = D = \mathit{diag}(\lambda_1, \ldots, \lambda_m)$ and set $w = (w_1, \ldots, w_m) = Cy$. Thus

$$\left(1 + \frac{1}{m-1}\right) \lambda_m^2 \leq \sum_i \lambda_i^2 = \left(1 + \frac{1}{m-1}\right) \sum_i \left(\lambda_i \frac{w_i}{|w|}\right)^2 \leq \left(1 + \frac{1}{m-1}\right) \lambda_m^2. \tag{1.6}$$

It follows that the equality holds in (1.5) which in turn forces, according to (1.4) and (the equality case in) Schwarz's inequality,

$$\lambda_1 = \cdots = \lambda_{m-1} = \mu; \quad \lambda_m = -(m-1)\mu,$$

for some $\mu < 0$. On the other hand, (1.6) gives

$$\sum_{i=1}^{m-1} \lambda_i^2 \frac{w_i^2}{|w|^2} + \lambda_m^2 \left(\frac{w_m^2}{|w|^2} - 1\right) = 0$$

proving that $w \in \operatorname{span}\{(0, \ldots, 0, 1)^t\}$ and therefore it is an eigenvector of D belonging to the multiplicity 1 eigenvalue λ_m. It follows that $y = C^t w$ is an eigenvector of M belonging to the multiplicity 1 eigenvalue $\lambda_m = -(m-1)\,\mu$. Obviously, $y^\perp$ is the eigenspace corresponding to the multiplicity $(m-1)$ eigenvalue μ.

Now let α be any positive integer. We note that

$$\sum_\alpha \|M^\alpha\|^2 - \frac{\left|\sum_\alpha M^\alpha y^\alpha\right|^2}{\sum_\alpha |y^\alpha|^2} \geq \sum_\alpha \|M^\alpha\|^2 - \frac{\left(\sum_\alpha |M^\alpha y^\alpha|\right)^2}{\sum_\alpha |y^\alpha|^2}.$$

Applying the first part of the proof we get, for every $\alpha = 1, \ldots, n$,

$$|M^\alpha y^\alpha| \leq \sqrt{\frac{m-1}{m}}\, \|M^\alpha\|\, |y^\alpha| \tag{1.7}$$

which in turn, used in the above, gives

$$\sum_\alpha \|M^\alpha\|^2 - \frac{\left|\sum_\alpha M^\alpha y^\alpha\right|^2}{\sum_\alpha |y^\alpha|^2} \geq \sum_\alpha \|M^\alpha\|^2 - \frac{\left(\sum_\alpha \sqrt{\frac{m-1}{m}}\, \|M^\alpha\|\, |y^\alpha|\right)^2}{\sum_\alpha |y^\alpha|^2}$$
$$\geq \sum_\alpha \|M^\alpha\|^2 - \frac{m-1}{m} \frac{\sum_\alpha \|M^\alpha\|^2 \sum_\alpha |y^\alpha|^2}{\sum_\alpha |y^\alpha|^2} = \frac{1}{m} \sum_\alpha \|M^\alpha\|^2 .$$

Whence, rearranging and simplifying yields (1.3). To complete the proof, note that the equality in (1.3) forces equality in (1.7) and therefore the first part of the proof applies to M^α. □

1.2 The complex case

We now turn our attention to the complex case.

Definition 1.6. An almost complex manifold (M, J) is a (real) manifold together with a (smooth) tensor field $J \in \Gamma(T^*M \otimes TM)$ of endomorphisms of TM such that

$$J_p^2 = -\mathrm{id}_p \tag{1.8}$$

for every $p \in M$.

Note that (1.8) implies $\dim T_pM = 2s$.

