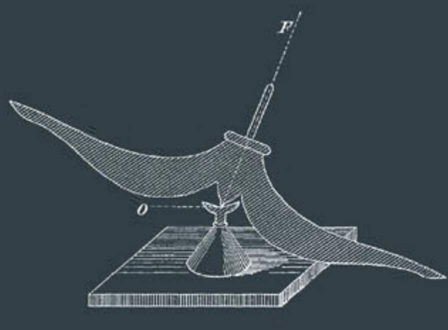
BIRKHÄUSER

Felix Klein
Arnold Sommerfeld

The Theory of the Top
Volume I

Introduction to the Kinematics and Kinetics of the Top



Translated by
Raymond J. Nagem
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Translators

Preface by Michael Eckert

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ISBN: 978-0-8176-4720-9 e-ISBN: 978-0-8176-4721-6
DOI: 10.1007/978-0-8176-4721-6

Library of Congress Control Number: 2008926224

Mathematics Subject Classification (2000): 01A75, 70E05, 70E15, 70E17, 70E50

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

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Preface

The *Theory of the Top* attained its great fame from both its monumental scope and its outstanding authors. In the early twentieth century, Felix Klein was known as a mathematician of world fame; Arnold Sommerfeld, Klein's disciple, had acquired his reputation as a rising star of theoretical physics. By 1910, when the final volume of this treatise was published, the names of Klein and Sommerfeld would signal to a student that a matter as complex as the top was presented in a most authoritative manner, from the perspective of both mathematics and physics. The work also stands out in other regards: by its sheer extent—four volumes comprising a total of almost a thousand pages—and by the time lag of about fifteen years between inception and completion. Klein himself regarded the final result as somewhat disjointed. Its "idiosyncratic disposition," he reflected in 1922, may be understood only by taking into account the historic circumstances at its inception in 1895; the developments between the first and last parts derailed the project from its intended course, so that for the technical applications described in Volume IV "almost no use was made of the theoretical framework developed at the beginning" [Klein 1922, p. 659].

It seems appropriate, therefore, to recall the historical circumstances under which this treatise was conceived and pursued. Felix Klein was not only a renowned mathematician, but also an entrepreneurial and ambitious university professor striving for a broader acknowledgment of mathematics as a cultural asset. During the Wilhelmian Era, when Germany was struggling for recognition as a great power, cultural affairs were no longer innocent bystanders of national politics. Friedrich Althoff, a powerful reformer at the Prussian Ministry of Culture, attempted to form centers of excellence at certain universities [Brocke 1980]. Klein had become professor in Göttingen University in 1886.

After a few years of frustration and uncertain prospects, Klein persuaded Althoff that Göttingen would assume the desired rank only if rising stars like David Hilbert and Hermann Minkowski were called to the university as his colleagues. As a result of Klein's strategy, backed by the almighty Althoff, Göttingen became a mecca of mathematics [Rowe 1989].

But Klein's ambitions were not restricted to local affairs at his university. As a part of his attempts to gain widespread recognition for mathematics, he began to edit an *Encyclopedia of Mathematical Sciences*, an enterprise that lasted until the 1920s and encompassed, in addition to pure mathematics, a broad spectrum of mathematical applications to mechanics, physics, and astronomy. Klein also established contacts with the Association for the Advancement of Mathematical and Scientific Education, in order to gain influence on high school teaching of mathematics. Furthermore, he displayed considerable interest in the scientific training of engineers, which was traditionally the realm of technical universities, and therefore made Klein the enemy of engineering professors who regarded his tendencies as an unwelcome interference in their own affairs. In 1895, for example, Klein conceived a memorandum in which he suggested the foundation of a new institute in Göttingen University for the education of the "general staff" of technology, whereas the training of "front officers" could be left to the technical universities [Rowe 1989, p. 203].

Such was the broader context for the birth of the *Theory of the Top.* Under this title Klein announced a special lecture in the winter semester of 1895/96, addressed to high school teachers who wished to keep in touch with advanced mathematical subjects. In the preceding semester, Klein had held another special lecture for the same audience on *Elementary Geometry.* One of his assistants was charged with elaborating the manuscript of this lecture into a booklet, which Klein presented to the high school teachers association as a special gift by which he intended to prepare the ground for his further engagement in aiming at a general reform of high school teaching. By lecturing on the top in the winter of 1895/96, Klein attempted to demonstrate that his university teaching was not an ivory tower activity but had relations to technological as well as educational affairs. Like the *Elementary Geometry* of the preceding semester, the *Theory of the Top* was meant to be printed afterward as a small booklet and presented as a gift to his extramural clients. Klein remarked in an autobiographical note in 1913 that the *Top* was a tactic intended as a second dedicatory publication [Jacobs 1977, p. 18].

