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Simplicial Homotopy Theory

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PREFACE

The origin of simplicial homotopy theory coincides with the beginning of algebraic topology almost a century ago. The thread of ideas started with the work of Poincaré and continued to the middle part of the 20th century in the form of combinatorial topology. The modern period began with the introduction of the notion of complete semi-simplicial complex, or simplicial set, by Eilenberg-Zilber in 1950, and evolved into a full blown homotopy theory in the work of Kan, beginning in the 1950s, and later Quillen in the 1960s.

The theory has always been one of simplices and their incidence relations, along with methods for constructing maps and homotopies of maps within these constraints. As such, the methods and ideas are algebraic and combinatorial and, despite the deep connection with the homotopy theory of topological spaces, exist completely outside any topological context. This point of view was effectively introduced by Kan, and later encoded by Quillen in the notion of a closed model category. Simplicial homotopy theory, and more generally the homotopy theories associated to closed model categories, can then be interpreted as a purely algebraic enterprise, which has had substantial applications throughout homological algebra, algebraic geometry, number theory and algebraic K-theory. The point is that homotopy is more than the standard variational principle from topology and analysis: homotopy theories are everywhere, along with functorial methods of relating them.

This book is, however, not quite so cosmological in scope. The theory has broad applications in many areas, but it has always been quite a sharp tool within ordinary homotopy theory — it is one of the fundamental sources of positive, qualitative and structural theorems in algebraic topology. We have concentrated on giving a modern account of the basic theory here, in a form that could serve as a model for corresponding results in other areas.

This book is intended to fill an obvious and expanding gap in the literature. The last major expository pieces in this area, namely [33], [67], [61] and [18], are all more than twenty-five years old. Furthermore, none of them take into account Quillen's ideas about closed model structures, which are now part of the foundations of the subject.

We have attempted to present an account that is as linear as possible and inclusive within reason. We begin in Chapter I with elementary definitions and examples of simplicial sets and the simplicial set category **S**, classifying objects, Kan complexes and fibrations, and then proceed quickly through much of the classical theory to proofs of the fundamental organizing theorems of the subject which appear in Section 11. These theorems assert that the category of simplicial sets satisfies Quillen's axioms for a closed model category, and that the associated homotopy category is equivalent to that arising from topological spaces. They are delicate but central results, and are the basis for all that follows.

Chapter I contains the definition of a closed model category. The foundations of abstract homotopy theory, as given by Quillen, start to appear in the first section of Chapter II. The "simplicial model structure" that most of the closed model structures appearing in nature exhibit is discussed in Sections 2–7. A simplicial model structure is an enrichment of the underlying category to simplicial sets which interacts with the closed model structure, like function spaces do for simplicial sets; the category of simplicial sets with function spaces is a standard example. Simplicial model categories have a singular technical advantage which is used repeatedly, in that weak equivalences can be detected in the associated homotopy category (Section 4). There is a detection calculus for simplicial model structures which leads to homotopy theories for various algebraic and diagram theoretic settings: this is given in Sections 5–7, and includes a discussion of cofibrantly generated closed model categories in Section 6 — it may be heavy going for the novice, but homotopy theories of diagrams almost characterize work in this area over the past ten years, and are deeply implicated in much current research. The chapter closes on a much more elementary note with a description of Quillen's non-abelian derived functor theory in Section 8, and a description of proper closed model categories, homotopy cartesian diagrams and gluing and cogluing lemmas in Section 9. All subsequent chapters depend on Chapters I and II.

Chapter III is a further repository of things that are used later, although perhaps not quite so pervasively. The fundamental groupoid is defined in Chapter I and then revisited here in Section III.1. Various equivalent formulations are presented, and the resulting theory is powerful enough to show, for example, that the fundamental groupoid of the classifying space of a small category is equivalent to the free groupoid on the category, and give a quick proof of the Van Kampen theorem. The closed model structure for simplicial abelian groups and the Dold-Kan correspondence relating simplicial abelian groups to chain complexes (ie. they're effectively the same thing) are the subject of Section 2. These ideas are the basis of most applications of simplicial homotopy theory and of closed model categories in homological algebra. Section 3 contains a proof of the Hurewicz theorem: Moore-Postnikov towers are introduced here in a self-contained way, and then treated more formally in Chapter VII. Kan's Ex^{∞}-functor is a natural, combinatorial way of replacing a simplicial set up to weak equivalence by a Kan complex: we give updated proofs of its main properties in Section 4, involving some of the ideas from Section 1. The last section presents the Kan suspension, which appears later in Chapter V in connection with the loop group construction.

Chapter IV discusses the homotopy theory, or more properly homotopy theories, for bisimplicial sets and bisimplicial abelian groups, with major applications. Basic examples and constructions, including homotopy colimits and the diagonal complex, appear in the first section. Bisimplicial abelian groups, the subject of Section 2, are effectively bicomplexes, and hence have canonical associated spectral sequences. One of the central technical results is the generalized Eilenberg-Zilber theorem, which asserts that the diagonal and total complexes of a bisimplicial abelian group are chain homotopy equivalent. Three different closed model structures for bisimplicial sets, all of which talk about the same homotopy theory, are discussed in Section 3. They are all important, and in fact used simultaneously in the proof of the Bousfield-Friedlander theorem in Section 4, which gives a much used technical criterion for detecting fibre sequences arising from maps of bisimplicial sets. There is a small technical innovation in this proof, in that the so-called π_*-Kan condition is formulated in terms of certain fibred group objects being Kan fibrations. The chapter closes in Section 4 with proofs of Quillen's "Theorem B" and the group completion theorem. These results are detection principles for fibre sequences and homology fibre sequences arising from homotopy colimits, and are fundamental for algebraic K-theory and stable homotopy theory.

From the beginning, we take the point of view that simplicial sets are usually best viewed as set-valued contravariant functors defined on a category $\mathbf{\Delta}$ of ordinal numbers. This immediately leads, for example, to an easily manipulated notion of simplicial objects in a category $\mathcal{C}$: they're just functors $\mathbf{\Delta}^{op} \to \mathcal{C}$, so that morphisms between them become natural transformations, and so on. Chapter II contains a detailed treatment of the question of when the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has a simplicial model structure.