Let $TM^{\mathbb{C}}$ denote the complexified tangent bundle of M whose fibers are $\mathbb{C} \otimes_{\mathbb{R}} T_pM$, $p \in M$. Here, $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{R}} T_pM) = 2s$. The smooth field J can be pointwise extended $\mathbb{C}$-linearly to $T_p^{\mathbb{C}}M$ so that, again, it satisfies (1.8). It follows that J_p has eigenvalues $\imath$ and $-\imath$ and

$$T_pM^{\mathbb{C}} = T_pM^{(1,0)} \oplus T_pM^{(0,1)} \tag{1.9}$$

where $T_pM^{(1,0)}$ and $T_pM^{(0,1)}$ are the eigenspaces of the eigenvalues $\imath$ and $-\imath$, respectively, Furthermore, $v' \in T_pM^{(1,0)}$ and $v'' \in T_pM^{(0,1)}$ if and only if there exist $u, w \in T_pM$ such that

$$v' = u - iJ_pu, \quad v'' = w + iJ_pw.$$

The above decomposition induces a dual decomposition

$$T_p^*M^{\mathbb{C}} = T_p^*M^{(1,0)} \oplus T_p^*M^{(0,1)}. \tag{1.10}$$

Note that (1.9) and (1.10) hold at the bundle level. Similar decompositions are induced on tensor products and in particular on the Grassmann bundle

$$\Lambda^kT^*M^{\mathbb{C}} = \sum_{i+j=k} \Lambda^{(i,j)}T^*M^{\mathbb{C}}.$$

As we have just seen, the existence of J as in Definition 1.6 induces restrictions on M and, for instance, one can, according to the previous discussion, easily prove that an almost complex manifold (M, J) is even-dimensional and orientable. However, these conditions are not sufficient to guarantee the existence of J. Indeed, C. Ehreshmann and H. Hopf (see [154] page 217) have shown that S^4 cannot be given an almost complex structure J.

Definition 1.7. An almost Hermitian manifold $(M, \langle\, , \rangle, J)$ is an almost complex manifold (M, J) with a Riemannian metric $\langle\, , \rangle$ with respect to which J is an isometry, that is, for every $p \in M$ and every $v, w \in T_pM$,

$$\langle J_pv, J_pw\rangle = \langle v, w\rangle\,.$$

In what follows, we extend $\langle\, , \rangle$ complex-bilinearly to $T_pM^{\mathbb{C}}$.

Definition 1.8. The Kähler form of an almost Hermitian manifold $(M, \langle\, , \rangle, J)$ is the $(1, 1)$-form defined by

$$\mathcal{K}\,(X, Y) = \langle X, JY\rangle$$

for each $X, Y \in TM^{\mathbb{C}}$.

Note that $d\mathcal{K} \in \Lambda^3T^*M^{\mathbb{C}}$ can be split into types according to the decomposition in (1.10).

Definition 1.9. An almost Hermitian manifold $(M, \langle\, , \rangle, J)$ is said to be $(1, 2)$-symplectic if

$$d\mathcal{K}^{(1,2)} = 0.$$

Similarly, if

$$d\mathcal{K} = 0$$

or

$$\delta\mathcal{K} = 0$$

where $\delta = - * d*$ is the co-differential acting on 2-forms (see Appendix B), the almost Hermitian manifold is said to be symplectic and co-symplectic, respectively.

Definition 1.10. Let $(M, \langle\,,\rangle\,, J)$ be a (symplectic) almost Hermitian manifold. If the almost complex structure J is induced by a complex structure on M, that is, J is the multiplication by $\imath$ in the charts of a holomorphic atlas, then $(M, \langle\,,\rangle\,, J)$ is called a (Kähler) Hermitian manifold.

Note that there are manifolds which cannot be given a Kählerian structure, for instance the Hopf and Calabi-Eckmann manifolds; see [35] page 69.

Given an almost complex manifold (M, J) the *Nijenhuis tensor* N is the tensor field of type $(1, 2)$ given by

$$N(X, Y) = 2\{[JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]\}$$

for each vector field $X, Y \in \Gamma(TM)$, and where $[\,,\,]$ denotes the Lie bracket.

By the Newlander-Nirenberg theorem, [118], an almost complex structure J is induced by a complex structure if and only if the Nijenhuis tensor vanishes identically.

