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Viability Theory

Jean-Pierre Aubin

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With 14 Illustrations

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THIS BOOK IS DEDICATED TO
HÉLÈNE FRANKOWSKA

with love

Epigraph

Viability theory is a mathematical theory that offers *mathematical metaphors*[1] *of evolution of macrosystems* arising in biology, economics, cognitive sciences, games, and similar areas, as well as in nonlinear systems of control theory.

We shall specifically be concerned with three main common features:

- A nondeterministic (or contingent) engine of evolution, providing several (and even many) opportunities to explore the environment,
- Viability constraints that the state of the system must obey at each instant under "death penalty",
- An inertia principle stating that the "controls" of the system are changed only when viability is at stake.

The first two features are best summarized by the deeply intuitive statement attributed to Democritus by Jacques Monod: *"Everything that exists in the Universe is due to Chance and Necessity"*. The inertia principle is a mathematical formulation of the concept of *punctuated equilibrium* introduced recently in paleontology by Elredge and Gould. It runs against the teleological trend assigning aims to

[1]Like other means of communications (languages, painting, music, etc.), mathematics provides *metaphors* that can be used to explain a given phenomenon by associating it with some other phenomenon that is more familiar, or at least is felt to be more familiar. This feeling of familiarity, individual or collective, inborn or acquired, is responsible for the inner conviction that this phenomenon is understood.

be achieved (in even an optimal way) by the state of the system and the belief that actors control the system for such purposes.

— **Nondeterminism:** We shall mean by this term that *les jeux ne sont jamais faits*, in the sense that at each instant, there are several available, or feasible, evolutions which depend upon the state, or even the history of the evolution of the state of the system up to this time. Therefore, the concept of evolution borrowed from Newtonian mechanics is no longer adequate for such systems. It has led to the misleading identification of mathematics with a *deterministic* paradigm, which implies that *the evolution of macrosystems can be predicted*. Even if we were to accept the existence of deterministic mechanisms[2] underlying the evolution of biological, economic and social macrosystems, we know that such systems often can be inherently unstable - and this places the actual computation of their solutions beyond the capabilities of even the most sophisticated of present-day computers! To "run" models which have some inbuilt structural instability can serve no useful purpose.

Thus, we suppose here that the dynamics responsible for the evolution are not deterministic. This lack of determinism has many different features: it may be due to nonstochastic "uncertainty"[3], to "disturbances"and "perturbations" of various kinds, or to errors in modeling due to the impossibility of a comprehensive description of the dynamics of the system.

In several instances, the dynamics of the system are related to certain "controls", which, in turn, are restricted by state-dependent constraints (closed systems.) Such controls, which we do not dare to call *regulees* instead of controls, are typically

1. *prices or other fiduciary goods* in economics (when the evolution of commodities and services is regulated by Adam Smith's invisible hand or the market, the planning bureau, ...),

[2]And now we discover that some of our "perfectly deterministic" models can exhibit all sorts of different trajectories. These are *chaotic* systems, making prediction virtually impossible.

[3]No a priori knowledge of an underlying probability law on the state of events is made. *Fuzzy viability* provides models where the available velocities can be ranked through a membership cost function to take into account that some velocities are more likely to be chosen than others.

2. *genotypes* or *fitness matrices* in genetics and population genetics (when the evolution of *phenotypes* of a population is regulated by sexual reproduction and mutations),

3. *conceptual controls* or *synaptic matrices* in pattern recognition mechanisms and neural networks (when the sensory-motor state is regulated by learning processes),

4. *affinity matrices* in immunological systems,

5. *strategies* in differential games (when the state of the system is regulated by the decision rules for the players),

6. *coalitions* in cooperative games,

7. *cultural codes* in sociology (when the evolution of societies is regulated by every individual believing and obeying such codes), etc..