Simplicial groups is one such category, and is the subject of Chapter V. We establish, in Sections 5 and 6, the classical equivalence of homotopy theories between simplicial groups and simplicial sets having one vertex, from a modern perspective. The method can the be souped up to give the Dwyer-Kan equivalence between the homotopy theories of simplicial groupoids and simplicial sets in Section 7. The techniques involve a new description of principal G-fibrations, for simplicial groups G, as cofibrant objects in a closed model structure on the category of G-spaces, or simplicial sets with G-action (Section 2). Then the classifying space for G is the quotient by the G-action of any cofibrant model of a point in the category of G-spaces (Section 3); the classical $\overline{W}G$ construction is an example, but the proof is a bit interesting. We give a new treatment of $\overline{W}G$ as a simplicial object of universal cocycles in Section 4; one advantage of this method is that there is a completely analogous construction for simplicial groupoids, which is used for the results of Section 7. Our approach also depends on a specific closed model structure for simplicial sets with one vertex, which is given in Section 6. That same section contains a definition and proof of the main properties of the Milnor FK-construction, which is a functor taking values in simplicial groups that gives a model for loops suspension $\Omega\Sigma X$ of a given space X.

The first section of Chapter V contains a discussion of skeleta in the category of simplicial groups which is later used to show the technical (and necessary) result that the Kan loop group functor outputs cofibrant simplicial groups. Skeleta for simplicial sets first appear in a rather quick and dirty way in Section I.2. Skeleta for more general categories appear in various places: we have skeleta for simplicial groups in Chapter V, skeleta for bisimplicial sets in Section IV.3, and then skeleta for simplicial objects in more general categories later, in Section VII.1. In all cases, skeleta and coskeleta are left and right adjoints of truncation functors.

Chapter VI collects together material on towers of fibrations, nilpotent spaces, and the homotopy spectral sequence for a tower of fibrations. The first section describes a simplicial model structure for towers, which is used in Section 3 as a context for a formal discussion of Postnikov towers. The Moore-Postnikov tower, in particular, is a tower of fibrations that is functorially associated to a space X; we show, in Sections 4 and 5, that the fibrations appearing in the tower are homotopy pullbacks along maps, or k-invariants, taking values in homotopy colimits of diagrams of Eilenberg-Mac Lane spaces, which diagrams are functors defined on the fundamental groupoid of X. The homotopy pullbacks can be easily refined if the space is nilpotent, as is done in Section 6. The development includes an introduction of the notion of covering system of a connected space X, which is a functor defined on the fundamental groupoid and takes values in spaces homotopy equivalent to the covering space of X. The general homotopy spectral sequence for a tower of fibrations is introduced, warts and all, in Section 2 — it is the basis for the construction of the homotopy spectral sequence for a cosimplicial space that appears later in Chapter VIII.

Chapter VII contains a detailed treatment of the Reedy model structure for the category of simplicial objects in a closed model category. This theory simultaneously generalizes one of the standard model structures for bisimplicial sets that is discussed in Chapter IV, and specializes to the Bousfield-Kan model structures for the category of cosimplicial objects in simplicial sets, aka. cosimplicial spaces. The method of the application to cosimplicial spaces is to show that the category of simplicial objects in the category $\mathbf{S}^{op}$ has a Reedy model structure, along with an adequate notion of skeleta and an appropriate analogue of realization, and then reverse all arrows. There is one tiny wrinkle in this approach, in that one has to show that a cofibration in Reedy's sense coincides with the original definition of cofibration of Bousfield and Kan, but this argument is made, from two points of view, at the end of the chapter.

The standard total complex of a cosimplicial space is dual to the realization in the Reedy theory for simplicial objects in $\mathbf{S}^{op}$, and the standard tower of fibrations tower of fibrations from [14] associated to the total complex is dual to a skeletal filtration. We begin Chapter VIII with these observations, and then give the standard calculation of the E_2 term of the resulting spectral sequence. Homotopy inverse limits and p-completions, with associated spectral sequences, are the basic examples of this theory and its applications, and are the subjects of Sections 2 and 3, respectively. We also show that the homotopy inverse limit is a homotopy derived functor of inverse limit in a very precise sense, by introducing a "pointwise cofibration" closed model structure for small diagrams of spaces having a fixed index category.

The homotopy spectral sequence of a cosimplicial space is well known to be "fringed" in the sense that the objects that appear along the diagonal in total degree 0 are sets rather than groups. Standard homological techniques therefore fail, and there can be substantial difficulty in analyzing the path components of the total space. Bousfield has created an obstruction theory to attack this problem. We give here, in the last section of Chapter VII, a special case of this theory, which deals with the question of when elements in bidegree $(0, 0)$ in the E_2-term lift to path components of the total space. This particular result can be used to give a criterion for maps between mod p cohomology objects in the category of unstable algebras over the Steenrod algebra to lift to maps of p-completions.

Simplicial model structures return with a vengeance in Chapter IX, in the context of homotopy coherence. The point of view that we take is that a homotopy coherent diagram on a category I in simplicial sets is a functor $X : \mathcal{A} \to \mathbf{S}$ which is defined on a category enriched in simplicial sets and preserves the enriched structure, subject to the object $\mathcal{A}$ being a resolution of I in a suitable sense. The main results are due to Dwyer and Kan: there is a simplicial model structure on the category of simplicial functors $\mathbf{S}^{\mathcal{A}}$ (Section 1), and a large class of simplicial functors $f : \mathcal{A} \to \mathcal{B}$ which are weak equivalences induce equivalences of the homotopy categories associated to $\mathbf{S}^{\mathcal{A}}$ and $\mathbf{S}^{\mathcal{B}}$ (Section 2). Among such weak equivalences are resolutions $\mathcal{A} \to I$ — in practice, I is the category of path components of $\mathcal{A}$ and each component of $\mathcal{A}$ is contractible. A realization of a homotopy coherent diagram $X : \mathcal{A} \to \mathbf{S}$ is then nothing but a diagram $Y : I \to \mathbf{S}$ which represents X under the equivalence of homotopy categories. This approach subsumes the standard homotopy coherence phenomena, which are discussed in Section 3. We show how to promote some of these ideas to notions of homotopy coherent diagrams and realizations of same in more general simplicial model categories, including chain complexes and spectra, in the last section.

Frequently, one wants to take a given space and produce a member of a class of spaces for which homology isomorphisms are homotopy equivalences, without perturbing the homology. If the homology theory is mod p homology, the p-completion works in many but not all examples. Bousfield's mod p homology localization technique just works, for all spaces. The original approach to homology localization [8] appeared in the mid 1970's, and has since been incorporated into a more general theory of f-localization. The latter means that one constructs a minimal closed model structure in which a given map f becomes invertible in the homotopy category — in the case of homology localization the map f would be a disjoint union of maps of finite complexes which are homology isomorphisms. The theory of f-localization and the ideas underlying it are broadly applicable, and are still undergoing frequent revision in the literature. We present one of the recent versions of the theory here, in Sections 1–3 of Chapter X. The methods of proof involve little more than aggressive cardinal counts (the cogniscenti will note that there is no mention of regular cardinals): this is where the wide applicability of these ideas comes from — morally, if cardinality counts are available in a model category, then it admits a theory of localization. We describe Bousfield's approach to localization at a functor in Section 4, and then show that it leads to the Bousfield-Friedlander model for the stable category.

