Progress in Mathematics
Volume 243

Studies in Lie Theory

Dedicated to A. Joseph
on his Sixtieth Birthday

Joseph Bernstein
Vladimir Hinich
Anna Melnikov
Editors

Birkhäuser
Boston • Basel • Berlin

Joseph Bernstein
Tel Aviv University
School of Mathematical Sciences
Ramat Aviv, Tel Aviv 69978
Israel

Vladimir Hinich
Anna Melnikov
University of Haifa
Department of Mathematics
Mount Carmel, Haifa 31905
Israel

Mathematics Subject Classifications (2000): 12A50, 13N10, 14L30, 14R20, 16K40, 16S30, 16S32, 16S36, 16S40, 16W35, 16W50, 17B35, 17B37, 17B56, 17B65, 17B69, 20C08, 20G10, 22D20, 22E35, 22E50, 32C38, 33C45, 35A27, 53D17

Library of Congress Control Number: 2005936729

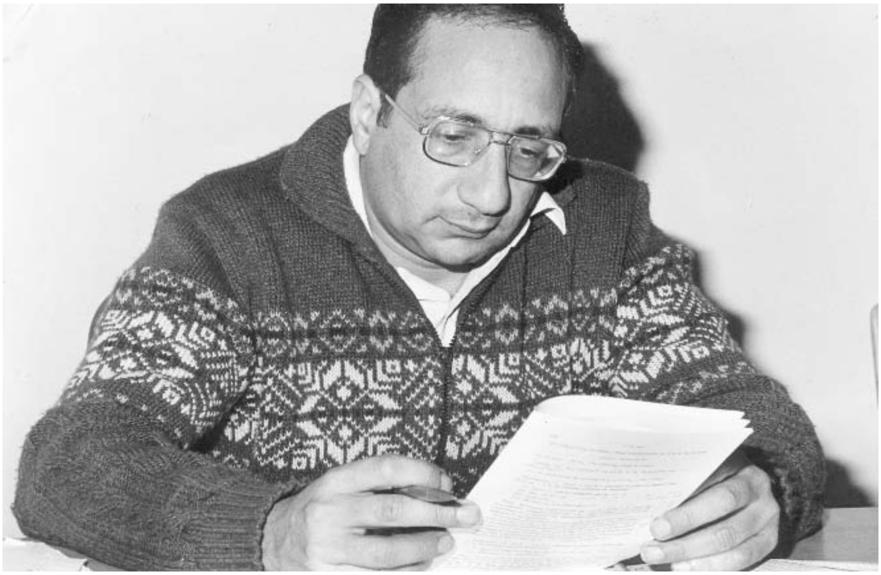
ISBN-10: 0-8176-4342-7   e-ISBN: 0-8176-4478-4
ISBN-13: 978-0-8176-4342-3

Printed on acid-free paper.

Printed in the United States of America.   (KeS/EB)

9 8 7 6 5 4 3 2 1

www.birkhauser.com


Anthony Joseph (shortly after joining the Weizmann Institute)


With Denise at Zichron Ya'acov, June 2002

Participants of the workshop on Representations of Lie algebras in honour of Anthony Joseph

First row from left to right: M. P. Malliavin, F. Fauquant-Millet, A. Braverman, G. Letzter, D. Joseph, A. Joseph, A. Melnikov, L. Makar-Limanov, V. Berkovich, V. Hinich;

Second row from left to right: J. Beck, D. Gaitsgory, D. Kazhdan, S. Arkhipov, O. Gabber, D. Todoric, R. Rentschler, N. Shomron, A. Elashvili, D. Vogan, M. Duflo, A. Stolin, G. Perets;

Third row from left to right: Ya. Greenstein, N. Papalexiou, C. Stroppel, B. Kostant, A. Kostant, I. Mirkovic, M. Finkelberg, V. Ginzburg, Yu. Bazlov, A. Retakh;

Last row from left to right: Sh. Zelikson, A. Juhasz, V. Vologodsky, B. Noyvert, E. Perelman, A. Vershik, R. Bezrukavnikov, M. Losik, J. Alev, V. Ostapenko, S. Khoroshkin, A. Reznikov.

Contents

Preface

This volume is dedicated to Anthony Joseph on the occasion of his 60th birthday. A conference entitled *Representations of Lie Algebras* was held in his honour at the Weizmann Institute, Rehovot, in July 2002. Subsequently, distinguished experts in representation theory and related areas were invited to contribute survey and research articles, which comprise this volume.

The focus here is on semisimple Lie algebras and quantum groups, the central subjects in representation theory to which the contribution of Tony Joseph is difficult to overestimate. For over three decades the impact of his work has been seminal and has changed the face of the subject.

The introductory part of the volume consists of a short note by Jacques Dixmier describing the beginnings of Tony's entry into mathematics, followed by the speech of Denise at the dinner honouring her husband. From Denise, the participants got a glimpse into another side of Tony's personality.

The scientific part of the volume begins with two surveys which give an overview of the central topics in representation theory to which Tony Joseph made his mark: the first, written by W. McGovern, describes Joseph's main input into the theory of primitive ideals in semisimple Lie algebras; the second, coauthored by D. Farkas and G. Letzter, is devoted to the study made by A. Joseph of quantized enveloping algebras. Thereafter, 16 research articles cover a number of different topics in representation theory.