The course of subsequent events, however, precluded a smooth realization of these plans. Klein entrusted Arnold Sommerfeld, who had become his assistant in the autumn of 1894, with much more than the elaboration of this lecture. Sommerfeld had just accomplished his habilitation (the German ritual to acquire the right to lecture in a university) on the theory of diffraction [Sommerfeld 1896] and was busy with the elaboration of Klein's *Number Theory*, a lecture that Klein held in two parts in the winter of 1895/96 and summer of 1896 [Klein 1896a; Klein 1896b]. When Sommerfeld finally started to work on the theory of the top in the autumn of 1896, he did so without great enthusiasm. He worked on several projects at the same time, all of them related to one or another of Klein's activities, such as a register for the *Mathematische Annalen* (a journal edited by Klein) or a review article on partial differential equations for Klein's *Encyclopedia of Mathematical Sciences*. Furthermore, he discovered that the methods developed in his habilitation work proved to be more fertile than he had originally anticipated. Writing papers on his own research appeared more interesting to him than editing Klein's lecture on the theory of the top.

Although the first parts of the lecture advanced to the state of proof reading by the spring of 1897, its completion was dragging on. In March of 1897 Sommerfeld wrote to Klein that "the number of boundary value problems that I am able to solve by my extension of Thomson's mirror method is very considerable." He felt sure that Klein would appreciate the temporary neglect of the top, because his method for solving physical differential equations was completely in line with Klein's tendencies: "I hope you will enjoy it yourself. But I still have several days to do with it. If you could arrange for this work to be published soon in an English journal, such as the London Math. Soc. [*Proceedings of the London Mathematical Society*], I would be very happy." To please Klein he added some remarks about his elaboration of the theory of the top, but finally revealed that this had a rather low priority on his to-do list. "Unfortunately, I have to admit that in the meantime the top has been in the nonetheless very interesting 'sleeping top' state" [Sommerfeld 1897a].

Working under Klein must have been quite demanding. "I really cannot write to you each day," Sommerfeld once apologized to his fiancée Johanna Höpfner. "Klein's bullwhip is rather close behind me" [Sommerfeld 1897b]. At some point in 1897, Klein must have decided to split the publication into several parts. Klein did not leave the theory of the top in the state in which he had presented it in his lecture of 1895/96. In the summer semester of 1896 he lectured on technical mechanics. In October and November of 1896, he chose the theory of the top as a theme for guest lectures at Princeton University [Klein 1897]. In view of Klein's goal of demonstrating the uses of mathematics to engineers at technical universities, he must have regarded it expedient to include more applied matters and charged Sommerfeld to work out the details.

Under these premises, the mathematical foundations as laid out in earlier lectures were published in 1897 as Volume I. As Sommerfeld prepared the subsequent volume, progress became slow because he struggled with problems that were the subject of controversial debates. "With regard to the equilibrium stability H.[adamard] does not go one step further than Lyapunov"; such remarks in the correspondence between Sommerfeld and Klein [Sommerfeld 1898a] illustrate that subjects like stability, dealt with in a chapter of the second volume, could easily give rise to new debates and prevent rapid publication. Nevertheless, Sommerfeld completed the second volume without much delay, so that it appeared just a year after the first volume in 1898. As the first reactions made evident, the more subtle parts of the book such as the chapter on stability provoked criticism: "I would have a number of remarks about your definition of stability," Heinrich Burkhardt commented after the appearance of Volume II. "But I would need a day or two to formulate them clearly and precisely, which I do not now have. It seems to me that in your definition stability is the rule, instability the exception ... I tend to guess that all motions of the top are stable according to this definition, except those whose instability you have proven" [Burkhardt 1898].

Such reactions cautioned against rushing to publication—all the more because the plan for the remaining parts addressed subjects beyond the realm of mathematics proper: gyroscopic phenomena in geophysics, astronomy, and technology. Employing mathematical virtuosity in these fields was easier proclaimed than done. But Sommerfeld was not afraid to meet this challenge. For example, he corresponded extensively with the naval engineer Carl Diegel in Kiel, the German naval base, about the application of the theory to the gyroscopic guidance of torpedos [Diegel 1898]. In December of 1898 he wrote to Klein that "My letters to D.[iegel] have the extent of treatises. I may travel from Göttingen back to Cl.[austhal] via Kiel. In any case this correspondence gives rise to a nice paragraph about 'applications of theory in technology' " [Sommerfeld 1898b].

Since the autumn of 1897, Sommerfeld had been professor of mathematics at the mining academy in Clausthal, so that his communication with Klein had to occur via the exchange of letters and at occasional meetings, which further retarded the project. In addition, Klein persuaded Sommerfeld in 1898 to join his *Encyclopedia* project as an editor for the planned volumes on physics, a capacity that contributed to the derailment of Sommerfeld's career as a mathematician; he transformed more and more from a mathematician into a theoretical physicist, a metamorphosis that is also reflected by the choice of his research papers at the turn of the century. "Unfortunately I had no time for the top," he apologized in a letter to Klein in November 1899. "I have to get rid of my [paper on] X-rays before I can deal with something else" [Sommerfeld 1899]. Klein responded that he was "thinking of our top with silent sorrow" [Klein 1899]. In 1900, Sommerfeld exchanged his position as a professor of mathematics in Clausthal for a professorship of mechanics at the technical university in Aachen. Although this brought him into closer contact with technological applications of gyroscopic theory, it did not accelerate the publication of the pending volumes. "When I will have time to resume the top?" he responded to Klein's urging in November 1900. "The entire next week there are examinations without interruption. I will hope for the best but promise nothing" [Sommerfeld 1900].