There are ten chapters in all; we use Roman numerals to distinguish them. Each chapter is divided into sections, plus an introduction. Results and equations are numbered consecutively within each section. The overall referencing system for the monograph is perhaps best illustrated with an example: Lemma 8.8 lives in Section 8 of Chapter II — it is referred to precisely this way from within Chapter II, and as Lemma II.8.8 from outside. Similarly, the corresponding section is called Section 8 inside Chapter II and Section II.8 from without.

Despite the length of this tome, much important material has been left out: there is not a word about traditional simplicial complexes and the vast modern literature related to them (trees, Tits buildings, Quillen's work on posets); the Waldhausen subdivision is not mentioned; we don't discuss the Hausmann-Husemoller theory of acyclic spaces or Quillen's plus construction; we have avoided all of the subtle aspects of categorical coherence theory, and there is very little about simplicial sheaves and presheaves. All of these topics, however, are readily available in the literature, and we have tried to include a useful bibliography.

This book should be accessible to mathematicians in the second year of graduate school or beyond, and is intended to be of interest to the research worker who wants to apply simplicial techniques, for whatever reason. We believe that it will be a useful introduction both to the theory and the current literature.

That said, this monograph does not have the structure of a traditional text book. We have, for example, declined to assign homework in the form of exercises, preferring instead to liberally sprinkle the text with examples and remarks that are designed to provoke further thought. Everything here depends on the first two chapters; the remaining material often reflects the original nature of the project, which amounted to separately written self contained tracts on defined topics. The book achieved its current more unified state thanks to a drive to achieve consistent notation and referencing, but it remains true that a more experienced reader should be able to read each of the later chapters in isolation, and find an essentially complete story in most cases.

This book had a lengthy and productive gestation period as an object on the Internet. There were many downloads, and many comments from interested readers, and we would like to thank them all. Particular thanks go to Frans Clauwens, who read the entire manuscript very carefully and made numerous technical, typographical, and stylistic comments and suggestions. The printed book differs substantially from the online version, and this is due in no small measure to his efforts.

Work on this monograph was partly supported by grants from the National Science Foundation and the Natural Sciences and Engineering Research Council of Canada, as well as a NATO Collaborative Research grant, and we would like to thank all of these agencies. The writing project involved multiple visits by each author to the other's home institution, and we would jointly like to thank Northwestern University, the University of Washington and the University of Western Ontario for their hospitality and support.

CONTENTS

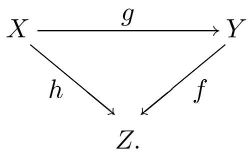
Chapter I Simplicial sets

This chapter introduces the basic elements of the homotopy theory of simplicial sets. Technically, the purpose is twofold: to prove that the category of simplicial sets has a homotopical structure in the sense that it admits the structure of a closed model category (Theorem 11.3), and to show that the resulting homotopy theory is equivalent in a strong sense to the ordinary homotopy theory of topological spaces (Theorem 11.4). Insofar as simplicial sets are algebraically defined, and the corresponding closed model structure is combinatorial in nature, we obtain an algebraic, combinatorial model for standard homotopy theory.

The substance of Theorem 11.3 is that we can find three classes of morphisms within the simplicial set category **S**, called cofibrations, fibrations and weak equivalences, and then demonstrate that the following list of properties is satisfied:

CM1: **S** is closed under all finite limits and colimits.

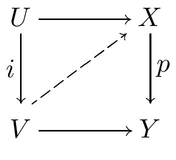
CM2: Suppose that the following diagram commutes in **S**:



If any two of f, g and h are weak equivalences, then so is the third.

CM3: If f is a retract of g in the category of maps of **S**, and g is a weak equivalence, fibration or cofibration, then so is f.

CM4: Suppose that we are given a commutative solid arrow diagram



where i is a cofibration and p is a fibration. Then the dotted arrow exists, making the diagram commute, if either i or p is also a weak equivalence.

CM5: Any map $f : X \to Y$ may be factored:

(a) $f = p \cdot i$ where p is a fibration and i is both a cofibration and a weak equivalence, and

(b) $f = q \cdot j$ where q is a fibration and a weak equivalence, and j is a cofibration.

The fibrations in the simplicial set category are the Kan fibrations, which are defined by a lifting property that is analogous to the notion of Serre fibration. The cofibrations are the monomorphisms, and the weak equivalences are morphisms which induce homotopy equivalences of CW-complexes after passage to topological spaces. We shall begin to investigate the consequences of this list of axioms in subsequent chapters — they are the basis of a great deal of modern homotopy theory.

Theorem 11.3 and Theorem 11.4 are due to Quillen [76], but the development given here is different: the results are proved simultaneously, and their proofs jointly depend fundamentally on Quillen's later result that the realization of a Kan fibration is a Serre fibration [77]. The category of simplicial sets is historically the first full algebraic model for homotopy theory to have been found, but the verification of its closed model structure is still one of the more difficult proofs of abstract homotopy theory. These theorems and their proofs effectively summarize all of the classical homotopy theory of simplicial sets, as developed mostly by Kan in the 1950's. Kan's work was a natural outgrowth of the work of Eilenberg and Mac Lane on singular homology theory, and is part of a thread of ideas that used to be called "combinatorial homotopy theory" and which can be traced back to the work of Poincaré at the beginning of the twentieth century.

We give here, in the proof of the main results and the development leading to them, a comprehensive introduction to the homotopy theory of simplicial sets. Simplicial sets are defined, with examples, in Section 1, the functorial relationship with topological spaces via realization and the singular functor is described in Section 2, and we start to describe the combinatorial homotopical structure (Kan fibrations and Kan complexes) in Section 3. We introduce the Gabriel-Zisman theory of anodyne extensions in Section 4: this is the obstruction-theoretic machine that trivializes many potential difficulties related to the function complexes of Section 5, the notion of simplicial homotopy in Section 6, and the discussion of simplicial homotopy groups for Kan complexes in Section 7. The fundamental groupoid for a Kan complex is introduced in Section 8, by way of proving a major result about composition of simplicial sets maps which induce isomorphisms in homotopy groups (Theorem 8.2). This theorem, along with a lifting property result for maps which are simultaneously Kan fibrations and homotopy groups isomorphisms (Theorem 7.10 — later strengthened in Theorem 11.2), is used to demonstrate in Section 9 (Theorem 9.1) that the collection of Kan complexes and maps between them satisfies the axioms for a category of fibrant objects in the sense of Brown [15]. This is a first axiomatic approximation to the desired closed model structure, and is the platform on which the relation with standard homotopy theory is constructed with the introduction of minimal fibrations in Section 10. The basic ideas there are that every Kan fibration has a "minimal model" (Proposition 10.3 and Lemma 10.4), and the Gabriel-Zisman result that minimal fibrations induce Serre fibrations after realization (Theorem 10.9). It is then a relatively simple matter to show that the realization of a Kan fibration is a Serre fibration (Theorem 10.10).