— **Viability:** For a variety of reasons, not all evolutions are possible. This amounts to saying that the state of the system must obey constraints, called *viability constraints*. These constraints include homeostatic constraints in biological regulation, scarcity constraints in economics, state constraints in control, power constraints in game theory, ecological constraints in genetics, sociability constraints in sociology, etc. Therefore, the goal is to select solutions which are *viable in the sense that they satisfy, at each instant, these constraints.*

Viability theorems thus yield selection procedures of viable evolutions, i.e., characterize the connections between the dynamics and the constraints for guaranteeing the existence of at least one viable solution starting from any initial state. These theorems also provide the *regulation processes (feedbacks*[4]*)* that maintain viability, or, even as time goes by, *improve* the state according to some *preference relation.*

Contrary to *optimal control theory*, viability theory does not require any single decision-maker (or actor, or player) to "guide" the

[4]thus providing the central concept of cybernetics as a *solution* to the regulation problem.

system by optimizing an *intertemporal* optimality criterion[5].

Furthermore, the choice (even conditional) of the controls is not made *once and for all* at some initial time, but *they can be changed at each instant so as to take into account possible modifications of the environment of the system*, allowing therefore for *adaptation* to viability constraints.

Finally, by not appealing to intertemporal criteria, *viability theory does not require any knowledge of the future*[6] (even of a stochastic nature.) This is of particular importance when experimentation[7] is not possible or when the phenomenon under study is not periodic. For example, in biological evolution as well as in economics and in the other systems we shall investigate, *the dynamics of the system disappear and cannot be recreated.*

Hence, *forecasting or prediction of the future are not the issues which we shall address in this book.*

However, the conclusions of the theorems allow us to reduce the choice of possible evolutions, or to single out impossible future events, or to provide explanation of some behaviors which do not fit any reasonable optimality criterion.

Therefore, instead of using intertemporal optimization[8] that involves the future, viability theory provides selection procedures of *viable evolutions* obeying, at each instant, state constraints which depend upon the *present or the past.* (This does not exclude *anticipations*, which are extrapolations of past evolutions, constraining in the last analysis the evolution of the system to be a function of its history.)

[5]the choice of which is open to question even in static models, even when multicriteria or several decision makers are involved in the model.

[6]Most systems we investigate do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

[7]Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to "know" the future evolution of the system.

[8]which can be traced back to Sumerian mythology which is at the origin of Genesis: one Decision-Maker, deciding what is good and bad and choosing the best (fortunately, on an intertemporal basis, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week...

Nonetheless, selection through viability constraints may not be discriminating enough. Starting from any state at any instant, several viable solutions may be implemented by the system, including equilibria, which are stationary evolutions[9].

Thus further selection mechanisms need to be devised or discovered. We advocate here a third feature to which a selection procedure must comply, the *Inertia Principle*.

— **Inertia Principle:** which states that *"the controls are kept constant as long as viability of the system is not at stake"*.

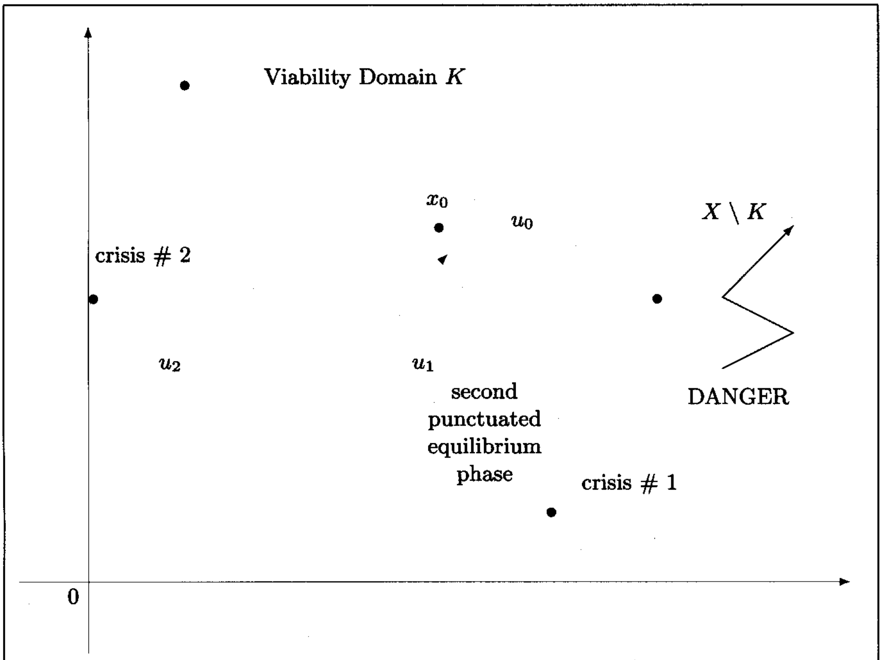
Indeed, as long as the state of the system lies in the interior of the viability set (the set of states satisfying viability constraints), any regularity control will work. Therefore, the system can maintain the control inherited from the past. This happens if the system obeys the inertia principle. Since the state of the system may evolve while the control remains constant, it may reach the viability boundary with an "outward" velocity. This event corresponds to a period of *crisis*: To survive, the system must find another regulatory control such that the new associated velocity forces the solution back inside the viability set. (See Figure 1.) Alternatively, if the viability constraints can evolve, another way to resolve the crisis is to relax the constraints so that the state of the system lies in the interior of the new viability set. When this is not possible, *strategies for structural change fail*: by design, this means the solution leaves the viability set and "dies".