J. Alev and F. Dumas study the invariants of the Weyl skew field $D_n(k)$ under the action of a subgroup G of $GL_n(k)$. The authors proved in some cases that the skew field of invariants is again a Weyl skew field $D_m(K)$ where K is a purely transcendental extension of k.

A. Beilinson presents a "spectral decomposition" of the category of Heisenberg modules (i.e., modules over a Heisenberg extension of a commutative Lie algebra of formal loops on a torus T). The "spectral parameters" form the moduli stack of $T^\vee$-local systems on Spec $k((t))$.

A. Braverman and P. Etingof study a generating function defined by certain equivariant integrals along a moduli space of framed G-bundles on P^2. They prove the conjecture of Nikita Nekrasov claiming that the leading term of asymptotics of this generating function is given by the instanton part of the Seiberg–Witten prepotential of the affine Toda system associated to the Langlands dual Lie algebra.

The paper of I. Cherednik is devoted to the study of polynomial representations of double affine Hecke algebras. It is proved that the quotient of such a representation by the radical of the duality pairing is irreducible if it is finite dimensional.

D. Gaitsgory and D. Kazhdan study representations of groups over two-dimensional local fields. Here it makes sense to consider representations in pro-vector spaces and work with the central extension of the original group. The authors construct the functor of "semiinfinite invariants" which pairs representations corresponding to two levels that sum up to the critical level.

A. Joseph studies a 20-year-old conjecture claiming that the tensor product of a Demazure module with a one-dimensional Demazure module admits a Demazure flag. This conjecture had already been proved for finite type. The present paper proves the result for the quantized universal enveloping algebra of a Kac–Moody Lie algebra with a simply-laced symmetrizable Cartan matrix in any characteristic.

M. Kashiwara, P. Schapira with their coauthors F. Ivorra and I. Waschkies construct a microlocalization functor

$$\mu_X : D^b(I(K_X)) \to D^b(I(K_{T^*X}))$$

from the derived category of Ind-sheaves of vector spaces on a C^∞-manifold X to the similar category constructed for the cotangent bundle of X. The classical microlocalization is expressed now as $\mu hom(A, B) = \mathrm{R}\mathcal{H}om(\mu_X(A), \mu_X(B))$.

D. Kazhdan and Y. Varshavsky study the endoscopic decomposition for supercuspidal level zero representations of a reductive group over a local nonarchimedean field.

A. Kirillov and L. Rybnikov introduce and study "odd analogues" of a special family of algebras ("family algebras") defined earlier by A. Kirillov.

B. Kostant and W. Wallach study a Poisson analog $J(n)$ of the Gelfand–Zetlin algebra. This is a maximal Poisson-commutative subalgebra of the algebra of polynomial functions on $\mathfrak{g} = M_n(\mathbb{C})$.

In the paper of S. Kumar and K.-H. Neeb the authors study a connection between the cohomology of an algebraic group with that of its Lie algebra. They prove an analog of the Van Est theorem and also study extensions of an algebraic group by an abelian algebraic group.

T. Levasseur and T. Stafford study the ring $D(X)$ of global differential operators on the "basic affine space" $X = G/U$ where G is a complex semisimple Lie group and U is a maximal unipotent subgroup. They prove that the cohomology $H(X, O_X)$ considered as a $D(X)$-module decomposes into a sum of non-isomorphic simple $D(X)$-modules indexed by the elements of the Weyl group.

G. Lusztig proves a remarkable identity in the Hecke algebra of type A generalizing an identity of Wallach in the group ring of the symmetric group.

L. Makar-Limanov studies centralizers of elements in a quantum space which is the $\mathbb{C}$-algebra generated by $x_1, \ldots, x_n$ subject to relations $x_i x_j = q_{ij} x_j x_i$ for $i < j$. In case the coefficients q_{ij} are "in general position," the centralizer of any non-constant element is a subalgebra of a polynomial ring in one variable. An example shows that the centralizer need not be integrally closed and that there is no upper bound on the number of generators of the centralizer.

M. Nazarov and A. Sergeev present a centralizer construction for the Yangian of the queer Lie superalgebra $\mathfrak{q}(N)$.

In his linguistically refreshing paper V. Schechtman presents a new proof of the theorem claiming that the vertex algebroid structures on a Lie algebroid T form a gerbe whose class coincides with the Chern–de Rham class of T. This had been proved earlier in a recent *Inventiones* paper by Gorbunov, Malikov and Schechtman.

Acknowledgments: The conference was supported by the Arthur and Rochelle Belfer Institute of Mathematics and Computer Science, the Maurice and Gabriella Goldshleger Conference Foundation, the Albert Einstein Minerva Center for Theoretical Physics at the Weizmann Institute of Science. Further support came from the TMR programme "Algebraic Lie Representations" of the European Union, and Minerva foundation, Germany.

The Editors would like to thank W. McGovern, D. Farkas and G. Letzter for their overview papers as well as all the authors of the research papers. The Editors further extend their thanks to Ann Kostant of Birkhäuser for her personal involvement in the project. Special thanks are due to Raanan Michael, the administrator of the Faculty of Mathematics and Computer Science of the Weizmann Institute, and his secretaries Michele Bensimon and Meira Hadar for being the most skillful, devoted and efficient team in all the organizational matters of the conference.