The main theorems are proved in the final section, but Section 10 is the heart of the matter from a technical point of view once all the definitions and elementary properties have been established. We have not heard of a proof of Theorem 11.3 or Theorem 11.4 that avoids minimal fibrations. The minimality concept is very powerful wherever it appears, but not much has yet been made of it from a formal point of view.

I.1. Basic definitions.

Let $\mathbf{\Delta}$ be the category of finite ordinal numbers, with order-preserving maps between them. More precisely, the objects for $\mathbf{\Delta}$ consist of elements $\mathbf{n}$, $n \geq 0$, where $\mathbf{n}$ is a string of relations

$$0 \to 1 \to 2 \to \cdots \to n$$

(in other words $\mathbf{n}$ is a totally ordered set with $n+1$ elements). A morphism $\theta : \mathbf{m} \to \mathbf{n}$ is an order-preserving set function, or alternatively a functor. We usually commit the abuse of saying that $\mathbf{\Delta}$ is the *ordinal number category*.

A *simplicial set* is a contravariant functor $X : \mathbf{\Delta}^{op} \to \mathbf{Sets}$, where $\mathbf{Sets}$ is the category of sets.

EXAMPLE 1.1. There is a standard covariant functor

$$\underset{\mathbf{n} \mapsto |\Delta^n|}{\mathbf{\Delta} \to \mathbf{Top}}.$$

The topological *standard n-simplex* $|\Delta^n| \subset \mathbb{R}^{n+1}$ is the space

$$|\Delta^n| = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^{n} t_i = 1, t_i \geq 0\},$$

with the subspace topology. The map $\theta_* : |\Delta^n| \to |\Delta^m|$ induced by $\theta : \mathbf{n} \to \mathbf{m}$ is defined by

$$\theta_*(t_0, \dots, t_m) = (s_0, \dots, s_n),$$

where

$$s_i = \begin{cases} 0 & \theta^{-1}(i) = \emptyset \\ \sum_{j \in \theta^{-1}(i)} t_j & \theta^{-1}(i) \neq \emptyset \end{cases}$$

One checks that $\theta \mapsto \theta_*$ is indeed a functor (exercise). Let T be a topological space. The *singular set* $S(T)$ is the simplicial set given by

$$\mathbf{n} \mapsto \hom(|\Delta^n|, T).$$

This is the object that gives the singular homology of the space T.

Among all of the functors $\mathbf{m} \to \mathbf{n}$ appearing in $\mathbf{\Delta}$ there are special ones, namely

$$\begin{aligned} d^i &: \mathbf{n-1} \to \mathbf{n} \qquad 0 \leq i \leq n \qquad \text{(cofaces)} \\ s^j &: \mathbf{n+1} \to \mathbf{n} \qquad 0 \leq j \leq n \qquad \text{(codegeneracies)} \end{aligned}$$

where, by definition,

$$d^i(0 \to 1 \to \cdots \to n-1) = (0 \to 1 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$$

(ie. compose $i-1 \to i \to i+1$, giving a string of arrows of length $n-1$ in $\mathbf{n}$), and

$$s^j(0 \to 1 \to \cdots \to n+1) = (0 \to 1 \to \cdots \to j \xrightarrow{1} j \to \cdots \to n)$$

(insert the identity 1_j in the j^{th} place, giving a string of length $n+1$ in $\mathbf{n}$). It is an exercise to show that these functors satisfy a list of identities as follows, called the *cosimplicial identities*:

$$\begin{cases} d^j d^i = d^i d^{j-1} & \text{if } i < j \\ s^j d^i = d^i s^{j-1} & \text{if } i < j \\ s^j d^j = 1 = s^j d^{j+1} & \\ s^j d^i = d^{i-1} s^j & \text{if } i > j+1 \\ s^j s^i = s^i s^{j+1} & \text{if } i \leq j \end{cases} \tag{1.2}$$

The maps d^j, s^i and these relations can be viewed as a set of generators and relations for $\mathbf{\Delta}$ (see [66]). Thus, in order to define a simplicial set Y, it suffices to write down sets Y_n, $n \geq 0$ (sets of n-simplices) together with maps

$$\begin{aligned} d_i &: Y_n \to Y_{n-1}, \qquad 0 \leq i \leq n \quad \text{(faces)} \\ s_j &: Y_n \to Y_{n+1}, \qquad 0 \leq j \leq n \quad \text{(degeneracies)} \end{aligned}$$

satisfying the *simplicial identities*:

$$\begin{cases} d_i d_j = d_{j-1} d_i & \text{if } i < j \\ d_i s_j = s_{j-1} d_i & \text{if } i < j \\ d_j s_j = 1 = d_{j+1} s_j & \\ d_i s_j = s_j d_{i-1} & \text{if } i > j+1 \\ s_i s_j = s_{j+1} s_i & \text{if } i \leq j \end{cases} \tag{1.3}$$

This is the classical way to write down the data for a simplicial set Y.

From a simplicial set Y, one may construct a simplicial abelian group $\mathbb{Z}Y$ (ie. a contravariant functor $\mathbf{\Delta}^{op} \to \mathbf{Ab}$), with $\mathbb{Z}Y_n$ set equal to the free abelian group on Y_n. The simplicial abelian group $\mathbb{Z}Y$ has associated to it a chain complex, called its *Moore complex* and also written $\mathbb{Z}Y$, with

$$\mathbb{Z}Y_0 \overset{\partial}{\leftarrow} \mathbb{Z}Y_1 \overset{\partial}{\leftarrow} \mathbb{Z}Y_2 \leftarrow \dots \qquad \text{and}$$

$$\partial = \sum_{i=0}^{n} (-1)^i d_i$$

in degree n. Recall that the integral singular homology groups $H_*(X;\mathbb{Z})$ of the space X are defined to be the homology groups of the chain complex $\mathbb{Z}SX$. The homology groups $H_n(Y, A)$ of a simplicial set Y with coefficients in an abelian group A are defined to be the homology groups $H_n(\mathbb{Z}Y \otimes A)$ of the chain complex $\mathbb{Z}Y \otimes A$.