At the cotangent bundle level, this is expressed by

$$d\omega = 0 \mod (1,0)\text{-forms}$$

for each form ω of type $(1, 0)$. In other words the ideal generated by the $(1, 0)$-forms is a differential ideal. Note that if $\dim_{\mathbb{R}} M = 2$ this is always true (the result is due to Korn and Lichtenstein). In a way similar to that of the definition of the Kähler form, we introduce the *Ricci form* $\mathcal{R}$, that is, for every $X, Y \in TM^{\mathbb{C}}$,

$$\mathcal{R}(X, Y) = \mathrm{Ric}(JX, Y).$$

Clearly, $\mathcal{R}$ is a $(1, 1)$ form and the Kähler manifold $(M, \langle\,,\rangle\,, J_M)$ is said to be *Kähler–Einstein* in case

$$\mathcal{R} = -\frac{\imath}{4m} S(x)\,\mathcal{K}$$

with $S(x)$ the scalar curvature.

Let $f : (M, \langle\,,\rangle\,, J_M) \to (N, (\,,)\,, J_N)$ be a smooth map between almost Hermitian manifolds. Then, df can be linearly extended to the complexified differential $df^{\mathbb{C}} : TM^{\mathbb{C}} \to TN^{\mathbb{C}}$. According to the decomposition

$$TN^{\mathbb{C}} = TN^{(1,0)} \oplus TN^{(0,1)}$$

we can write

$$df^{\mathbb{C}} = df^{(1,0)} + df^{(0,1)}.$$

Definition 1.11. A map $f : (M, \langle\,,\rangle\,, J_M) \to (N, (\,,)\,, J_N)$ between almost Hermitian manifolds is holomorphic if and only if

$$J_N \circ df = df \circ J_M.$$

This is immediately seen to be equivalent to the fact that $df^{\mathbb{C}}$ carries $(1,0)$ vectors into $(1,0)$ vectors or the pull-back of $(1,0)$ forms, under the complex linear extension $\left(f^{\mathbb{C}}\right)^*$, are $(1,0)$ forms or, finally, to the fact that $df^{(0,1)} = 0$.

On the other hand, f is said to be anti-holomorphic if

$$J_N \circ df = -df \circ J_M.$$

The basic relation between (anti-)holomorphic maps and harmonic maps is given by the following local result due to A. Lichnerowicz, [108].

Proposition 1.12. *Let* $(M, \langle\,,\rangle, J_M)$ *and* $(N, (\,,), J_N)$ *be almost Hermitian manifolds. If* M *is co-symplectic and* N *is* $(1,2)$*-symplectic, then any (anti-)holomorphic map* $f : M \to N$ *is harmonic.*

Note that, if M is symplectic, then it is also co-symplectic. We should also remark that some condition on M is necessary for a (anti-)holomorphic map to be harmonic, as an example of A. Grey shows. See [48], page 58.

We now consider the case where $(M, \langle\,,\rangle, J_M)$ is an almost Hermitian manifold and $(N, (\,,))$ is Riemannian. Given a map $f : M \to N$ we can split its generalized second fundamental tensor Ddf according to types in $T^*M^{\mathbb{C}} \otimes T^*M^{\mathbb{C}} \otimes f^{-1}TN$. We have

$$Ddf^{\mathbb{C}} = Ddf^{(2,0)} + Ddf^{(1,1)} + Ddf^{(0,2)}$$

where $Ddf^{\mathbb{C}}$ is the complex linear extension of Ddf.

Definition 1.13. The map $f : (M, \langle\,,\rangle, J_M) \to (N, (\,,))$ is said to be pluriharmonic, or $(1,1)$-geodesic, if $Ddf^{(1,1)} = 0$.

When $N = \mathbb{R}$, then $Ddf^{(1,1)}$ is a Hermitian form referred to as the Levi form of f.

Definition 1.14. We say that the function $f : (M, \langle\,,\rangle, J_M) \to \mathbb{R}$ is plurisubharmonic if all eigenvalues of its Levi form are non-negative.