J. Bernstein Rehovot, June 2005
V. Hinich
A. Melnikov
Editors

Publications of Anthony Joseph

1. (With C.A. Coulson), Spheroidal wave functions for the hydrogen atom, *Proc. Phys. Soc.*, **90** (1967), 887–893.
2. (With C.A. Coulson), A constant of the motion for the two-centre Kepler problem, *Int. J. Quant. Chem.*, **1** (1967), 337–347.
3. The influence of the mass on the bound states of molecular systems, *Proc. Phys. Soc.*, **91** (1967), 574–576.
4. The theory of local degeneracy, *Int. J. Quant. Chem.*, **1** (1967), 535–559.
5. On the determination of the exact number of bound states of a given potential, *Int. J. Quant. Chem.*, **1** (1967), 615–629.
6. The spin dipole, *Proc. Phys. Soc.*, **92** (1967), 847–852.
7. Self-adjoint ladder operators I, *Rev. Mod. Phys.*, **99** (1967), 829–837.
8. (With C.A. Coulson), Self-adjoint operators II, *Rev. Mod. Phys.*, **39** (1967), 838–849.
9. Self-adjoint ladder operators III, *Rev. Mod. Phys.*, **40** (1968), 845–871.
10. The theory of conditional invariance, *Proc. Roy. Soc.*, **A305** (1968), 405–427.
11. The hemispherical box: An example of virtual symmetry, *J. of Phys.*, **A2** (1969), 719.
12. (With A.I. Solomon), Global and infinitesimal non-linear chiral transformations, *J. Math. Phys.*, **11** (1970), 748.
13. Derivations of Lie brackets and canonical quantisation, *Commun. Math. Phys.*, **17** (1970), 210–232.
14. A second anticommutant theorem for symmetric ternary algebras, *Proc. Camb. Phil. Soc.*, **69** (1971), 25–52.
15. (With M.A. Jacobs, S. Nussinov and A.A. Rangwala), An improved upper bound on the imaginary part of elastic scattering, *Phys. Rev.*, **D2** (1970), 1970–1974.
16. A classification of local current algebras, *Commun. Math. Phys.*, **19** (1970), 106–118.
17. (With W.G. Sullivan), A commutant property of symmetric ternary algebras, *Proc. Camb. Phil. Soc.*, **70** (1971), 1–4.
18. Commuting polynomials in quantum canonical operators and realizations of Lie algebras, *J. Math. Phys.*, **13** (1972), 351–357.

19. Magnetic spin monopole from the Yang-Mills field, *Phys. Rev.*, **D5** (1972), 313–320.
20. Realizations of Lie algebras from the canonical anticommutation relations, *Nuovo Cimento*, **8A** (1972), 217–234.
21. On the dynamical solution to the Sugawara model, *Nucl. Phys.*, **B42** (1972), 418–436.
22. On the dynamical solution to the Sugawara model: II The Lie identities, *Ann. of Phys.*, **73** (1972), 417–438.
23. Recovery of the Lie identities of field theory from minimal conditions, *Nucl. Phys.*, **B43** (1972), 107–118.
24. An inequality relating cross sections to polarization from isospin conservation, *Phys. Letters*, **42B** (1972), 368.
25. Gelfand-Kirillov dimension for algebras associated with the Weyl algebra, *Ann. de l'Institut Henri Poincaré*, **17** (1972), 325–336.
26. Combinatorial analysis of the Lie identities in field theory, in *Statistical Mechanics and Field Theory*. Lectures given at the 1971 Haifa Summer School (Eds. R.N. Sen and C. Weil, Keter Publishing House, Jerusalem).
27. The Weyl algebra – semisimple and nilpotent elements, *Am. J. Math.*, **97** (1975), 597–615.
28. A characterization theorem for realizations of $s\ell(2)$, *Proc. Camb. Phil. Soc.*, **75** (1974), 119–131.
29. Proof of the Gelfand-Kirillov conjecture for solvable Lie algebras, *Proc. Amer. Math. Soc.*, **45** (1974), 1–10.
30. Symplectic structure in the enveloping algebra of a Lie algebra, *Bull. Soc. Math. France*, **102** (1974), 75–83.
31. Minimal realizations and spectrum generating algebras, *Commun. Math. Phys.*, **36** (1974), 325–338.
32. Realizations in classical and quantum mechanics, *Proc. of the Third Intern. Colloq. on Group Theoretical Methods in Physics* (Eds. H. Bacry and A. Grossman), 1974.
33. Infinite dimensional Lie algebras in mathematics and physics, *Proc. of the Third Intern. Colloq. on Group Theoretical Methods in Physics* (Eds. H. Bacry and A. Grossman), 1974.
34. A generalization of the Gelfand-Kirillov conjecture, *Amer. J. Math.*, **99** (1977), 1151–1165.
35. The algebraic method in representation theory, in *Group Theoretical Methods in Physics* (Ed. A. Janner), LN 50 (Physics), 95–109, Springer-Verlag, New York, 1976.
36. Second commutant theorems in enveloping algebras, *Amer. J. Math.*, **99** (1977), 1167–1192.
37. The minimal orbit in a simple Lie algebra and its associated maximal ideal, *Ann. Sci. Ec. Norm. Sup.*, **9** (1976), 1–30.
38. A preparation theorem for the prime spectrum of a semisimple Lie algebra, *J. of Algebra*, **48** (1977), 241–289.