To cut a long story short, it took five years after the appearance of Volume II in 1898 before the third volume of the *Theory of the Top* was published, and seven more years before the fourth and final volume appeared in 1910. In the meantime, gyroscopic theory itself had advanced or was made the subject of other reviews. In 1907, for example, Klein admonished Sommerfeld to pay attention to a recent article of Paul Stäckel, who was writing on the top for the *Encyclopedia* [Klein 1907]. In the foreword to the fourth volume, dated April 1910, Klein and Sommerfeld had to admit that during this long time span "the unity of substance and manner of presentation was lost." The loss of unity and coherence was caused not just by a turn from mathematical foundations to technological applications. Despite Sommerfeld's close contacts with technology, his presentation of applied subjects in Volumes III and IV was written from the perspective of a mathematician and theoretical physicist, so that it did not really address engineering concerns. With regard to the technology of the gyrocompass, for example, Sommerfeld admitted later that even in the part about the technical applications the text "nowhere addresses technical details" [Broelmann 2002, p. 138].

The *Theory of the Top*, therefore, is a strange monument of scientific literature from the turn of the nineteenth to the twentieth century: too heterogeneous to please one or another orientation, and yet outstanding in its scope and detail. Klein and Sommerfeld hoped that its versatility would be considered as a compensation for its "lack of systematicness." In the end, they confessed that the top was for them what it had been already for many natural philosophers in the nineteenth century: a target of opportunity for "awakening the sense for true mechanics," a "philosophical instrument."

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Deutsches Museum, Munich

Translators' Remarks

The ordinary difficulties of a translation are happily moderated, in our present case, by the extraordinary greatness of the original work and its authors. We could do no better than to preserve the form and notation of the text in a literal and unabridged rendering. Our notes are added separately, and are intended primarily to provide historical context. They are indicated by numerical superscripts; the footnotes of Klein and Sommerfeld are retained, as are their own supplementary notes to Volume I, which were published as addenda when Volume IV appeared in 1910.

More comments in this place are not necessary. It remains for us only to acknowledge the very fine editors at Birkhäuser Boston, and to return, with pleasure, to the preparation of Volume II.

FOREWORD

When F. K l e i n gave a two-hour lecture "On the Top" in the winter semester of 1895/96, he attempted, in the first place, to emphasize the direct and particularly English conception of mechanical problems, as opposed to the more abstract coloring of the German school, and, on the other hand, to make the particularly German methods of Riemann function theory fruitful in mechanics. The consideration of applications and physical reality would thus be outlined and forcefully advanced in a detailed example, but not yet carried out to full extent.

In the extensive printed edition originating from the pen of A. S o m m e r f e l d, interest in applications prevailed more and more, especially after his appointment to a teaching position in technical mechanics and later in physics. The astronomical, geophysical, and technical content added in this way required, in consequence, the necessity of a change, compared with the original lecture, in the mathematical point of view. While the approximation methods prepared in the first volumes (the method of small oscillations, the treatment of pseudo-regular precession) and the intuitive formulation of the principles of mechanics by means of the impulse concept were perfectly conformable to applications, the advanced function-theoretical methods, the exact representation of the motion by elliptic functions, etc., were later found to be dispensable. Thus, for example, the parameters α, β, γ, δ and their related quaternion quantities, whose geometric meaning was elaborated in the first volume and whose analytic importance was given special emphasis in the second volume, withdrew in the third and fourth volumes, naturally in complete agreement with K l e i n himself, whose interests had likewise turned more and more toward applications. In particular, the presentation of the technical top problems in the fourth volume used only the very simplest and most elementary law of top motion, which flows immediately from the concept of the impulse in the dynamics of rigid bodies, and which is briefly derived once more at the beginning of this volume.

We would not deny, that with the loss of the unity of time in the course of the fifteen years which have elapsed between the first plan and the present conclusion of the book, our work has also lost its unity of substance and manner of presentation; that what we often promised earlier with respect to the general, so-called analytic mechanics, especially in the advertisements of Vols. I and II, was later not kept; and that we have pursued many mathematical side roads which temporarily diverted us from our primary goal: the concrete understanding of dynamical problems. May the comprehensiveness of the content and the multiplicity of the engaged fields of interest be regarded as substitutes for the lack of systematicness and purposefulness of the presentation.

If we had to dispose of the collected subject material anew, we would probably present the actual mechanics of the top, including its applications, in a much smaller space, by pruning the analytic shoots that branch so joyously from the stem of mechanics. With this presentation we would address the large audience with scientific or technical interests in the theory of the top. The detailed analytic developments, which we would certainly not suppress simply on the basis of their special beauty, would be submitted in another presentation only to the more restricted mathematical circle. As for what pertains, finally, to the requirements of the completely unmathematical reader, and therefore to the difficult question of the popular explanation of the top phenomena, we have taken an extensively grounded critical position in the second volume, and at the beginning of the fourth volume have again pointed out the somewhat long but, it appears to us, only passable way that begins from the general impulse theorems of rigid body dynamics. The impulse theorems are either systematically developed from particle mechanics, or, should the occasion arise, illustrated only by experiments, and then postulated axiomatically; on the basis of these theorems, all the partly paradoxical facts of the theory of the top may be understood qualitatively as well-defined approximations, and their domains of validity delimited without want of clarity.