Note that any pluriharmonic map is harmonic and, if the almost Hermitian manifolds $(M, \langle\,,\rangle, J_M)$ and $(N, (\,,), J_N)$ are also $(1,2)$-symplectic, then any (anti-)holomorphic map $f : M \to N$ is pluriharmonic.

Thus, the notion of pluriharmonic map lies between those of harmonic and (anti-)holomorphic maps.

In case $(M, \langle\,,\rangle, J_M)$ is almost Hermitian and $(1,2)$-symplectic, and $(N, (\,,))$ is Riemannian, J. Rawnsley, [136], has given the following characterization.

Theorem 1.15. *A map* $f : (M, \langle\,,\rangle, J_M) \to (N, (\,,))$ *is pluriharmonic if and only if its restriction to every complex curve in* M *is harmonic.*

Note that, from this it follows that if $(M, \langle\,,\rangle, J_M)$ is Kähler, then the notion of pluriharmonic map does not depend on the choice of the Kähler metric $\langle\,,\rangle$ on M.

We also note that, if $(M, \langle\, , \rangle, J_M)$ and $(N, (\, ,), J_N)$ are Kähler and $f : M \to N$ is an isometry, then we can express holomorphicity of f via the system

$$\begin{cases} \mathrm{II}\,(X,Y) + \mathrm{II}\,(J_M X, J_M Y) = 0, \\ \mathrm{II}\,(X,Y) + J_N \mathrm{II}\,(X, J_M Y) = 0 \end{cases}$$

for all X, Y vector fields on M, where we have used the more familiar notation II for Ddf in the isometric case. Clearly, the first equation is nothing but the definition of a pluriharmonic map.

The notion of a pluriharmonic map has appeared in the literature in the context of the work of Y.T. Siu, [152], who used it as a bridge from harmonicity to (anti-)holomorphicity in the analysis of the strong rigidity of compact Kähler manifolds. Since then, it has been used in a variety of geometrical problems and it will be used below with the aim of providing extra geometric information.

1.3 Hermitian bundles

Later on we shall also be interested in *vector bundles* of rank q on a base manifold M. This means that we have a map

$$\pi : E \to M$$

such that the following conditions are satisfied:

(i) for each $x \in M$, $\pi^{-1}(x)$ is a real (or complex) vector space of dimension q.

(ii) E is locally a product, that is, for each $x \in M$, there exists an open neighborhood U of x and a bijection

$$\varphi_U : U \times V \to \pi^{-1}(U)$$

with V any fixed real (or complex) vector space of dimension q satisfying the condition

$$\pi \circ \varphi_U(x, v) = x,$$

for each $v \in V$.

(iii) For any two of the above neighborhoods U_1, U_2 such that $U_1 \cap U_2 \neq \emptyset$, there is a map

$$g_{U_1 U_2} : U_1 \cap U_2 \to Gl_q(\mathbb{R}) \;\; (\text{or } Gl_q(\mathbb{C}))$$

such that, for $x \in U_1 \cap U_2$, and for each $v, w \in V$,

$$\varphi_{U_1}(x, v) = \varphi_{U_2}(x, w)$$

if and only if

$$v = g_{U_1 U_2}(x)\, w.$$

Clearly, E can be given a (unique) topology and differentiable structure such that each $\left(\pi^{-1}(U), \varphi_U^{-1}\right)$ of (ii) is a local chart. The functions $g_{U_1U_2}$ are called *transition functions* of the bundle and they satisfy

$$\begin{aligned} g_{UU}(x) &= id \in Gl_q(\mathbb{R}), && \text{for each } x \in U, \\ g_{U_1U_2}g_{U_2U_1} &= id \in Gl_q(\mathbb{R}), && \text{for each } x \in U_1 \cap U_2, \\ g_{U_1U_2}g_{U_2U_3}g_{U_3U_1} &= id \in Gl_q(\mathbb{R}), && \text{for each } x \in U_1 \cap U_2 \cap U_3. \end{aligned}$$

It is well known that the transition functions relative to a covering of M completely determine the bundle.