39. A characteristic variety for the prime spectrum of a semisimple Lie algebra, in Non-commutative Harmonic Analysis (Ed. J. Carmona), LN 587, 102–118, Springer-Verlag, New York, 1977.
40. Sur les vecteurs de plus haut poids dans l'algèbre enveloppante d'une algèbre de Lie semisimple complexe, *Comptes Rendus A* **281**, (1975), 835–837.
41. Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie réductive, *Comptes Rendus A* **284** (1977), 425–427.
42. On the annihilators of the simple subquotients of the principal series, *Ann. Ec. Norm. Sup.*, **10** (1977), 419–440.
43. A wild automorphism of $Us\ell(2)$, *Math. Proc. Camb. Phil. Soc.*, **80** (1976), 61–64.
44. A generalization of Quillen's lemma and its application to the Weyl algebras, *Israel J. Math.*, **28** (1977), 177–192.
45. Gelfand-Kirillov dimension for the annihilators of simple quotients of Verma modules, *J. Lond. Math. Soc.*, **18** (1978), 50–60.
46. (With L.W. Small), An additivity principle for Goldie rank, *Israel J. Math.*, **31** (1978), 105–114.
47. Sur la classification des idéaux primitifs dans l'algèbre enveloppante de $s\ell(n+1,\mathbb{C})$, *Comptes Rendus Ser.*, A-287 (1978), 303–306.
48. W module structure in the primitive spectrum of the enveloping algebra of a semisimple Lie algebra, in Lecture Notes in Mathematics, **728**, Springer-Verlag, (1979), 116–135.
49. Dixmier's problem for Verma and principal series submodules, *J. Lond. Math. Soc.*, **20** (1979), 193–204.
50. Towards the Jantzen conjecture, *Compositio Math.*, **40** (1980), 35–67.
51. Towards the Jantzen conjecture II, *Compositio Math.*, **40** (1980), 69–78.
52. Towards the Jantzen conjecture III, *Compositio Math.*, **41** (1981), 23–30.
53. Goldie rank in the enveloping algebra of a semisimple Lie algebra I, *J. Algebra*, **65** (1980), 269–283.
54. Goldie rank in the enveloping algebra of a semisimple Lie algebra II, *J. Algebra*, **65** (1980), 284–306.
55. Kostant's problem, Goldie rank and the Gelfand-Kirillov conjecture, *Invent. Math.*, **56** (1980), 191–213.
56. On the Gelfand-Kirillov conjecture for induced ideals in the semisimple case, *Bull. Math. Soc. France*, **107** (1979), 139–159.
57. (With O. Gabber), On the Bernstein-Gelfand-Gelfand resolution and the Duflo sum formula, *Compos. Math.*, **43** (1981), 107–131.
58. (With O. Gabber), towards the Kazhdan-Lusztig conjecture, *Ann. Ec. Norm. Sup.*, **14** (1981), 261–302.
59. Goldie rank in the enveloping algebra of a semisimple Lie algebra III, *J. Algebra*, **73** (1981), 295–326.
60. The Enright functor in the Bernstein-Gelfand-Gelfand O category, *Invent. Math.*, **67** (1982), 423–445.
61. Completion functors in the O category, in Lecture Notes, **1020**, Springer-Verlag, 1983, 80–106.

62. On the classification of primitive ideals in the enveloping algebra of a semisimple Lie algebra, in Lecture Notes, **1024**, Springer-Verlag, 1983, 30–76.
63. Primitive ideals in enveloping algebras, ICM Proceedings, PWM – North Holland, Warsaw 1984, 403–414.
64. On the variety of a highest weight module, *J. Algebra*, **88** (1984), 238–278.
65. On the associated variety of a primitive ideal, *J. Algebra*, **93** (1985), 509–523.
66. (With J.T. Stafford), Modules of k-finite vectors over semisimple Lie algebras, *Proc. Lond. Math. Soc.*, **49** (1984), 361–384.
67. On the Demazure character formula, *Ann. Ec. Norm. Sup.*, **18** (1985), 389–419.
68. Three topics in enveloping algebras, Proceedings of Durham Symposium, 1983, (unpublished).
69. On the Demazure character formula II, *Compos. Math.*, **58** (1986), 259–278.
70. Kostant's problem and Goldie rank, in: Lecture Notes in Mathematics **880**, Springer-Verlag, Berlin, 1981, 249–266.
71. On the cyclicity of vectors associated with Duflo involutions, *Lecture Notes in Mathematics*, **1243**, 144–188, Springer-Verlag, Berlin, 1987.
72. A criterion for an ideal to be induced, *J. Algebra*, **110** (1987), 480–497.
73. On the multiplicities of the adjoint representation in simple quotients of an enveloping algebra of a semisimple Lie algebra, *Trans. Amer. Math. Soc.*, **316** (1989), 447–491.
74. Rings which are modules in the Bernstein-Gelfand-Gelfand O category, *J. Algebra*, **113** (1988), 110–126.
75. A sum rule for scale factors in the Goldie rank polynomials, *J. Algebra*, **118** (1988), 276–311.
76. A sum rule for scale factors in the Goldie rank polynomials, addendum, *J. Algebra*, **118** (1988), 312–321.
77. A surjectivity theorem for rigid highest weight modules, *Invent. Math.*, **92** (1988), 567–596.
78. The primitive spectrum of an enveloping algebra, *Astérisque*, **173–174** (1989), 13–53.
79. Characters for unipotent representations, *J. Algebra*, **130** (1990), 273–295.
80. On the characteristic polynomials for orbital varieties, *Ann. Ec. Norm. Sup.*, **22** (1989), 569–603.
81. Rings of $\mathfrak{b}$-finite endomorphisms of simple highest weight modules are Goldie, IMCP, Ring theory, 1989, The Weizmann Science Press, Jerusalem 1990, 124–134.
82. The surjectivity theorem, characteristic polynomials and induced ideals, in *The Orbit Method in Representation Theory*, Eds. M. Duflo, N.V. Pedersen, M. Vergne, Birkhäuser, Boston, 1990, 85–98.
83. (With T.J. Enright). An intrinsic analysis of unitarizable highest weight modules, *Math. Ann.*, **288** (1990), 571–594.
84. Annihilators and associated varieties of unitary highest weight modules, *Ann Ec. Norm. Sup.*, **25** (1992), 1–45.