The top is suitable above all other mechanical devices for awakening the sense for true mechanics. May it, in the presentation of our book, serve this purpose in elevated measure, and thus prove worthy in the future of the honorable surname formerly bestowed upon it by Sir John Herschel,[1] the name of a philosophical instrument!

Göttingen and München, April 1910.

F. KLEIN. A. SOMMERFELD.

Advertisement of the Book

(from the notices of B. G. Teubner publishing company in Leipzig).

The work owes its origin to a lecture given by Prof. Klein during the winter semester of 1895/96 in Göttingen University. The elaboration of the ideas set forth in the lecture and the rounding out of the subject matter have since been the primary responsibility of Dr. Sommerfeld.

The first part, which appears in July of this year, presents, after a preparatory chapter of kinematic content, the fundamental considerations on the principles of mechanics, in so far as they apply to the present topic. A singular character of this section is the authors' frequent return, in the spirit of the older writers, to impact forces, and, throughout, to the concept of the "impulse" (W. Thomson's terminology; Poinsot's couple d'impulsion); that is, the impact turning-force that is able to produce the actual motion instantaneously from rest. The theory of the top, and the mechanics of rigid bodies in general, thus acquire a higher degree of clarity and simplicity than that obtained by the exclusive use of continuously applied forces.

The second part treats in detail of the mathematical side of the theory, the explicit representation of the motion of the heavy top by means of elliptic functions. It is shown here that neither the commonly used asymmetric Euler angles nor the symmetric Euler parameters (quaternion quantities), but rather certain parameters arising from Riemann function theory, are the simplest building stones, in analytic respects, from which the general formulas for the motion of the top may be composed.

The third part contains, in addition to many supplements to the previous material (consideration of friction at the support point, criticism of the popular top literature, etc.), the manifold applications of the theory to astronomical and physical questions. The accumulated treasures of the English literature, and especially the Natural Philosophy of Thomson and Tait, are of particular value here in presenting the investigations of cyclic systems, gyrostats, etc., to the German public in a convenient readable form.

Originally conceived as a dedication to the Association for the Advancement of Mathematical and Scientific Education,[2] the book should also be understandable without difficulty to the more advanced research mathematician and physicist. Specific prior knowledge of analytic mechanics or function theory is not assumed. It is hoped, however, that the specialized mathematical circle will thus feel, without displeasure, a certain breadth and comfort in the presentation.

The tendency of the book may be characterized, finally, by a few sentences taken from the Introduction:

"The development of theoretical mechanics has taken, especially in Germany, an overly exclusive direction toward abstraction and formulas, which often detracts from a direct understanding. The student who learns well to derive the general principles of mechanics analytically does not always grasp their true mechanical meaning in a sufficiently lively sense, and often appears awkward when faced with obtaining the solution of a specific problem.

"We wish to oppose this recent and rising evil by a thorough treatment of our problem. We wish to establish not only a knowledge of mechanics, but also, so to speak, a feeling for it. Full clarity in the geometric aspects of motion is naturally a first prerequisite for this. ... Still more important for us, however, is full clarity concerning the forces that come into play as the mechanical causes of the motion. We will convey these forces as concretely as possible in space by vectors; we place special value on the development and consistent use of the impulse principle, etc. ... We do not intend, however, to minimize in any way the analytical side of our problem. The formula is ultimately the simplest and most concise description of the process of motion; it is indispensable, moreover, as the basis of actual numerical calculation. We will only demand, that our knowledge of mechanics be based not on formulas, but rather, on the contrary, that the analytic formulation appear of itself as the last consequence of a fundamental understanding of the mechanical principles."

Volume I

Introduction to the Kinematics and Kinetics of the Top

Introduction

We are obliged, in fairness, to establish at the beginning of this lecture what we mean by the word top, and what we do not wish to mean by this word.

By a top we mean — reserving a later generalization of the concept — *a rigid body subject to gravity, whose mass is symmetrically distributed around an axis of the body, and which, by means of an appropriate device, is fixed in space at one point of the symmetry axis.*

We denote the fixed point of the body as the *support point O*; this point divides the symmetry axis into two half-lines. We designate one of these half-lines, by an arbitrary choice, as the *figure axis*. The plane perpendicular to the figure axis through the point O is called the *equatorial plane of the top.*

The model illustrated below, which was made by the particularly noteworthy French engineer and experimentalist R o z é,[3] represents a top in the given sense, and will serve repeatedly in these lectures for the purpose of demonstration.

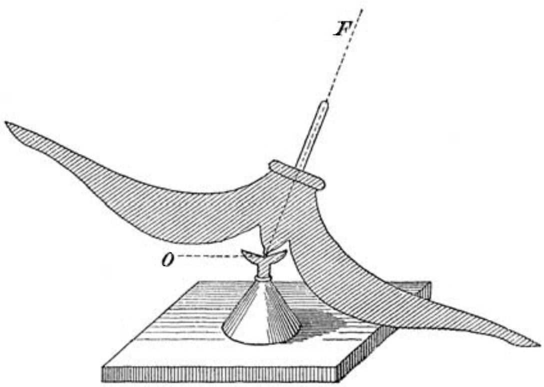

Fig. 1.