A *section* of $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = id_M$. The set $\Gamma(E)$ of smooth sections of E is a vector space over $\mathbb{R}$ (or $\mathbb{C}$)

A *connection on* E is a map

$$D : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

such that the following conditions are satisfied for each $s, t \in \Gamma(E)$, and for each $f \in C^\infty(M)$ (f either real- or complex-valued):

$$\begin{aligned} &(D_1) \quad D(s+t) = Ds + Dt, \\ &(D_2) \quad D(fs) = fDs + df \otimes s. \end{aligned}$$

Letting $X \in \Gamma(TM)$, $D_X s$ is the derivative of s in the direction of X. Note that $D_X s \in \Gamma(E)$.

It does make sense to define the *curvature transformation*

$$\tilde{K}(X,Y) : \Gamma(E) \to \Gamma(E)$$

where $X, Y \in \Gamma(TM)$ are any two vector fields of M, by setting

$$\tilde{K}(X,Y)s = D_X D_Y s - D_Y D_X s - D_{[X,Y]}s.$$

A *Riemannian vector bundle* is a smooth vector bundle with a *fibre metric* h and a *compatible connection* D, that is, if s and t are sections of $\pi : E \to M$, then, for each vector field $X \in \Gamma(TM)$,

$$Xh(s,t) = h(D_X s, t) + h(s, D_X t).$$

We will be mainly concerned with *Hermitian bundles*, that is, E is a Hermitian manifold with a connection, the Hermitian connection, which is compatible with the metric and uniquely determined by the next requirement (see, [35]).

Let $q = \dim_{\mathbb{R}} \pi^{-1}(x) = 2p$ be the real dimension of the fibres and let $\{e_a\}$, $1 \leq a, b, \ldots \leq p$, be a unitary $(1,0)$-type local frame of sections of E. Thus, indicating with $\{\mu^a\}$ the dual $(1,0)$ forms, we have

$$h = \sum_a \mu^a \otimes \bar{\mu}^a.$$

Then, the *Hermitian connection* on E is the unique connection whose connection forms μ^a_b are determined by the requirements

$$\begin{aligned} &\text{(i)}\ \ \mu^a_b + \bar{\mu}^b_a = 0, \\ &\text{(ii)}\ d\mu^a = -\mu^a_b \wedge \mu^b + \zeta^a, \end{aligned} \tag{1.11}$$

where the ζ^a are forms of $(2,0)$-type, and $^{-}$ denotes complex conjugation. The *curvature forms* M^a_b are then defined by the second structure equations

$$d\mu^a_b = -\mu^a_c \wedge \mu^c_b + M^a_b,$$

which are of type $(1,1)$ and satisfy

$$M^a_b + \bar{M}^b_a = 0.$$

Having set

$$M^a_b = A^a_{bc\bar{d}} \mu^c \wedge \bar{\mu}^d,$$

the metric of the bundle is said to be *Hermitian–Einstein* if

$$\sum_c A^a_{bc\bar{c}} = \lambda \delta^a_b$$

for some constant $\lambda \in \mathbb{C}$. Note that the matrix

$$\left(\sum_c A^a_{bc\bar{c}} \right)_{a,b}$$

is called the *mean curvature* and

$$\mathrm{scal}_h(x) = \sum_{a,c} A^a_{ac\bar{c}}$$

(in a unitary frame) is called the *scalar curvature* of the Hermitian bundle $\pi : E \to M$.