85. Some ring theoretic techniques and open problems in enveloping algebras, Proceedings of Berkeley Conference, July 1989, Ed. S. Montgomery and L. Small, Springer-Verlag, Berlin 1992, p. 27–67.
86. (With G. Letzter). Local finiteness of the adjoint action for quantized enveloping algebras. *J. Algebra*, **153** (1992), 289–318.
87. (With G. Letzter). Separation of variables for quantized enveloping algebras. *Amer. J. of Math.*, **116** (1994), 125–177.
88. (With G. Perets and P. Polo). Sur l'équivalence de catégories de Beilinson et Bernstein, *Comptes Rendus*, Paris, **313** (1991), 705–709.
89. (With G. Letzter). Verma module annihilators for quantized enveloping algebras, *Ann. Ec. Norm. Sup.*, **28** (1995), 493–526.
90. Faithfully flat embeddings for minimal primitive quotients of quantized enveloping algebras, *Israel Math. Conf. Proc.*, **7** (1993), 79–106.
91. Idéaux premiers et primitifs de l'algèbre des fonctions sur un groupe quantique, *Comptes Rendus*, **316** (1993), 1139–1142.
92. On the prime and primitive spectra of the algebra of functions on a quantum group, *J. Algebra*, **169** (1994), 441–511.
93. (With G. Letzter). Rosso's form and quantized Kac-Moody algebras, *Math. Zeit.*, **222** (1996), 543–571.
94. *Enveloping Algebras: Problems Old and New*, in Progress in Mathematics **123**, Birkhäuser, Boston, 1994, 385–413.
95. Some Remarks on the R-matrix, *J. Algebra*, **180** (1996), 412–430.
96. Orbital varieties, Goldie rank polynomials and unitary highest weight modules, In B. Orsted and H. Schlichtkrull (eds.), *Algebraic and Analytic Methods in Representation Theory*, Academic Press, London 1997, 53–98.
97. Sur une conjecture de Feigin, *Comptes Rendus*, Paris, **320** (1995), 1441–1444.
98. Sur les idéaux génériques sur l'algèbre des fonctions sur un groupe quantique, *Comptes Rendus*, Paris, **321** (1995), 135–140.
99. Preservation of Coxeter structure under finite group action, Acts du Colloque, Reims 1995, Colloques et Seminaires du SMF, 1997, 185–219.
100. (With A. Braverman). The minimal realization from deformation theory, *J. Algebra*, **205** (1998), 13–36.
101. The orbital varieties of the minimal orbit, *Ann. Ec. Norm. Sup., Serie 4*, **31** (1998), 17–45.
102. On a Harish-Chandra homomorphism, *CRAS*, **324(I)** (1997), 759–764.
103. Sur l'annulateur d'un module de Verma, In B. Broer (ed.), *Representation Theories and Algebraic geometry*, Kluwer Academic Publishers, NATO Scientific Affairs, 1998, 237–300.
104. On the mock Peter-Weyl theorem and the Drinfeld double of a double, *J. reine angew. Math.*, **507** (1999), 37–56.
105. The admissibility of bounded modules for an affine Lie algebra, *Alg. Rept. Theory*, **3** (2000), 131–149.
106. A completion of the quantized enveloping algebra of a Kac-Moody algebra, *J. Algebra*, **214** (1999), 235–275.