Naturally, there are several procedures for selecting a viable control when viability is at stake. For instance, the selection at each instant of the controls providing viable evolutions with *minimal velocity* is an example that obeys this inertia principle. They are called *"heavy" viable evolutions*[10] in the sense of heavy trends in economics.

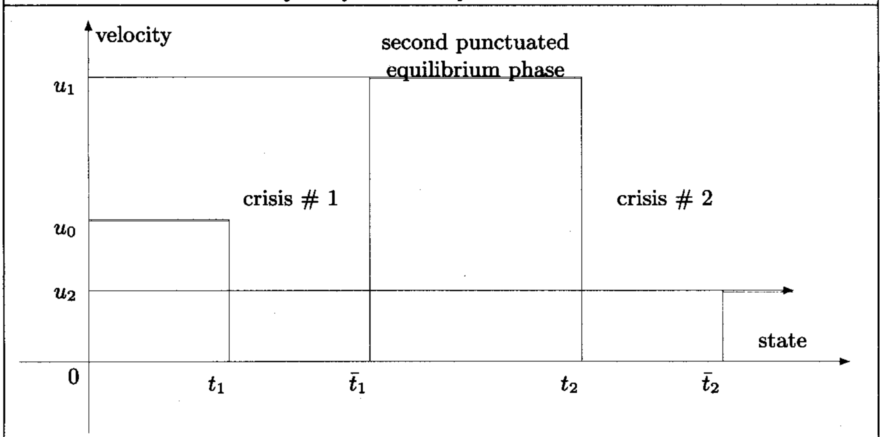
[9]This touches on another aspect of viability theory - that concerned with complexity and robustness: It may be observed that the state of the system becomes increasingly robust the further it is from the boundary of the viability set. Therefore, after some time has elapsed, only the parts of the trajectories furthest away from the viability boundary will remain. This fact may explain the apparent discontinuities ("missing links") and hierarchical organization arising from evolution in certain systems.

[10]When the controls are the velocities, heavy solutions are the ones with minimal acceleration, i.e., maximal inertia.

Figure 0.1: Heavy Viable Solutions



Trajectory of a Heavy Viable Solution



Punctuated Evolution of Velocities. Starting from x_0 with the constant velocity u_0, the solution evolves in K until time t_1, (first *punctuated equilibrium phase*) when the velocity. Then a *crisis* happens during which velocities also evolve (as slowly as possible) to maintain viability, until time $\bar{t}_1$ where the velocities are contingent to K during the *second punctuated equilibrium phase.*

Heavy viable evolutions can be viewed as providing mathematical metaphors for the concept of *punctuated equilibrium*[1] introduced recently in paleontology by Elredge and Gould.

In a nutshell, *the main purpose of viability theory is to explain the evolution of a system, determined by given nondeterministic dynamics and viability constraints, to reveal the concealed feedbacks which allow the system to be regulated and provide selection mechanisms for implementing them.*

It assumes implicitly an "opportunistic" and "conservative" behavior of the system: a behavior which enables the system to keep viable solutions as long as its potential for exploration (or its lack of determinism) — described by the availability of several evolutions — makes possible its regulation.