1.4 Complex geometry via moving frames

In what follows we shall always deal with the case where $(M, \langle\, , \rangle, J_M)$ is Kähler, while $(N, (\, ,))$ or $(N, (\, ,), J_N)$, the target manifolds of maps, will be Riemannian or Kählerian. Later on we shall also consider the case where $(N, (,) , J_N)$ is Hermitian. The situation, from our point of view, will be very similar to the Kähler case so, since in the Hermitian case the formalism is definitively heavier, we will not bother to provide details in derivation of the appropriate differential inequalities needed in some proofs of theorems in later chapters. We formalize the Kähler structure with a particular emphasis on its Riemannian counterpart and to do so we will use the method of moving frame. Thus, let $(M, \langle , \rangle , J_M)$ be a Kähler manifold with $s = \dim_{\mathbb{C}} M$ so that $m = 2s = \dim_{\mathbb{R}} M$. We fix the index convention $1 \le i, j, k \ldots \le s$. The Kähler structure of M is naturally described by a unitary coframe $\{\varphi^j\}$ of $(1, 0)$-type, 1-forms giving the metric

$$\langle , \rangle = \sum_j \varphi^j \otimes \bar{\varphi}^j$$

and the corresponding Kähler connection forms $\{\varphi^i_j\}$ characterized by the property

$$\varphi^i_j + \bar{\varphi}^j_i = 0$$

and by the structure equations

$$d\varphi^j = -\varphi^j_k \wedge \varphi^k. \tag{1.12}$$

Note that, comparing with (1.11), we are now requiring that the $(2, 0)$-forms ζ^a are identically zero. This can be seen to be equivalent to $d\mathcal{K} = 0$, i.e., to the condition that the complex manifold is Kähler (see Definition 1.9).

The Kähler curvature forms $\left\{\Phi^j_k\right\}$ are determined by the second structure equations

$$d\varphi^j_k = -\varphi^j_i \wedge \varphi^i_k + \Phi^j_k \tag{1.13}$$

and satisfy the symmetry relations

$$\Phi^j_k + \bar{\Phi}^k_j = 0. \tag{1.14}$$

The coefficients H^i_{jkt} of the Hermitian curvature tensor are determined by

$$\Phi^i_j = H^i_{jkt}\varphi^k \wedge \bar{\varphi}^t \tag{1.15}$$

and condition (1.14) becomes equivalent to

$$H^i_{jkt} = \overline{H^j_{itk}}.$$

Differentiating (1.12) we obtain the first complex Bianchi identities, that is,

$$\Phi^j_k \wedge \varphi^k = 0$$

while differentiating (1.13) we obtain the second complex Bianchi identities which we write in the form

$$d\Phi^j_i + \Phi^k_i \wedge \varphi^j_k - \varphi^k_i \wedge \Phi^j_k = 0.$$

We also recall that the Kähler form $\mathcal{K}$ and the Ricci form $\mathcal{R}$ are respectively given by

$$\mathcal{K} = \frac{\imath}{2} \sum_j \varphi^j \wedge \bar{\varphi}^j,$$
$$\mathcal{R} = \frac{1}{2} \sum_i H^i_{ikt} \varphi^k \wedge \bar{\varphi}^t,$$

so that $(M, \langle\, , \rangle\, , J_M)$ is Kähler–Einstein if and only if

$$\sum_i H^i_{ikt} = \frac{s}{8m} \delta_{kt}$$

with s the scalar curvature. In order to detect the underlying Riemannian structure we set

$$\varphi^j = \theta^j + \imath\theta^{s+j}, \tag{1.16}$$
$$\varphi^j_k = \theta^j_k + \imath\theta^{s+j}_k, \tag{1.17}$$
$$\theta^j_k = \theta^{s+j}_{s+k}, \qquad \theta^j_{s+k} = -\theta^{s+j}_k. \tag{1.18}$$

Then, the θ^j, θ^{s+j} give an orthonormal coframe for the metric $\langle\, , \rangle$ whose corresponding Levi–Civita connection forms are determined by (1.17), (1.18) and the usual skew symmetry conditions

$$\theta^a_b + \theta^b_a = 0,$$

where, from now on, we shall adhere also to the further index convention $1 \leq a, b, \ldots \leq m$. Analogously, setting

$$\Phi^k_j = \Theta^k_j + \imath\Theta^{s+k}_j, \tag{1.19}$$
$$\Theta^k_j = \Theta^{s+k}_{s+j}, \qquad \Theta^j_{s+k} = -\Theta^{s+j}_k, \tag{1.20}$$
$$0 = \Theta^a_b + \Theta^b_a, \tag{1.21}$$