107. On the Kostant-Parthasarathy-Ranga Rao-Varadarajan determinants, I. Injectivity and multiplicitie, *J. Algebra*, **241** (2001), 27–45.
108. (With G. Letzter). On the KPRV determinants, II. Construction of the KPRV determinants, *J. Algebra*, **241** (2001), 46–66.
109. (With G. Letzter and D. Todoric). On the KPRV determinants, III. Computation of the KPRV determinants, *J. Algebra*, **241** (2001), 67–88.
110. (With G. Letzter and S. Zelikson). On the Brylinski-Kostant filtration, *JAMS*, **14** (2000), 945–970.
111. (With F. Fauquant-Millet). Semi-invariants de l'algèbre enveloppante quantifiée d'une algèbre parabolique, *Transf. Groups.*, **6** (2001), 125–142.
112. On an affine KPRV determinant at $q = 1$, *Bull. Math. Sci.*, **125** (2001), 23–48.
113. (With J. Greenstein). A Chevalley-Kostant presentation of basic modules for $\mathfrak{sl}(2)^\wedge$ and the associated affine KPRV determinants at $q = 1$, *Bull. Math. Soc.*, **125** (2001), 85–108.
114. (With W. Borho). Sheets and topology of primitive spectra for semisimple Lie algebras, *J. Algebra*, **244** (2001), 76–167.
115. (With D. Todoric). On the quantum KPRV determinants for semisimple and affine Lie algebras, *Alg. Rept. Theory*, **5** (2002), 57–99.
116. (With G. Letzter). Evaluation of the quantum affine PRV determinant, *Math. Res. Lett.*, **2** (2002), no. 2–3, 307–322.
117. (With A. Melnikov). Quantization of Orbital Varieties in $\mathfrak{sl}_n$, Progr. in Math. Birkhäuser, **213** (2003), 165–196.
118. A Decomposition Theorem for Demazure Crystals, *J. Algebra*, **265** (2003), 562–578.
119. On the graded injectivity of the Conze embedding, *J. Algebra*, **265** (2003), 358–378.
120. (With W. Borho). Corrigendum to Sheets and topology of primitive spectra for semisimple Lie algebras, *J. Algebra*, **259** (2003), 310–311.

Communicated Preprints

121. (With V. Hinich). Orbital variety closures and the convolution product in Borel-Moore homology, to appear in *Selecta Mathematica, New Series*.
122. Modules with a Demazure flag, this volume.
123. (With F. Fauquant-Millet), Semi-centre de l'algèbre enveloppante d'une sous-algèbre parabolique d'une algèbre de Lie semi-simple, *Ann. Ec. Norm. Sup.,* **38**(2), (2005), 155–191.
124. The enigma of the missing invariants on the nilradical of a Borel, *Bull. Sci. Math.*, **128** (2004), 433–446.
125. Results and problems in enveloping algebras arising from quantum groups, to appear in volume dedicated to 70th birthday of A. N. Vershik.
126. Goldie rank ratios and quaternionic extensions, *J. Algebra* (to appear).

Book

Quantum Groups and Their Primitive Ideals, Springer-Verlag, Berlin 1995.

Edited Volumes

1. (With A. Connes, M. Duflo and R. Rentschler). *Operator Algebras, Unitary Representations, Enveloping Algebras and Invariant Theory*, Actes du colloque en l'honneur de Jacques Dixmier, Progr. in Math., Vol. 92 Birkhäuser, Boston 1990.
2. (With S. Shnider). *Quantum deformations of algebras and their representations*, Israel Mathematical Conference Proceedings Vol. 7, Bar-Ilan University, 1993.
3. (With F. Mignot, F. Murat, B. Prum and R. Rentschler). *First European Congress of Mathematics, Vol. I*, Progr. in Math. Vol. 119, Birkhäuser, Boston 1994.
4. (With F. Mignot, F. Murat, B. Prum and R. Rentschler). *First European Congress of Mathematics, Vol. II*, Progr. in Math. Vol. 120, Birkhäuser, Boston 1994.
5. (With F. Mignot, F. Murat, B. Prum and R. Rentschler). *First European Congress of Mathematics, Vol. III*, Progr. in Math. Vol. 121, Birkhäuser, Boston, 1994.
6. (With A. Melnikov and R. Rentschler). *Studies in Memory of Issai Schur*, Progr. in Math. Progr. in Math. Vol. 210, Birkhäuser, Boston, 2002.

Students of Anthony Joseph

S. Bamba
M. Hervé
P. Polo
G. Perets
E. Benlolo
V. Hinich
A. Melnikov
A. Zahid
A. El Alaoui
F. Fauquant-Millet
D. Todoric
M. Kébé
Sh. Zelikson
M. Gorelik
E. Lanzmann
J. Greenstein
Yu. Bazlov
E. Perelman

List of Summer Students

Anna Klugman (Melnikov), Technion
Gail Letzter, University of Chicago
Eti Gvirtz, Bar-Ilan University
Elise Benlolo, University of Haifa
Markus Stricker, ETH
Gadi Perets, Université Pierre et Marie Curie
Avital Frumkin, Hebrew University
Alexander Braverman, Tel Aviv University
Denis Gaitsgory, Tel Aviv University
Maria Gorelik, Moscow State University

Maxim Leenson, Moscow Independent University
Elena Perelman, St. Petersburg State University
Yuri Bazlov, St. Petersburg State University
Judita Preiss, Cambridge University
Tatiana Assing, Université Pierre et Marie Curie
Vera Zolataskai, Moscow State University
Shira Zerbiv, Technion
Zelijka Ljujic, University of Belgrade
Alexander Yom Dim, University of Haifa
Irina Bobkova, St. Petersburg State University

From Denise Joseph

First, I wish to thank all the guests who joined us tonight to celebrate Tony's 60th birthday and especially the scientists who came from abroad. They don't know how much we appreciate their coming to Israel in this difficult time and we thank them heartily for their courage, support and wish to separate science from politics. I also wish to thank Anna who worked so hard to make this conference a success, and all who also gave so much of their time to organize so many things.