Example 1.4. Suppose that $\mathcal{C}$ is a (small) category. The *classifying space* (or nerve) $B\mathcal{C}$ of $\mathcal{C}$ is the simplicial set with

$$B\mathcal{C}_n = \hom_{\mathbf{cat}}(\mathbf{n}, \mathcal{C}),$$

where $\hom_{\mathbf{cat}}(\mathbf{n}, \mathcal{C})$ denotes the set of functors from $\mathbf{n}$ to $\mathcal{C}$. In other words an n-simplex is a string

$$a_0 \xrightarrow{\alpha_1} a_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} a_n$$

of composeable arrows of length n in $\mathcal{C}$.

We shall see later that there is a topological space $|Y|$ functorially associated to every simplicial set Y, called the realization of Y. The term “classifying space” for the simplicial set $B\mathcal{C}$ is therefore something of an abuse – one really means that $|B\mathcal{C}|$ is the classifying space of $\mathcal{C}$. Ultimately, however, it does not matter; the two constructions are indistinguishable from a homotopy theoretic point of view.

Example 1.5. If G is a group, then G can be identified with a category (or groupoid) with one object $*$ and one morphism $g : * \to *$ for each element g of G, and so the classifying space BG of G is defined. Moreover $|BG|$ is an Eilenberg-Mac Lane space of the form $K(G, 1)$, as the notation suggests; this is now the standard construction.

Example 1.6. Suppose that $\mathcal{A}$ is an exact category, like the category $\mathcal{P}(R)$ of finitely generated projective modules on a ring R (see [79]). Then $\mathcal{A}$ has associated to it a category $Q\mathcal{A}$. The objects of $Q\mathcal{A}$ are those of $\mathcal{A}$. The arrows of $Q\mathcal{A}$ are equivalence classes of diagrams

$$\bullet \twoheadleftarrow \bullet \rightarrowtail \bullet$$

where both arrows are parts of exact sequences of $\mathcal{A}$, and composition is represented by pullback. Then $K_{i-1}(\mathcal{A}) := \pi_i|BQ\mathcal{A}|$ defines the K-groups of $\mathcal{A}$ for $i \geq 1$; in particular $\pi_i|BQ\mathcal{P}(R)| = K_{i-1}(R)$, the i^{th} *algebraic K-group* of the ring R.

EXAMPLE 1.7. The *standard n-simplex, simplicial* Δ^n in the simplicial set category $\mathbf{S}$ is defined by

$$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n}).$$

In other words, Δ^n is the contravariant functor on $\mathbf{\Delta}$ which is represented by $\mathbf{n}$.

A *map* $f : X \to Y$ *of simplicial sets* (or, more simply, a *simplicial map*) is a natural transformation of contravariant set-valued functors defined on $\mathbf{\Delta}$. We shall use $\mathbf{S}$ to denote the resulting category of simplicial sets and simplicial maps.

The Yoneda Lemma implies that simplicial maps $\Delta^n \to Y$ classify n-simplices of Y in the sense that there is a natural bijection

$$\hom_{\mathbf{S}}(\Delta^n, Y) \cong Y_n$$

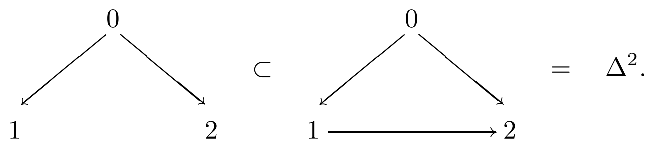
between the set Y_n of n-simplices of Y and the set $\hom_{\mathbf{S}}(\Delta^n, Y)$ of simplicial maps from Δ^n to Y (see [66], or better yet, prove the assertion as an exercise). More precisely, write $\iota_n = 1_{\mathbf{n}} \in \hom_{\mathbf{\Delta}}(\mathbf{n},\mathbf{n})$. Then the bijection is given by associating the simplex $\varphi(\iota_n) \in Y_n$ to each simplicial map $\varphi : \Delta^n \to Y$. This means that each simplex $x \in Y_n$ has associated to it a unique simplicial map $\iota_x : \Delta^n \to Y$ such that $\iota_x(\iota_n) = x$. One often writes $x = \iota_x$, since it's usually convenient to confuse the two.

Δ^n contains subcomplexes $\partial\Delta^n$ (*boundary* of Δ^n) and Λ^n_k, $0 \le k \le n$ (k^{th} *horn*, really the cone centred on the k^{th} vertex). The simplicial set $\partial\Delta^n$ is the smallest subcomplex of Δ^n containing the faces $d_j(\iota_n)$, $0 \le j \le n$ of the standard simplex ι_n. One finds that $\partial\Delta^n$ is specified in j-simplices by

$$\partial\Delta^n_j = \begin{cases} \Delta^n_j & \text{if } 0 \le j \le n-1, \\ \text{iterated degeneracies of elements of } \Delta^n_k, \\ \qquad 0 \le k \le n-1, \quad \text{if } j \ge n. \end{cases}$$

It is a standard convention to write $\partial\Delta^0 = \emptyset$, where $\emptyset$ is the "unique" simplicial set which consists of the empty set in each degree. The object $\emptyset$ is initial for the simplicial set category $\mathbf{S}$.

The k^{th} horn $\Lambda^n_k \subset \Delta^n$ $(n \ge 1)$ is the subcomplex of Δ^n which is generated by all faces $d_j(\iota_n)$ *except* the k^{th} face $d_k(\iota_n)$. One could represent Λ^2_0, for example, by the picture



I.2. Realization.

Let **Top** denote the category of topological spaces. To go further, we have to get serious about the realization functor $|\ | : \mathbf{S} \to \mathbf{Top}$. There is a quick way to construct it which uses the *simplex category* $\mathbf{\Delta} \downarrow X$ of a simplicial set X. The objects of $\mathbf{\Delta} \downarrow X$ are the maps $\sigma : \Delta^n \to X$, or simplices of X. An arrow of $\mathbf{\Delta} \downarrow X$ is a commutative diagram of simplicial maps

$$\begin{array}{ccc} \Delta^n & \xrightarrow{\sigma} & \\ {\scriptstyle\theta}\downarrow & & X \\ \Delta^m & \xrightarrow[\tau]{} & \end{array}$$

Observe that θ is induced by a unique ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$.

Lemma 2.1. *There is an isomorphism*

$$X \cong \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta} \downarrow X}} \Delta^n.$$

Proof: The proof is the observation that any functor $\mathcal{C} \to \mathbf{Sets}$, which is defined on a small category $\mathcal{C}$, is a colimit of representable functors. □

The *realization* $|X|$ of a simplicial set X is defined by the colimit

$$|X| = \varinjlim_{\substack{\Delta^n \to X \\ \text{in } \mathbf{\Delta} \downarrow X}} |\Delta^n|.$$

in the category of topological spaces. The construction $X \mapsto |X|$ is seen to be functorial in simplicial sets X, by using the fact that any simplicial map $f : X \to Y$ induces a functor $f_* : \mathbf{\Delta} \downarrow X \to \mathbf{\Delta} \downarrow Y$ by composition with f.