Our figure gives only a cut through a meridian of the bell-shaped form of the top; to obtain a spatial image of the top, we must imagine the drawing rotated about the figure axis OF. The lower end of the figure axis rests at O in a seat fixed to the pedestal,

R.J. Nagem, G. Sandri, *The Theory of the Top*, DOI: 10.1007/978-0-8176-4721-6_0,

so that it is essentially stationary during the motion. Because of the characteristic shape of the top, its center of gravity lies directly beneath the support point when the figure axis is vertical, so that the body is in stable equilibrium in this position. If the opposite is desired, this is effected by the addition of weights that are themselves rotating bodies placed axially on the upper part of the figure axis. A special feature of the R o z é top is the ingenious mechanism by which it is possible to give the top a lively rotation without damaging the sharp tip in which the support point terminates.[4] All parts of the model are made of solid metal. We recognize in our example the given properties of the top concept: *the rotational symmetry about the figure axis, the fixed position of one of its points, and the rigidity of the material.*

In contrast, the well-known child's toy commonly called a top does not, strictly speaking, represent a top in the given sense of the word, since the support point is not fixed in space, but rather is free to move in the horizontal plane of the surface of the Earth. Our mechanical analysis confronts us here in a more complicated form. For orientation, it may be remarked now that the complete analytic treatment of the *top with a moving support point* leads to *hyperelliptic functions*, while the general motion of the *top with a fixed support point* is represented by *elliptic functions.* In our lectures, we will be able to consider the more complicated problem only in an appendix.

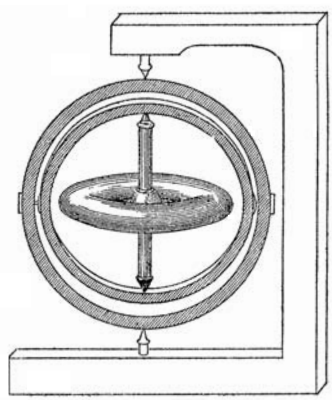

Fig. 2.

Our concept of a top applies even less, strictly speaking, to the apparatus represented in the adjacent figure, which is usually called a *Bohnenberger machine*[5] (or also a *Foucault gyroscope*[6]). The primary component of the apparatus is a rotor whose axis is freely mounted in an inner ring. The inner ring is free to move about an axis that is perpendicular to the axis of the rotor, and has its bearing in an outer ring. This outer ring is pivoted, in turn, about an axis perpendicular to the axis of the inner ring. By this mechanism, one point (the middle point of the rotor) does indeed remain fixed in space during the motion of the system. The apparatus is not, however, a single rigid body, since the rings can move relative to the rotor; moreover, the rotational symmetry about the axis of the rotor is also destroyed by the masses of the rings. If we wish, nevertheless, to draw upon this apparatus occasionally as an example for our analysis of the top, then we must add the explicit assumption that the mass of the rotor is very large compared to the masses of the inner and outer rings, and must therefore neglect the latter compared to the former. We have then to consider in the mechanical treatment of the apparatus only the rotor, which represents a single rigid rotating body. We see, however, that without this simplifying assumption, the theory of the apparatus will be considerably more complicated than that of our top.

Indeed, it is self-evident that no real body whatsoever corresponds, strictly speaking, to the initially given definition. It is not possible to completely fix a material point in space by means of a mechanical apparatus, nor does there exist anywhere in nature an absolutely rigid body. But it is equally self-evident that we can never do complete justice to reality with our analysis. Mathematics always treats of idealized problems; we must constantly simplify reality considerably through the abstraction of all types of secondary circumstances before we can think of its mathematical treatment.

In this regard, there arises the question of how well actual phenomena may coincide with our idealized mathematical models. To reach a conclusion here, one will seek to determine the individual influences of the circumstances not included in the calculation, and will add the resulting deviations as correction terms to the solution of the idealized problem. In this spirit, we will later investigate the friction of the top in the supporting seat; we will also take into consideration, at least qualitatively, the elasticity of the base that bears the top. An entire series of other circumstances—the elasticity of the top material itself, the entrainment of the surrounding air, etc., etc.—will, being of lesser importance and entirely too complicated, remain excluded from consideration.

Perhaps it is necessary to explain why we single out, from the abundance of problems in mechanics, such a special subject as the motion of the top.

First, the top offers a particular interest in itself. Its motions are in some respects very well known, and yet contain paradoxes and apparent contradictions to general mechanical principles. To resolve these contradictions is an attractive exercise from the point of view of mechanics. The interest that our topic may claim is witnessed by the numerous old and new monographs on the subject. (Cf. the following literature review.) In addition, the top plays an important role in the neighboring fields of astronomy and theoretical physics. A special study of our problem appears to be indicated on these grounds as well. Finally, the theory of the top is, at least historically, of singular interest from the standpoint of pure mathematics. Indeed, it was our problem, with reference to the included problem of the oscillation of the pendulum, that motivated the early development of the theory of elliptic functions.