On the mathematical side, viability theory contributed to vigorous renewed interest in the field of "differential inclusions", as well as an engine for the development of a differential calculus of set-valued maps[2]. Indeed, as it often occurs in mathematics, these techniques have already found applications to other domains, for instance, to nonlinear systems theory (tracking, zero dynamics, local controllability and observability[3], control under state constraints, etc.) and

[1]Excavations at Kenya's Lake Turkana have provided clear evidence of evolution from one species to another. The rock strata there contain a series of fossils that show every small step of an evolution journey that seems to have proceeded in fits and starts. Examination of more than 3,000 fossils by P. Williamson showed how 13 species evolved. The record indicated that the animals stayed much the same for immensely long stretches of time. But twice, about two million years ago and then, 700,000 years ago, the pool of life seemed to explode — set off, apparently, by a drop in the lake's water level. Intermediate forms appeared very quickly, new species evolving in 5,000 to 50,000 years, after millions of years of constancy, leading paleontologists to challenge the accepted idea of continuous evolution.

[2]One can say that by now the main results of functional analysis have their counterpart in what can be called *Set-Valued Analysis*. Only the results needed in this book will be presented. An exposition of Set-Valued Analysis can be found in the companion monograph SET-VALUED ANALYSIS by Hélène Frankowska and the author.

[3]These topics will be not developed here. The forthcoming monograph CONTROL OF NONLINEAR SYSTEMS AND DIFFERENTIAL INCLUSIONS by Hélène

Artificial Intelligence (qualitative physics, learning processes, etc.) These techniques can be efficiently used as mathematical tools and have been related to other questions (such as Lyapunov's second method, variational differential equations, etc..)

This is a book of *motivated mathematics*[4], which searches for new sources of mathematical metaphors.

Unfortunately, the length of the theoretical part of viability theory did not allow us to include in this volume the discussion of the motivating problems. Some problems arising in Artificial Intelligence, economics, game theory, biology, cognitive sciences, etc., which have spawned many of the mathematical questions treated below, will be investigated in forthcoming additional volumes.

By looking at common features of otherwise very different systems and looking at shared consequences, it was necessary to set our mathematical metaphors at a fairly high level of abstraction, yielding an amount of information inversely proportional to the height of this level so to speak.

For the time being at least, this theory is still far from providing an ideal description of the evolution of macrosystems. Some potential users (economists, biologists, ...) should not be disappointed or discouraged by the results obtained so far — for it is too early for such a theory to be "applied" in the engineering sense.

However, the available results may explain a portion of "reality"

Frankowska provides an exhaustive treatment of Control Theory using set-valued analysis and differential inclusions.

[4]We have already mentioned a mathematical metaphor as a means of associating a particular mathematical theory with a certain observed phenomenon. This association can arise in two different ways. The first possibility is to look for an existing mathematical theory which seems to provide a good explanation of the phenomenon under consideration. This is usually regarded as the domain of applied mathematics. However, it is also possible to approach the problem from the opposite direction. Other fields provide mathematicians with metaphors, and this is the domain of what can be called "motivated mathematics".

The ancients divided *analysis* into two forms: *zetetic*, which corresponds to what we mean by motivated mathematics or modeling, and *poristic*, which corresponds to applied mathematics, a procedure by which the validity of the model is confirmed. It is much later, in 1591, that F. Viète added a third form, *rhetic* or *exegetic*, which would correspond to our pure mathematics.

in the extent where *the degree of reality for a social group at a given time is understood in terms of the consensus*[5] *interpretations of the group member's perceptions of their physical, biological, social and cultural environments.*

I hope that this book may help readers from different scientific areas to find a common ground for comparing the behaviors of the systems they study and for asking new questions. Anyhow, whatever the ultimate outcome, the motivation provided by the viability problems has already benefited mathematics by suggesting new concepts and lines of argument, by giving some inkling of possible solutions, or by developing new modes of intuition, leading many mathematicians to revive and enrich the theory of dynamical systems and set-valued analysis. The history of mathematics is full of instances in which mathematical techniques motivated by problems encountered in one scientific field have found applications in many others. *It is this "universality" which renders mathematics so fascinating.*