the Θ^a_b's defined in (1.19), (1.20), (1.21) coincide with the corresponding curvature forms. Thus, letting R^a_{bcd} be the coefficients of the Riemannian curvature tensor (obeying the usual symmetries), for which

$$\Theta^a_b = \frac{1}{2} R^a_{bcd} \theta^c \wedge \theta^d \tag{1.22}$$

form (1.20), we obtain the Kähler symmetry relations

$$R^k_{jab} = R^{s+k}_{s+j\ ab}, \qquad R^j_{s+k\ ab} = -R^{s+j}_{kab}. \tag{1.23}$$

We use (1.15) and (1.19), (1.22) to relate Hermitian and Riemannian curvatures. We obtain

$$H^i_{jkt} = \frac{1}{2}\left(R^i_{jkt} + R^{s+i}_{j\ k+s\ t}\right) + \frac{\imath}{2}\left(R^i_{jk\ s+t} + R^{s+i}_{jkt}\right). \tag{1.24}$$

Extending $\mathbb{C}$-linearly the ((4, 0)-version of the) Riemannian curvature tensor we obtain

$$R_{abcd}\theta^a \otimes \theta^b \otimes \theta^c \otimes \theta^d = R_{i\bar{j}k\bar{l}}\varphi^i \otimes \bar{\varphi}^j \otimes \varphi^k \otimes \bar{\varphi}^l + R_{i\bar{j}\bar{k}l}\varphi^i \otimes \bar{\varphi}^j \otimes \bar{\varphi}^k \otimes \varphi^l$$
$$+ R_{\bar{i}j\bar{k}l}\overline{\varphi}^i \otimes \varphi^j \otimes \overline{\varphi}^k \otimes \varphi^l + R_{\bar{i}jk\bar{l}}\overline{\varphi}^i \otimes \varphi^j \otimes \varphi^k \otimes \overline{\varphi}^l$$

where

$$R_{\bar{i}j\bar{k}l} = \overline{R_{i\bar{j}k\bar{l}}}, \qquad R_{\bar{i}jk\bar{l}} = \overline{R_{i\bar{j}\bar{k}l}}, \tag{1.25}$$
$$R_{i\bar{j}\bar{k}l} = -R_{i\bar{j}l\bar{k}} = -R_{\bar{j}i\bar{k}l}, \qquad R_{i\bar{j}\bar{k}l} = R_{\bar{k}li\bar{j}}, \tag{1.26}$$

the remaining coefficients, for instance $R_{\bar{i}\bar{j}kl}$, being null. From (1.25), (1.26) and (1.24) we deduce

$$H^i_{jkl} = R_{i\bar{j}\bar{k}l}. \tag{1.27}$$

Recalling the first (Riemannian) Bianchi identities

$$R^a_{bcd} + R^a_{cdb} + R^a_{dbc} = 0,$$

with the aid of (1.23) we obtain

$$\sum_k R^{s+i}_{j\ s+k\ k} = \mathrm{Ric}_{s+i\ s+j} = \mathrm{Ric}_{ij}.$$

Hence, tracing (1.24) twice we obtain that the scalar curvature s is given by

$$s = 4\sum_{k,i} H^i_{ikk}.$$

Furthermore

$$\mathrm{Ric}_{s+i\ j} = -\mathrm{Ric}_{i\ s+j} = -\mathrm{Ric}_{s+j\ i} = -\mathrm{Ric}_{j\ s+i} \tag{1.28}$$

in particular, for each fixed $i = 1, \ldots, s$,

$$\mathrm{Ric}_{s+i\ i} = 0.$$

Finally, the "Ricci curvature" of the Kähler manifold has components given by the Hermitian matrix

$$R_{i\bar{j}} = \sum_k H^i_{jkk} = \frac{1}{2}\mathrm{Ric}_{ij} + \frac{\imath}{2}\mathrm{Ric}_{s+i\ j}. \tag{1.29}$$