In the following, however, we consider the top as an example of general mechanics, and hope to enliven the comprehension of the general directly through the presentation of the particular. The development of mechanics has taken, especially in Germany, an overly exclusive direction toward abstraction and formulas, which often detracts from a direct understanding. The student who learns well to derive the general principles of mechanics analytically does not always grasp their true mechanical meaning in a sufficiently lively sense, and appears awkward when faced with obtaining the solution of a specific problem.

We wish to oppose this evil state of affairs by a thorough treatment of our problem. We wish to establish not only a knowledge of mechanics, but also, so to speak, a feeling for it. Full clarity in the geometric aspects of motion is naturally a first prerequisite for this. We thus propose to enliven the geometric perception through numerous figures—in contrast to Lagrange, the greatest advocate of the abstract direction in mechanics, who stressed with special predilection that in his Analytic Mechanics not one figure is to be found.[7] Still more important for us, however, is full clarity concerning the forces that come into play as the mechanical causes of the motion. We will convey these forces as concretely as possible in space by vectors; we place special value on the development and consistent use of the impulse principle, in which we conceive the impact force, or the system of impact forces, that is able to produce the actual motion instantaneously from rest. We do not intend, however, to minimize in any way the analytical side of our problem. The formula is ultimately the simplest and most concise description of the process of motion; it is indispensable, moreover, as the basis of actual numerical calculation. We will only demand, that our knowledge of mechanics be based not on formulas, but rather, on the contrary, that the analytic formulation appear of itself as the last consequence of a fundamental understanding of the mechanical principles.

The tendency expressed here for mechanical conception over formulas is particularly prominent in the English textbooks. We naturally cite, in the first place, the ingenious work of Thomson and Tait, the Treatise on Natural Philosophy*),[8] and, further, the work of Routh**),[9] which, while generally not sufficiently well known in Germany, may be more appropriate as a textbook, since it is more systematically worked through and not as difficult to understand. In addition to the widespread French textbooks***), the presentations of Voigt†)[11] and Budde††)[12] are particularly suitable for us, while the famous *Mechanik* by Kirchhoff[13] appears one-sidedly systematic and too abstract. The demands advocated here were raised first and most forcefully by Poinsot, and particularly in the context of our rotation problem.[14] We will cultivate with special pleasure the beautiful method of Poinsot in these lectures. We do not wish, however, to go as far as Poinsot, who banishes coordinate calculations from his considerations as much as possible, and thus closes his access to the more difficult problems. We prefer to regard the considerations of Poinsot only as a

*) Only the first edition of this work (published in Cambridge, 1867) is translated at the present time into German (Braunschweig 1871); the second edition, which appeared in 1883–86 in two parts, is much more comprehensive; citations in the following always refer to this second edition.

**) Dynamics of a system of rigid bodies, 2 Vols., 5th ed., London 1891; a German translation by B. G. Teubner will appear soon.

***) By Duhamel, Despeyrous-Darboux, Appell, etc.[10]

†) Voigt, Elementare Mechanik, Leipzig 1891.

††) Budde, Allegemeine Mechanik der Punkte und starren Systeme, 2 Bde., Berlin 1890.

first and very valuable introduction to the theory of rotation problems, wishing, however, to complete them through analysis where they alone cease to lead to the goal or become too complicated. Thus we will begin the first chapter, for example, with geometric investigations after the model of Poinsot, but soon go over to analytic considerations.

We cannot, naturally, give a systematic development of mechanics in these lectures; we must assume a general knowledge of the broad subject. In the same way, we will not be able to forgo a certain familiarity with the methods of function theory. Nevertheless, the mechanical as well as the function-theoretical concepts are explained briefly when they appear in the example of our top, so that these lectures can also serve as a first orienting introduction to the field of elliptic functions and to the higher regions of the mechanics of more general systems.

Finally, a word about the treatment of the infinitesimal calculus. We do not, by any means, intend to proceed in this presentation with the rigor in the infinitesimal calculus that is possible and often customary today. We will, rather, make use of all the simplifications that come with the use of infinitesimal quantities and the interchange of limit processes. The meaning of the modern sharpening of the infinitesimal calculus is obviously not that one should be obstructed at every step by doubts on these matters, but rather that one should dispatch these doubts once and for all to the foundation of the subject, so as to be able to proceed afterward without hesitation. Whoever knows the more exact methods of differential calculus will always be easily able to add a somewhat desirable increase of precision to our manner of expression. We omit this not because of any difficulty, but rather because we would thus make the presentation unnecessarily slow, and divert attention from the true difficulties of the problem.

Chapter I

The kinematics of the top

§1. Geometric treatment of the kinematics

We begin with a chapter of geometric content that treats of the *kinematics* of the top. In opposition to kinematics, we use, after the suggestion of Thomson and Tait, the word *kinetics*.[15] While kinematics operates merely with space and time and investigates motions only according to their geometric possibility, kinetics adds the concepts of mass and force, and treats of motions with regard to their mechanical possibility.