Jean-Pierre Aubin
Paris, May 12, 1990

[5]Since our brains are built according to the same biological blueprint, and since the general acceptance of local cultural codes seems to be an innate and universal phenomenon, it is highly probable that the individuals comprising a social group arrive at a consensus wide enough for a reasonably believable concept of reality to emerge. However, the prophets and scholars of each group continually question the validity of the metaphors on which this consensus is based, while the high priests and other guardians of ideological purity ultimately try to transform it into dogma and impose it on the other members of the group. (It often happens that the prophets and scholars themselves eventually become high priests ; movement in the reverse direction is much less common.) It is through this permanent struggle that knowledge evolves. But there is an important difference between the metaphors of science and those of, say, religion or ideology : a metaphor that claims scientific validity must be limited, even narrow, in scope. The more "applied" a scientific study, the narrower it must necessarily be. Scientific theories — scientific metaphors — must be capable of logical refutation (as in mathematics) or of experimental falsification (which of course requires that theories be falsifiable.) Ideologies escape these requirements : the "broader" they are, the more seductive they appear, the more dangerous they can be.

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The **CEREMADE** (Centre de Recherches de Mathématiques de la Décision), UNIVERSITÉ DE PARIS-DAUPHINE, has offered me for twenty years a unique scientific environment. The pervasive innovative spirit shared by my colleagues[1] encouraged me to devote the last 10 years to the development of viability theory and set-valued analysis and their applications.

I am also thankful to the **S.D.S.** (Systems and Decision Sciences) Program of IIASA (International Institute for Applied Systems Analysis), especially, its head, A. Kurzhanski, with whom I share common views on the evolution of systems under uncertainty. It offered me opportunities to collaborate during many summers with a great number of colleagues, A. Anderson, C. Byrnes, A. Isidori, N. Keyfitz, A. Krener, G. Leitmann, A. Nagy, C. Olech, R. T. Rockafellar, D. Saari, K. Sigmund and R. Wets among many others, who contributed to shaping the ideas exposed in this book.

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This book is dedicated to Hélène Frankowska, for many reasons. She actually should have been the coauthor of this book as well as of SET-VALUED ANALYSIS, thanks to her contributions (many of the results are due to her), daily remarks and continuous ideas, corrections, suggestions and challenges to improve the redaction of this monograph. Without her, this book would never have reached this stage. With her, it would have been much better.

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[1] among whom my old friends A. Bensoussan, I. Ekeland, the members of the group VIABILITÉ ET CONTRÔLE who contributed so much to this book, without forgetting more transient colleagues, P. Bernhard, A. Cellina, F. Clarke to name a few.

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Introduction

Consider the evolution of a control system with (multivalued) feedbacks:

$$\begin{cases} i) & x'(t) = f(x(t), u(t)) \\ ii) & u(t) \in U(x(t)) \end{cases}$$

where the state $x(\cdot)$ ranges over a finite dimensional vector-space X and the control $u(\cdot)$ ranges over another finite dimensional vector-space Z. Here, the first equation describes how the control — regarded as an *input* to the system — yields the state of the system[1] — regarded as an *output* — whereas the second inclusion shows how the state-output "feeds back" to the control-input. The set-valued map $U : X \rightsquigarrow Z$ may be called an "a priori feedback". It describes the *state-dependent constraints on the controls*. A solution to this system is a function $t \to x(t)$ satisfying this system for some control $t \to u(t)$.

Viability constraints are described by a closed subset[2] K of the state space: These are intended to describe the "viability" of the system because outside of K, the state of the system is no longer viable.

A subset K *is viable* under the control system described by f and U if for every initial state $x_0 \in K$, there exists at least one solution to the system starting at x_0 which is *viable* in the sense that

$$\forall\, t \geq 0, \quad x(t) \in K$$

[1]once the initial state is fixed.

[2]We shall naturally investigate in the book the cases when K depends upon the time, the state, the history of the evolution of the space. We shall also cover the case of solutions *which improve a reference preorder when time evolves*.

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The first task is to characterize the subsets having this property. To be of value, this task must be done without solving the system and then checking the existence of viable solutions for each initial state.

An immediate intuitive idea jumps to the mind: at each point on the boundary of the viability set, where the viability of the system is at stake, there should exist a velocity which is in some sense *tangent* to the viability domain and serves to allow the solution to bounce back and remain inside it. This is, in essence, what the Viability Theorem states. But, first, the mathematical implementation of the concept of tangency must be made.