Proposition 2.2. *The realization functor is left adjoint to the singular functor in the sense that there is an isomorphism*

$$\hom_{\mathbf{Top}}(|X|, Y) \cong \hom_{\mathbf{S}}(X, SY)$$

which is natural in simplicial sets X and topological spaces Y.

Proof: There are isomorphisms

$$\begin{aligned} \hom_{\mathbf{Top}}(|X|, Y) &\cong \varprojlim_{\Delta^n \to X} \hom_{\mathbf{Top}}(|\Delta^n|, Y) \\ &\cong \varprojlim_{\Delta^n \to X} \hom_{\mathbf{S}}(\Delta^n, S(Y)) \\ &\cong \hom_{\mathbf{S}}(X, SY)). \end{aligned}$$
□

Note that $\mathbf{S}$ has all colimits and the realization functor $|\ |$ preserves them, since it has a right adjoint.

Proposition 2.3. *$|X|$ is a CW-complex for each simplicial set X.*

Proof: Define the n^{th} *skeleton* $\operatorname{sk}_n X$ of X be the subcomplex of X which is generated by the simplices of X of degree $\leq n$. Then X is a union

$$X = \bigcup_{n \geq 0} \operatorname{sk}_n X$$

of its skeleta, and there are pushout diagrams

$$\begin{array}{ccc} \bigsqcup_{x \in NX_n} \partial\Delta^n & \longrightarrow & \operatorname{sk}_{n-1} X \\ \big\downarrow & & \big\downarrow \\ \bigsqcup_{x \in NX_n} \Delta^n & \longrightarrow & \operatorname{sk}_n X \end{array}$$

of simplicial sets, where $NX_n \subset X_n$ is the set of *non-degenerate simplices* of degree n. In other words,

$$NX_n = \{x \in X_n | x \text{ not of the form } s_i y \text{ for any } 0 \leq i \leq n-1 \text{ and } y \in X_{n-1}\}.$$

The realization of Δ^n is the space $|\Delta^n|$, since $\mathbf{\Delta} \downarrow \Delta^n$ has terminal object $1 : \Delta^n \to \Delta^n$. Furthermore, one can show that there is a coequalizer

$$\bigsqcup_{0 \leq i < j \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

given by the relations $d^j d^i = d^i d^{j-1}$ if $i < j$ (exercise), and so there is a coequalizer diagram of spaces

$$\bigsqcup_{0 \leq i < j \leq n} |\Delta^{n-2}| \rightrightarrows \bigsqcup_{i=0}^{n} |\Delta^{n-1}| \to |\partial\Delta^n|$$

Thus, the induced map $|\partial\Delta^n| \to |\Delta^n|$ maps $|\partial\Delta^n|$ onto the $(n-1)$-sphere bounding $|\Delta^n|$. It follows that $|X|$ is a filtered colimit of spaces $|\operatorname{sk}_n X|$ where $|\operatorname{sk}_n X|$ is obtained from $|\operatorname{sk}_{n-1} X|$ by attaching n-cells according to the push-out diagram

$$\begin{array}{ccc} \bigsqcup_{x \in NX_n} |\partial\Delta^n| & \longrightarrow & |\operatorname{sk}_{n-1} X| \\ \big\downarrow & & \big\downarrow \\ \bigsqcup_{x \in NX_n} |\Delta^n| & \longrightarrow & |\operatorname{sk}_n X|. \end{array}$$

□

In particular $|X|$ is a *compactly generated Hausdorff space*, and so the realization functor takes values in the category **CGHaus** of all such. We shall interpret $|\ |$ as such a functor. Here is the reason:

Proposition 2.4. *The functor* $|\ | : \mathbf{S} \to \mathbf{CGHaus}$ *preserves finite limits.*

We won't get into the general topology involved in proving this result; a demonstration is given in [33]. Proposition 2.4 avoids the problem that $|X \times Y|$ may not be homeomorphic to $|X| \times |Y|$ in general in the ordinary category of topological spaces, in that it implies that

$$|X \times Y| \cong |X| \times_{Ke} |Y|$$

(Kelley space product = product in **CGHaus**). We lose no homotopical information by working **CGHaus** since, for example, the definition of homotopy groups of a CW-complex does not see the difference between **Top** and **CGHaus**.

I.3. Kan complexes.

Recall the "presentation"

$$\bigsqcup_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n$$

of $\partial\Delta^n$ that was mentioned in the last section. There is a similar presentation for Λ^n_k.

Lemma 3.1. *The "fork" defined by the commutative diagram*

$$\begin{array}{ccccc} \Delta^{n-2} & \xrightarrow{d^{j-1}} & \Delta^{n-1} & & \\ {\scriptstyle in_{i<j}}\downarrow & & \downarrow{\scriptstyle in_i} & \searrow{\scriptstyle d^i} & \\ \bigsqcup_{0\le i<j\le n} \Delta^{n-2} & \rightrightarrows & \bigsqcup_{i\neq k}\Delta^{n-1} & \longrightarrow & \Lambda^n_k \\ {\scriptstyle in_{i<j}}\uparrow & & \uparrow{\scriptstyle in_j} & \nearrow{\scriptstyle d^j} & \\ \Delta^{n-2} & \xrightarrow[d^i]{} & \Delta^{n-1} & & \end{array}$$

is a coequalizer in **S**.

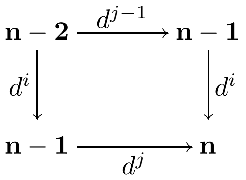
Proof: There is a coequalizer

$$\bigsqcup_{i<j} \Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1} \rightrightarrows \bigsqcup_{\substack{i \neq k \\ 0 \le i \le n}} \Delta^{n-1} \to \Lambda^n_k.$$

But the fibre product $\Delta^{n-1} \times_{\Lambda^n_k} \Delta^{n-1}$ is isomorphic to

$$\Delta^{n-1} \times_{\Delta^n} \Delta^{n-1} \cong \Delta^{n-2}$$

since the diagram

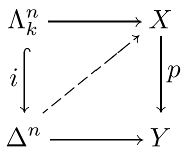


is a pullback in $\mathbf{\Delta}$. In effect, the totally ordered set $\{0 \dots \hat{\imath} \dots \hat{\jmath} \dots n\}$ is the intersection of the subsets $\{0 \dots \hat{\imath} \dots n\}$ and $\{0 \dots \hat{\jmath} \dots n\}$ of $\{0 \dots n\}$, and this poset is isomorphic to $\mathbf{n-2}$. □

The notation $\{0 \dots \hat{\imath} \dots n\}$ means that i isn't there.