From (1.27) and (1.28) we deduce

$$\mathrm{Ric}_{s+i\ j} = \imath\sum_k \left(R_{\bar{i}jk\bar{k}} - R_{i\bar{j}\bar{k}k}\right),$$
$$\mathrm{Ric}_{ij} = \sum_k \left(R_{i\bar{j}\bar{k}k} + R_{\bar{i}jk\bar{k}}\right).$$

In particular

$$\sum_k H^i_{jkk} = R_{i\bar{j}\,\bar{k}\,k} = R_{i\bar{j}},$$

and the Ricci form can be expressed as

$$\mathcal{R} = \frac{1}{2}\sum_i R_{i\bar{j}}\varphi^k \wedge \overline{\varphi}^l.$$

Note that an orthogonal transformation U_z of $T_zM^{\mathbb{C}}$ is unitary if and only if it commutes with J_z (see, e.g., [88], p. 116), and we may therefore diagonalize the Hermitian matrix $R_{k\bar{j}}$ with a $(1,0)$-basis of the form $E_k = e_k - \imath J_z e_k = e_k - \imath e_{k+s}$, where $\{e_k,\, e_{k+s}\}$ is the orthonormal basis of T_zM dual to $\{\theta^i, \theta^{s+i}\}$. If λ_k are the corresponding eigenvalues of $R_{k\bar{j}}$, then

$$R_{k\bar{j}} = \lambda_k \delta_{kj} = \frac{1}{2}\Big(\mathrm{Ric}_{kj} + \imath \mathrm{Ric}_{k+sj}\Big)$$

which implies that

$$\mathrm{Ric}_{kj} = 2\lambda_k\delta_{kj} \;\text{ and }\; \mathrm{Ric}_{k+sj} = 0 \quad \forall k, j.$$

Further, since $\mathrm{Ric}_{k+sj+s} = \mathrm{Ric}_{kj} = 2\lambda_k\delta_{kj}$, we conclude that $2\lambda_k$ is an eigenvalue of Ric. This shows that, if

$$2R_{i\,\bar{j}}u^i\bar{u}^j \geq -\rho|u|^2$$

holds for every $u \in \mathbb{C}^m$, then inequality

$$\mathrm{Ric}_{i\,j}v^iv^j \geq -\rho|v|^2$$

holds for every $v \in \mathbb{R}^m$. Since the reverse implication is obviously true, we conclude that two conditions are in fact equivalent.

Let $\{e_a\}$ be the dual frame to $\{\theta^a\}$. For each $i, k = 1, \dots, s$, we consider the holomorphic 2-planes Π and $\widehat{\Pi}$ spanned by e_i,$Je_i = e_{s+i}$, and e_k, $Je_k = e_{s+k}$, respectively. Then the holomorphic bisectional curvature of Π and $\widehat{\Pi}$ is defined by

$$H^i_{ikk} = \frac{1}{4}R_{i\;s+i\;k\;s+k},$$

where, in this case, there is no summation over repeated indices. In particular, if $\Pi = \widehat{\Pi}$ we obtain the holomorphic sectional curvature of the 2-plane Π, namely,

$$H^i_{iii} = \frac{1}{4}R_{i\;s+i\;i\;s+i} = \frac{1}{4}\mathrm{Sect}\,(\Pi)$$

where, as above, there is no summation over repeated indices, and where Sect (Π) is the (Riemannian) sectional curvature of Π.

We say that the holomorphic bisectional curvature of M is bounded above by a function $k\,(z)$ if, for all $(1,0)$ vectors $\zeta = \xi^kE_k$, $\eta = \eta^jE_j$, at z, we have

$$\frac{1}{2}\frac{H^i_{jkl}\xi^i\overline{\xi}^j\eta^k\overline{\eta}^l}{\sum \xi^k\overline{\xi}^k \sum \eta^k\overline{\eta}^k} \leq k\,(z)\,.$$