I asked Tony what I should speak about and he said I should tell how difficult it is to be the wife of a mathematician, I would then have the sympathy of all the wives and husbands and persons living with mathematicians. But I shall not follow his advice.

Last week we went to the BA graduation ceremony of our son in the Faculty of Agriculture which is situated opposite the Weizmann Institute. On a big screen it was written, "60^{th} birthday". Then we were shown a film of when the Faculty was inaugurated in 1942 with just one building and one department of Agriculture, and we could see how it developed so beautifully with a big campus and 14 departments. Tony said, "you imagine it is as old as me". I cannot speak of Tony's mathematical achievements although he has tried many times to explain to me on what subject he was working.

I know that Tony is not only hard on himself but also on his children and I am sure on his students. I was nicely surprised to find out when I met some of his students not only how highly they regard him but that they also like him.

Tony loves sport, tennis, skiing and windsurfing. I always admired his determination to be a good windsurfer although in the beginning he used to fall all the time. One day he said, "I am going to learn to play the flute". I was astonished since Tony had never played any musical instrument. I asked, "are you going to take lessons?" He replied, "of course not". He bought some books and after a week or two I was amazed to hear nice musical rhythms of known songs.

Tony likes to build and repair things in his spare time. I still remember when we moved some twenty years ago to our house he built a wooden table for the garden. The neighbour was so surprised; she said to me, "I thought people who use their brain are not good with their hands".

Let us raise our glasses and drink to the health happiness and long life — to Tony.

From Jacques Dixmier: A Recollection of Tony Joseph

When I saw Tony for the first time, he had published a number of papers about physics and even chemistry. He wanted to discuss my 1968 paper concerning the Weyl algebra A_1. He explained to me why physical arguments permitted a deeper understanding. But at some other points, he said “this is a typical mathematical idea”. So maybe I played a little role in Tony’s transition towards mathematics. If so, I think the mathematical world is greatly indebted to me!

After that, Tony often wrote to me (when he wasn’t in Paris). In spite of my bad organization, I rediscovered the following old letter. Thank you, Tony, for your wonderful work, and my warmest encouragements for the future.

7/7/72.

Dear Professor Dixmier,

I must first thank you for the manuscripts you have sent me. I have been working on some of the problems you posed in your first paper on the Weyl algebra, in particular that on ~~the~~ the question of the existence of endomorphisms which are not automorphisms. Your lemma 7.3 is central to this question. By slightly extending its validity and ~~by~~ through its systematic application, I have been able to show that if Q, P are of degree less than 147, then they generate A_1. Hence ~~every~~ such an endomorphism is an automorphism. For polynomials of higher degree, I am able to partially ~~to~~ construct Q, P; but unfortunately I have been as yet unable to exhibit an endomorphism. As far as I can see it is not possible to push your leading term analysis further.

In another direction I have studied the question of the existence of polynomial relations between commuting elements of the quotient field and have obtained some partial results. There is a quite simple and explicit method computing the common divisor of two elements based on the fact that the elements of the zero eigenspace ~~of the~~ ad $q_i p_i$: $i = 1, 2 .. n$, commute. This is quite valuable in situations where one wishes to control the degree of the denominator (or numerator).

I hope to spend a few days in Paris at IHES with Professor Michel, ~~In that case~~ during the last week of August. In that case I hope that I might be able to drop in on you.

Yours sincerely,

Tony Joseph.

Part I

Survey and Review

The work of Anthony Joseph in classical representation theory

W. McGovern

Department of Mathematics
University of Washington
Seattle, WA 98195
USA

Over some three decades Joseph's groundbreaking work in classical representation theory has changed the face of the subject. Joseph has not only introduced many beautiful ideas of his own but also has shown a gift for reinterpreting the work of others in an entirely new light, establishing connections between quite disparate aspects of the subject. What follows is a survey of this part of Joseph's work, beginning with his papers on primitive ideals in the enveloping algebra of a semisimple Lie algebra. Such ideals were the primary focus of Joseph's work from the mid-1970s until the mid-1990s and he is responsible for the lion's share of the advances in their theory during that time.

Most of Dixmier's classic 1974 text on enveloping algebras of Lie algebras $\mathfrak{g}$ is devoted to the case where $\mathfrak{g}$ is solvable. What really put primitive ideals in enveloping algebras of semisimple Lie algebras on the map was Duflo's fundamental theorem that any such ideal is the annihilator of a very special kind of simple module, namely a highest weight module. Thus one can study such ideals via abstract ring theory or highest weight theory. Joseph used both of these tools in his first papers on primitive ideals. In [39, 41] he introduced the notion of the characteristic variety of a primitive ideal, using the idea behind the Harish-Chandra homomorphism, and used it to detect many inclusions and equalities among annihilators of simple highest weight modules.