COROLLARY 3.2. *The set* $\hom_{\mathbf{S}}(\Lambda^n_k, X)$ *of simplicial set maps from* Λ^n_k *to* X *is in bijective correspondence with the set of* n*-tuples* $(y_0, \dots, \hat{y}_k, \dots, y_n)$ *of* $(n-1)$*-simplices* y_i *of* X *such that* $d_i y_j = d_{j-1} y_i$ *if* $i < j$, *and* $i, j \neq k$.

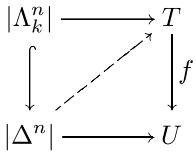
We can now start to describe the internal homotopy theory carried by $\mathbf{S}$. The central definition is that of a fibration of simplicial sets. A map $p : X \to Y$ of simplicial sets is said to be a *fibration* if for every commutative diagram of simplicial set homomorphisms



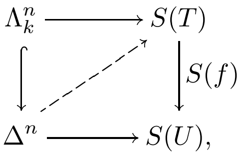
there is a map $\theta : \Delta^n \to X$ (the dotted arrow) making the diagram commute. The map i is the inclusion of the subcomplex Λ^n_k in Δ^n.

This requirement was called the *extension condition* at one time (see [58], [67], for example), and fibrations were (and still are) called *Kan fibrations*. The condition amounts to saying that if $(x_0 \dots \hat{x}_k \dots x_n)$ is an n-tuple of simplices of X such that $d_i x_j = d_{j-1} x_i$ if $i < j$, $i, j \neq k$, and there is an n-simplex y of Y such that $d_i y = p(x_i)$, then there is an n-simplex x of X such that $d_i x = x_i$, $i \neq k$, and such that $p(x) = y$. It is usually better to formulate it in terms of diagrams.

The same language may be used to describe Serre fibrations: a continuous map of spaces $f : T \to U$ is said to be a *Serre fibration* if the dotted arrow exists in each commutative diagram of continuous maps



making it commute. By adjointness (Proposition 2.2), all such diagrams may be identified with diagrams



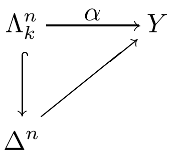
so that $f : T \to U$ is a Serre fibration if and only if $S(f) : S(T) \to S(U)$ is a (Kan) fibration. This is partial motivation for the definition of fibration of simplicial sets. The simplicial set $|\Lambda^n_k|$ is a strong deformation retract of $|\Delta^n|$, so that we've proved

Lemma 3.3. *For each space X, the map $S(X) \to *$ is a fibration.*

The notation $*$ refers to the simplicial set Δ^0, as is standard. It consists of a singleton set in each degree, and is therefore a terminal object in the category of simplicial sets.

A *fibrant* simplicial set (or *Kan complex*) is a simplicial set Y such that the canonical map $Y \to *$ is a fibration. Alternatively, Y is a Kan complex if and only if one of the following equivalent conditions is met:

K1: Every map $\alpha : \Lambda^n_k \to Y$ may be extended to a map defined on Δ^n in the sense that there is a commutative diagram



K2: For each n-tuple of $(n-1)$-simplices $(y_0 \dots \hat{y}_k \dots y_n)$ of Y such that $d_i y_j = d_{j-1} y_i$ if $i < j$, $i, j \neq k$, there is an n-simplex y such that $d_i y = y_i$.

The standard examples of fibrant simplicial sets are singular complexes, as we've seen, as well as classifying spaces BG of groups G, and simplicial groups. A *simplicial group* H is a simplicial object in the category of groups; this means that H is a contravariant functor from $\mathbf{\Delta}$ to the category **Grp** of groups. We generally reserve the symbol e for the identities of the groups H_n, for all $n \geq 0$.

Lemma 3.4 (Moore). *The underlying simplicial set of any simplicial group H is fibrant.*

Proof: Suppose that $(x_0, \dots, x_{k-1}, x_{\ell-1}, x_\ell, \dots, x_n)$, $\ell \geq k+2$, is a family of $(n-1)$-simplices of H which is compatible in the sense that $d_i x_j = d_{j-1} x_i$ for $i < j$ whenever the two sides of the equation are defined. Suppose that there is an n-simplex y of H such that $d_i y = x_i$ for $i \leq k-1$ and $i \geq \ell$. Then the family

$$(e, \dots, e, \overset{\ell-1}{x_{\ell-1} d_{\ell-1}(y^{-1})}, e, \dots, e)$$

is compatible, and $d_i(s_{\ell-2}(x_{\ell-1} d_{\ell-1} y^{-1}) y) = x_i$ for $i \leq k-1$ and $i \geq \ell-1$. This is the inductive step in the proof of the lemma. □

Recall that a *groupoid* is a category in which every morphism is invertible. Categories associated to groups as above are obvious examples, so that the following result specializes to the assertion that classifying spaces of groups are Kan complexes.

Lemma 3.5. *Suppose that G is a groupoid. Then BG is fibrant.*

Proof: If $\mathcal{C}$ is a small category, then its nerve $B\mathcal{C}$ is a 2-coskeleton in the sense that the set of simplicial maps $f : X \to B\mathcal{C}$ is in bijective correspondence with commutative (truncated) diagrams

$$\begin{array}{ccc}
X_2 & \xrightarrow{f_2} & B\mathcal{C}_2 \\
\downarrow\uparrow & & \downarrow\uparrow \\
X_1 & \xrightarrow{f_1} & B\mathcal{C}_1 \\
\downarrow\uparrow & & \downarrow\uparrow \\
X_0 & \xrightarrow{f_0} & B\mathcal{C}_0
\end{array}$$

in which the vertical maps are the relevant simplicial structure maps. It suffices to prove this for $X = \Delta^n$ since X is a colimit of simplices. But any simplicial map $f : \Delta^n \to B\mathcal{C}$ can be identified with a functor $f : \mathbf{n} \to \mathcal{C}$, and this functor is completely specified by its action on vertices (f_0), and morphisms (f_1), and the requirement that f respects composition (f_2, and $d_i f_2 = f_1 d_i$). Another way of saying this is that a simplicial map $X \to B\mathcal{C}$ is completely determined by its restriction to $\operatorname{sk}_2 X$.