We cannot be content with viability sets that are smooth manifolds, because inequality constraints would thereby be ruled out. So, we need to "implement" the concept of a direction v tangent to K at $x \in K$, which should mean that starting from x in the direction v, we do not go too far from K.

To convert this intuition into mathematics, we shall choose from among the many ways there are to translate what it means to be "not too far" the one suggested by Bouligand fifty years ago: a direction v is *contingent to K at $x \in K$* if it is a limit of a sequence of directions v_n such that $x + h_n v_n$ belongs to K for some sequence $h_n \to 0+$. The collection of such directions, which are in some sense "inward", constitutes a closed cone $T_K(x)$, called the *contingent cone*[3] to K at x. Naturally, except if K is a smooth manifold, we lose the fact that the set of contingent vectors is a vector-space.

We then associate with the dynamical system (described by f and U) and with the viability constraints (described by K) the *(set-valued) regulation map R_K*. It maps any state x to the subset $R_K(x)$ consisting of controls $u \in U(x)$ which are *viable* in the sense that

$$f(x, u) \quad \text{is contingent to } K \text{ at } x$$

If, for every $x \in K$, there exists at least one viable control $u \in R_K(x)$, we then say that K is a *viability domain* of the control system with dynamics described by both f and U.

[3]replacing the linear structure underlying the use of tangent spaces by the contingent cone is at the root of *Set-Valued Analysis*.

The Viability Theorem we mentioned earlier holds true for a rather large class of systems, called *Marchaud systems*: Beyond imposing some weak technical conditions, the only severe restriction is that, for each state x, the set of velocities $f(x, u)$ when u ranges over $U(x)$ is *convex*[4]. From now on, we assume that the systems under investigation are Marchaud systems.

The basic viability theorem states that for such systems,

a closed subset K is viable under a Marchaud system if and only if K is a viability domain of this system.

Many of the traditional interesting subsets such as *equilibrium points, trajectories of periodic solutions, the ω-limit sets of solutions, are examples of closed viability domains.* Actually, equilibrium points $\overline{x}$, which are solutions to

$$f(\overline{x}, \overline{u}) = 0 \text{ for some } \overline{u} \in U(\overline{x})$$

are the smallest viability domains, the ones reduced to a single point. This is because being *stationary states*, the velocities $f(\overline{x}, \overline{u})$ are equal to zero. Furthermore, there exists a basic and curious link between viability theory and general equilibrium theory:

every compact convex viability domain contains an equilibrium point.

This statement is an equivalent version of the 1910 *Brouwer Fixed Point Theorem*, the cornerstone of nonlinear analysis, which finds here a particularly relevant formulation (viability implies stationarity.)

What happens if a closed subset K is not a viability domain?

First, we characterize the points of the boundary from which some, or all solutions enter or leave the subset (anatomy of a set).

---

[4]This happens for the class of control systems of the form

$$x'(t) = f(x(t)) + G(x(t))u(t)$$

where $G(x)$ are linear operators from the control space to the state space and when the control set U (or the images $U(x)$) are convex.

Second, we also look for closed subsets of K which are viability domains. We shall prove that

there exists a largest closed viability domain contained in K.

This domain will be denoted Viab(K) and called the *viability kernel*[5] of K. It may be empty (in this case, the subset K is some kind of "repeller".) Furthermore, every closed subset of the viability kernel is contained in a minimal viability domain, called *viability envelope.*

Third, one can also keep the set of constraints and change the dynamics, as it is done in mechanics of unilateral constraints (variational differential equations).

The Viability Theorem also provides a *regulation law* for regulating the system in order to maintain the viability of a solution: The viable solutions $x(t)$ are regulated by viable "open loop controls" $u(t)$ through the regulation law:

$$\text{for almost all } t, \quad u(t) \in R_K(x(t))$$

The multivaluedness of the regulation map (this means that several controls $u(t)$ may exist in $R_K(x(t))$) is an indicator of the "robustness" of the system: *The larger the set* $R_K(x(t))$, *the larger the set of disturbances which do not destroy the viability of the system !*

Observe that solutions to a control system are solutions to the differential inclusion $x'(t) \in F(x(t))$ where, for each state x, $F(x) := f(x, U(x))$ is the subset of feasible velocities, Conversely, a differential inclusion is an example of a control system in which the controls are the velocities ($f(x,u) = u$ & $U(x) = F(x)$.)