Among the properties of the top postulated in the Introduction, only the rigidity of the material and the fixed position of the support point come into consideration here, since the mass distribution of the body is completely irrelevant in kinematics. The following investigations are therefore valid for an *arbitrary rigid body with a fixed support point. We wish to denote such a body as a "generalized top," in contrast to the "symmetric top" defined in the Introduction.* In the immediately following chapters we also refer, on occasion, to this "generalized top," while in the later chapters we must limit ourselves entirely to the "symmetric top."

From the most general point of view, problems of kinematics are classified according to the *number of degrees of freedom.* The meaning of this expression, also introduced by Thomson and Tait, will be illustrated through the following small tabulation.

A freely moving point in space has *three degrees of freedom* (its position is determined by three independent coordinates).

A freely moving rigid body in space has *six degrees of freedom* (the position and orientation of the body are fixed through the specification of six appropriate independent parameters).

A rigid body with one point held fixed has again *three degrees of freedom.*

R.J. Nagem, G. Sandri, *The Theory of the Top*, DOI: 10.1007/978-0-8176-4721-6_I,

Our *top with a fixed point*, correspondingly, has *three degrees of freedom*, in so far as we may treat it as a rigid body. The *moving top* whose support point runs in a horizontal plane has *five degrees of freedom.* We can also construct a top with *one* or *two* degrees of freedom if we place the figure axis in a fixed frame or in a ring that rotates, in turn, about a fixed axis. On the other hand, our top has *infinitely many degrees of freedom* as soon as we wish to consider the *elastic deformation of the material.*

In the following considerations, we will begin with a freely moving rigid body in space, of which the top with a fixed support point is a special case. Since this is a very simple and well-known subject, it is enough to recall the relevant theorems briefly, without deriving them in detail. The proofs may be found, if necessary, in the previously cited textbooks.

We consider two different positions of a moving rigid body, and pose the problem of giving the motion that leads from the initial position to the final position in the simplest way. The position of the body is completely determined if the positions of any three of its points, say O, P, Q, are known. The initial positions of the points may be denoted by O_1, P_1, Q_1, and the final positions by O_2, P_2, Q_2. We can first transport the point O_1 to O_2 by a parallel displacement of the body; the points P_1, Q_1 are thus transformed into P_1', Q_1', respectively. We then connect P_1' with P_2 and Q_1' with Q_2, and construct at the center of the connecting lines the normal planes to the lines. These planes intersect in an axis that passes through O_2. We now rotate the body about this axis through an appropriate angle, so that P_1' is brought to P_2 and Q_1' is brought to Q_2. We can therefore transport the triad OPQ, and thus also the rigid body, from its initial position to its final position through a combination of a parallel displacement and a rotation.[16] Thus the theorem:

The most general change of position of a freely moving rigid body can always be replaced by a combination of a rotation and a translation.

If we take into consideration, further, that a parallel displacement is equivalent to a rotation about an infinitely distant axis, or a so-called rotation-pair (that is, two rotations about parallel axes with the same rotation angle but opposite sense[17]), then we can also give the previous theorem in the following form, which is of interest with regard to the corresponding theorem for the statics of a rigid system:[18]

The most general change of position of a rigid body can be replaced by a single rotation and a rotation-pair.

Our construction can obviously be changed in a great variety of ways by replacing the initially chosen point O with some other point. We may designate the chosen point O as the "reference point," and can ask whether we can simplify the result of our construction through an appropriate choice of the reference point. In this respect, it results that one can always choose the reference point so that the direction of the translation and the axis of rotation are parallel. The combination of a rotation and a parallel displacement along the rotation axis is commonly called a *screw* (more precisely, a "motion-screw")*). The magnitude of the parallel displacement together with the magnitude of the rotation determine the pitch; the magnitude, axis, and sense of the rotation give the rotation angle, the rotation axis, and the rotation sense of the screw.[19] We can thus say:

The most general change of position of a rigid body can, by appropriate choice of the reference point, be replaced by a screw with a specific axis, a specific rotation angle and rotation sense, and a specific pitch.

Our screw-motion naturally coincides with the actual motion of the body only in the initial and final positions; the intermediate positions of the actual motion can be entirely different from the intermediate positions of the imagined screw-motion. Let us consider, however, an infinitesimal motion of the rigid body (that is, the limiting case of a finite motion during an infinitely diminished time interval) and the corresponding infinitesimal screw (that is, the limiting case of the corresponding finite screw-motion). Here we can no longer speak of intermediate states; consequently, we will regard an infinitesimal motion as directly identical to the constructed screw-motion and can state concisely:

Every infinitesimal motion of a rigid body is a screw-motion.

*) Cf. Sir Robert Ball. The theory of screws. Dublin 1876. (German edition by Gravelius, Berlin 1889).

In the following, we will characterize an infinitesimal screw not by its (infinitesimal) rotation angle, but rather by its (assumed to be finite) rotational velocity.

We now enter into the special circumstances of our top (that is, naturally, of the generalized top). Here we will place the reference point O at the fixed support point. The pitch of the screw is then zero; the screw-motion becomes a simple rotation about an axis passing through O. We thus have the theorems:

An arbitrary motion of our top can be replaced, with respect to its final result, by a rotation with a specific axis, a specific rotation angle, and a specific rotation sense;

and

Every instantaneous (infinitesimal) motion of the top is a rotation with a specific axis, a specific rotational velocity, and a specific sense.