The inclusion $\Lambda^n_k \subset \Delta^n$ induces an isomorphism

$$\operatorname{sk}_{n-2} \Lambda^n_k \cong \operatorname{sk}_{n-2} \Delta^n.$$

To see this, observe that every simplex of the form $d_i d_j \iota_n$, $i < j$, is a face of some $d_r \iota_n$ with $r \neq k$: if $k \neq i, j$ use $d_i(d_j \iota_n)$, if $k = i$ use $d_k(d_j \iota_n)$, and if $k = j$ use $d_i(d_k \iota_n) = d_{k-1}(d_i \iota_n)$. It immediately follows that the extension problem

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\alpha} & BG \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

is solved if $n \geq 4$, for in that case $\operatorname{sk}_2 \Lambda^n_k = \operatorname{sk}_2 \Delta^n$.

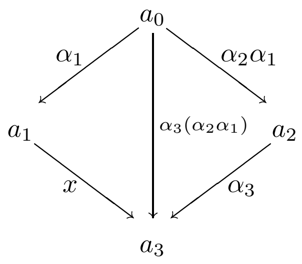
Suppose that $n = 3$, and consider the extension problem

$$\begin{array}{ccc} \Lambda^3_0 & \xrightarrow{\alpha} & BG \\ \downarrow & \nearrow & \\ \Delta^3 & & \end{array}$$

Then $\operatorname{sk}_1 \Lambda^3_0 = \operatorname{sk}_1 \Delta^3$ and so we are entitled to write $\alpha_1 : a_0 \to a_1$, $\alpha_2 : a_1 \to a_2$ and $\alpha_3 : a_2 \to a_3$ for the images under the simplicial map α of the 1-simplices $0 \to 1$, $1 \to 2$ and $2 \to 3$, respectively. Write $x : a_1 \to a_3$ for the image of $1 \to 3$ under α. Then the boundary of $d_0 \iota_3$ maps to the graph

$$\begin{array}{ccc} a_1 & \xrightarrow{\alpha_2} & a_2 \\ & {\scriptstyle x}\searrow \quad \swarrow{\scriptstyle \alpha_3} & \\ & a_3 & \end{array}$$

in the groupoid G under α, and this graph bounds a 2-simplex of BG if and only if $x = \alpha_3\alpha_2$ in G. But the images of the 2-simplices $d_2\iota_3$ and $d_1\iota_3$ under α together determine a commutative diagram



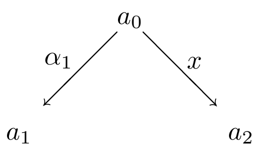
in G, so that

$$x\alpha_1 = \alpha_3(\alpha_2\alpha_1),$$

and $x = \alpha_3\alpha_2$, by right cancellation. It follows that the simplicial map $\alpha : \Lambda^3_0 \to BG$ extends to $\partial\Delta^3 = \mathrm{sk}_2\,\Delta^3$, and the extension problem is solved.

The other cases corresponding to the inclusions $\Lambda^3_i \subset \Delta^3$ are similar.

If $n = 2$, then, for example, a simplicial map $\alpha : \Lambda^2_0 \to BG$ can be identified with a diagram



and α can be extended to a 2-simplex of BG if and only if there is an arrow $\alpha_2 : a_1 \to a_2$ of G such that $\alpha_2\alpha_1 = x$. But $\alpha_2 = x\alpha_1^{-1}$ does the trick. The other cases in dimension 2 are similar. □

The standard n-simplex $\Delta^n = B\mathbf{n}$ fails to be fibrant for $n \geq 2$, precisely because the last step in the proof of Lemma 3.5 fails in that case.

I.4. Anodyne extensions.

The homotopy theory of simplicial sets is based on the definition of fibration given in the last section. Originally, all statements involving fibrations were expressed in terms of the extension condition, and this often led to some rather difficult combinatorial manipulations based on the standard subdivision of a prism.

The algorithms involved in these manipulations are actually quite formal, and can be encoded in the Gabriel-Zisman theory of anodyne extensions [33].

This theory suppresses or engulfs most of the old combinatorial arguments, and is a basic element of the modern theory. We describe the Gabriel-Zisman theory in this section.

A class M of (pointwise) monomorphisms of $\mathbf{S}$ is said to be *saturated* if the following conditions are satisfied:

A: All isomorphisms are in M.

B: M is closed under pushout in the sense that, in a pushout square

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ B & \longrightarrow & B \cup_A C, \end{array}$$

if $i \in M$ then so is i_* (Exercise: Show that i_* is monic).

C: Each retract of an element of M is in M. This means that, given a commutative diagram

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "1"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B' \arrow[r] \arrow[rr, bend right, "1"'] & B \arrow[r] & B', \end{tikzcd}$$

of simplicial set maps, if i is in M then so is i'.

D: M is closed under countable compositions and arbitrary direct sums, meaning respectively that:

D1: Given

$$A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \dots$$

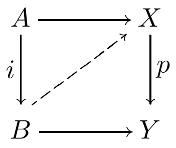
with $i_j \in M$, the canonical map $A_1 \to \varinjlim A_i$ is in M.

D2: Given $i_j : A_j \to B_j$ in M, $j \in I$, the map

$$\sqcup\, i_j : \bigsqcup_{j \in I} A_j \to \bigsqcup_{j \in I} B_j$$

is in M.

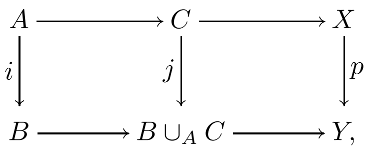
A map $p : X \to Y$ is said to have the *right lifting property* (RLP is the standard acronym) with respect to a class of monomorphisms M if in every solid arrow diagram



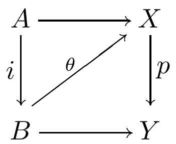
with $i \in M$ the dotted arrow exists making the diagram commute.

Lemma 4.1. *The class M_p of all monomorphisms which have the left lifting property (LLP) with respect to a fixed simplicial map $p : X \to Y$ is saturated.*

Proof: (trivial) For example, we prove the axiom **B**. Suppose given a commutative diagram



where the square on the left is a pushout. Then there is a map $\theta : B \to X$ such that the "composite" diagram



commutes. But then θ induces the required lifting $\theta_* : B \cup_A C \to X$ by the universal property of the pushout. □

The *saturated class M_B generated by a class of monomorphisms B* is the intersection of all saturated classes containing B. One also says that M_B is the *saturation* of B.

Consider the following three classes of monomorphisms:

$\mathbf{B_1}$:= the set of all inclusions $\Lambda^n_k \subset \Delta^n$, $0 \le k \le n$, $n > 0$

$\mathbf{B_2}$:= the set of all inclusions

$$(\Delta^1 \times \partial\Delta^n) \cup (\{e\} \times \Delta^n) \subset (\Delta^1 \times \Delta^n), \qquad e = 0, 1$$