As far as servomechanisms are concerned, the question arises of how to build mechanisms for selecting a *unique* control $\widehat{u}(x)$ in $R_K(x)$ for each state x. Such a map $\widehat{u}(\cdot)$, associating with every x a single control $\widehat{u}(x)$ is called a *closed loop* control (or single-valued feedback.) This is because it allows the system to *automatically associate with*

[5]This concept of viability kernel happens to be a quite efficient mathematical tool that we shall use often.

It is also closely related to the concept of *zero dynamics* introduced recently by Byrnes and Isidori in control theory.

any state $x(t)$ the control $\widehat{u}(x(t))$ which produces a viable solution through the differential equation

$$x'(t) = f(x(t), \widehat{u}(x(t)))$$

An interesting example of closed loop control is provided by *slow solutions*. These are the solutions regulated by the controls $u^0(x) \in R_K(x)$ with minimal norm. Despite the fact that $u^0(\cdot)$ is not necessarily continuous, we shall prove that the above differential equation still has solutions. For instance, when the controls are the velocities of the system, viable solutions with *velocities of minimal norm* are implemented by such a selection procedure. This is why they are called *slow solutions*.

Such selection procedures by closed loop controls answer many engineering control problems. But they may not be adequate for the type of systems arising in economic, social, biological and cognitive sciences, as well as in some areas of engineering where the controls must evolve continuously. Here, we are looking for selection procedures which obey the *inertia principle*: keep the control constant as long as the viability of the system is not at stake.

We can reformulate the inertia principle by saying that if the derivative of a viable open loop control $u(\cdot)$ is equal to 0, then this control is the one which is chosen and implemented.

This raises several questions.

— The first one concerns controls which are smooth (at least, differentiable almost everywhere.) (This issue may be relevant for engineering problems, where the lack of continuity of controls $u(t) := \widehat{u}(x(t))$ can be damaging.)

— The second one deals with the problem of differentiating the regulation law.

— The third is to find selections (called *dynamical closed loops*) of the derivative of the regulation map, with which we obtain a system of differential equations which govern the *smooth* viable evolution of both the state and the control.

— The fourth is to find some feedback controls as solutions to systems of first-order partial differential inclusions.

We see at once that this programme requires a concept of derivative of a set-valued map and a chain rule formula in order to differentiate the regulation law.

The idea behind the construction of a differential calculus of set-valued maps is simple and goes back to the very origins of differential calculus, when Pierre de Fermat introduced in the first half of the seventeenth century the concept of a tangent to the graph of a function:

the tangent space to the graph of a function f at a point (x, y) of its graph is the line of slope $f'(x)$, i.e., the graph of the linear function $u \mapsto f'(x)u$

Consider now a set-valued map $F : X \rightsquigarrow Y$, which is characterized by its graph (the subset of pairs (x, y) such that y belongs to $F(x)$.)

The contingent cone to the graph of F at the point (x, y) of its graph is the graph of the contingent derivative of the set-valued map F at a point (x, y)

The contingent derivative at (x, y) is a set-valued map from X to Y denoted by $DF(x, y)$.

Contingent derivatives keep enough properties of the derivatives of smooth functions to be quite efficient. They enjoy a rich calculus, and they enable such basic theorems of analysis as the inverse function theorem to be extended to the set-valued case.

The chain rule is an example of a property which is still true in this framework: Assume that we start from a "smooth state", producing a viable solution $x(t)$ and a viable control $u(t)$ which are both differentiable (almost everywhere.) Then we can "differentiate" the regulation law to obtain a "first order regulation law":

$$\text{for almost all } t, \quad u'(t) \in DR_K(x(t), u(t))(x'(t))$$

Heavy viable solutions are the ones regulated by the controls whose velocities have minimal norm in the set

$$DR_K(x(t), u(t))(f(x(t), u(t)))$$

For instance, when the controls are the velocities of the system, we choose viable solutions with *acceleration of minimal norm*, i.e.,