There may next follow some remarks about the composition of rotations, in which we need consider, with respect to the top, only rotations whose axes pass through O. We suppose that the top is given two successive finite rotations. According to the previous theorem, we can effect the result of these two rotations by a single rotation. We obtain the properties of this single rotation from the following theorem:*) if we rotate space successively about the three edges of a three-sided corner, each rotation through twice the corresponding edge angle, then we return to the initial position.[20] This theorem yields the following construction for the single resultant of two given rotations: we place a unit sphere around O, connect its intersection points with the axes of the individual given rotations by a great circle, and apply to this great circle the *half*-angles of the respective individual rotations. *The third corner of the resulting spherical triangle then gives the axis of the resultant rotation, and the adjacent exterior angle gives the half-angle of the resultant rotation.*[21] This construction operates in a remarkable way with the half-angles of rotation, so that the value of the half-angle of the rotation that follows from the construction is determined up to an additive multiple of 2π (that is, the value of the entire angle of the rotation is determined modulo 4π).

We now speak of infinitesimal rotations or rotational velocities. As usual, we assign the infinitesimal rotation a geometric representation

*) Cf. Schell: *Theorie der Bewegung*, Leipzig 1879 II. Teil, Kap. II, §9. The theorem plays a large role in Hamilton's *Lectures on quaternions* (art. 217 and ff.).

by the following procedure: we extend from O, along the axis of rotation, a line segment that represents the magnitude of the rotational velocity, and extends, in particular, in the direction from which the rotation appears to occur in the clockwise sense. We call the resulting geometric counterpart of the infinitesimal rotation a *rotation vector*. If this rotation vector is known, then the axis, velocity, and sense of the infinitesimal rotation follow in an unambiguous way.

We have only to add a convention regarding the unit of measure with which we extend the line, and the system of units with which we wish to measure the angular velocity. It is simplest, here and throughout the following, to adopt the so-called *"absolute system of measure,"* and thus to measure length in centimeters and time in seconds. An angular velocity will always be expressed in arc measure; thus, for example, by the arc of a circle, measured in cm, that a point extending 1 cm from the rotation axis would describe during one second of uniform rotation. In the absolute system of measure, every rotation has, in this sense, a specific numerical value, say n. We determine our representative line segment by extending n cm directly on the rotation axis in the manner given above.

The relevant theorem for the composition of two infinitesimal rotations is now simply:

Two infinitesimal rotations are composed according to the parallelogram law of forces; that is, the corresponding rotation vectors add geometrically (as line segments or vectors).[22]

This fundamental and very well known theorem justifies after the fact the introduction of the word rotation *vector*, and shows, moreover, that the resultant of two infinitesimal rotations is *independent* of their order, and that *infinitesimal rotations are thus interchangeable operations*. For the proof of this it is enough to consider the figure of the parallelogram. We remark that, in contrast, the resultant of two finite rotations changes if we reverse the order of the two rotations, and thus *finite rotations are not interchangeable*. The proof follows from the construction that is indicated on page 10, in which the defining elements of the two rotations are used in an asymmetric manner.[23]

We now consider the moving top in an entire series of different positions, therefore taking a first, second, third, ... of these positions into view. We replace the motions that lead from the first position to the second, from the second to the third, etc., by single rotations, and thus obtain a series of different rotation axes passing through O. Here we distinguish, as usual in kinematics, a *moving frame* and a *fixed frame*. The moving frame is our top, and the fixed frame is ideal space.

We remark, further, that the distinction between one frame and the other that lies in the words "moving" and "fixed" is actually unjustified from the standpoint of pure kinematics, and that it would be more correct to speak, for example, of a *first* and a *second* frame. Namely, every motion is as equally valid geometrically as its inversion, in which the roles of the moving and fixed frames are interchanged. *Kinematics*, therefore, always treats only of *relative motion*. In *kinetics* it is different. The necessary forces for the generation of a motion change completely if we interchange the moving frame and the fixed frame. In kinetics, therefore, the *direct* and the *inverse motions* have, in general, completely different characters. We will later emphasize an exception to this rule, when we become acquainted with the theorem that the inversion of the motion of the top has, under special circumstances, the same kinetic character as the direct motion.

We wish to mark, in the moving frame and in the fixed frame, the positions of the axes of the above rotations that bring the top from the first position to the second, from the second position to the third, etc. We thus obtain, if we join the successive axes by planes, a *pyramid*[24] fixed in space and a *pyramid* fixed in the top, with equal respective side angles. In the first rotation, the respective first edges of the two pyramids coincide. The moving frame turns around this edge until the second edges become coincident. In the second rotation, the moving frame turns around the second edge. The magnitude of the rotation is such that the respective third edges must coincide at the end of this rotation. So it continues. We can describe the entire rotation process succinctly in the following manner:

The moving pyramid rolls on the fixed pyramid.

This motion must, naturally, necessarily coincide with the actual motion of the top only at each of the final positions of the individual rotations. The intermediate positions can be very different in the two cases. To attain in this manner a complete reproduction of the actual