In [42] he identified the annihilators of simple Harish-Chandra bimodules for complex groups as annihilators of explicit highest weight modules. In [38] he used localization techniques from noncommutative ring theory to decompose the primitive spectrum of an enveloping algebra as the union of spectra of more manageable rings; along the way he exhibited many subalgebras of a semisimple Lie algebra whose enveloping algebras have centers that are polynomial rings (see also [40]). He used these results in an unpublished preprint of 1976 to classify the primitive spectrum in types A_2 and B_2 before Duflo had proved his theorem. Later he used his work on characteristic varieties together with Duflo's theorem to classify primitive ideals in type A, via the well-known Robinson–Schensted bijection between permutations of n letters and pairs of standard Young n-tableaux of the same shape [50, 51, 47]. More recently

Barbasch–Vogan and Garfinkle have extended this classification to the other classical types, working with standard domino tableaux rather than standard Young tableaux.

The connection between primitive ideals and highest weight modules was further strengthened by the discovery of an equivalence between a large category of bimodules over an enveloping algebra, including all of its primitive quotients and the Bernstein–Gelfand–Gelfand category $\mathcal{O}$, which includes all simple highest weight modules. This discovery is due independently to Joseph, Bernstein–Gelfand, and Enright. Both Joseph and Bernstein–Gelfand used it to give an algebraic proof of Duflo's theorem ([49]; see also *Comp. Math.* **41** (1980), 245–285). Joseph started out by establishing a bijection that had been conjectured by Dixmier between ideals containing a fixed minimal primitive ideal and submodules of a Verma module. (Enright approached the category equivalence quite differently from Bernstein–Gelfand and Joseph: he introduced functors which essentially built up the bimodule structure in category $\mathcal{O}$ from scratch. Later Joseph studied these functors in [60, 61], establishing the connection between them and the techniques of [49].)

At about this time Joseph also refined his earlier work on inclusions and equalities among primitive ideals, showing (modulo a conjecture later proved by Vogan) that the set of primitive ideals of a fixed regular integral central character spans a vector space carrying a natural action of the Weyl group [48]; it is determined by the multiplicities of composition factors in Verma modules. This paper was one of the inspirations for Kazhdan and Lusztig's fundamental paper in *Inv. Math.* **53**, in which they formulated their famous conjecture for computing these multiplicities. In turn Gabber and Joseph related this conjecture to the hereditarity of the Jantzen filtration on Verma modules in [58]. This filtration was used to study composition factors of primitive quotients in [94]. Gabber and Joseph also studied natural questions arising from the category equivalence in [57].

Joseph first combined the tools of abstract ring theory and highest weight theory in [45], where he related the Gelfand–Kirillov dimension of a simple highest weight module L to that of U/I, I the annihilator of L in the enveloping algebra U. He did so again in an especially beautiful way in the papers [53, 54], which studied Goldie ranks of primitive quotients. He showed that the Goldie ranks of primitive quotients obtained from a fixed one P by translation functors are given by a polynomial function which moreover completely determines P once the central character of this quotient is given. In addition this polynomial is determined up to a multiplicative scalar by the composition factor multiplicities in Verma modules, which are given by the Kazhdan–Lusztig polynomials evaluated at $q = 1$. Goldie rank polynomials attached to the set of all primitive ideals with a fixed regular integral central character λ span a vector space which carries a natural action of the Weyl group. As a consequence the number of primitive ideals of central character λ equals the sum of the dimensions of certain Weyl group representations. Barbasch and Vogan later identified just which Weyl group representations occur in this way in *Math. Ann.* **259** (1982) and *J. Alg.* **80** (1983). In type A they all do, as was already clear from Joseph's earlier work; in general, exactly the special ones in Lusztig's sense do. Thus the picture of the primitive spectrum first introduced in [48] was brought to a very satisfactory culmination. In the course of his work on Goldie ranks Joseph also showed that they satisfy an additivity property which he generalized to a wider setting in joint work with Small [46].

It remained (and in fact still remains) to compute the multiplicative scalars in the Goldie rank polynomials. Joseph launched a program for doing this in [59] which was continued in [71, 75, 76]. In these papers an important topic is the relationship between a typical primitive quotient U/I and the Kostant ring of ad-finite maps from L to itself, where L is a simple highest weight module with annihilator I; Joseph had already studied this ring in [55, 70] and proved his Goldie rank theorems in a special case. In [59, 71, 75, 76] he studies numerical invariants attached to a Kostant ring and the quotient U/I that it contains and shows that their ratio is a positive integer. He goes on to establish beautiful relations among these ratios. They (and the multiplicative scalars) can be computed exactly if one can construct sufficiently many primitive quotients with known Goldie rank. It turns out that Goldie ranks are the easiest to compute (and the most interesting) when they are 1, or equivalently when the primitive quotient is completely prime.

Motivated by his earlier work in theoretical physics, Joseph had already introduced an important completely prime primitive quotient attached to the minimal nilpotent orbit in [37] (which he revisited with A. Braverman in [100], using deformation theory of Koszul algebras); now he proved a positivity property for Goldie rank polynomials which shows that completely prime primitive quotients are quite rare [80, 82]. This property was refined and extended in [114], where he and Borho extended Dixmier's old notion of a sheet of adjoint orbits to primitive ideals and showed that that there are only finitely many sheets of primitive ideals with Goldie rank bounded above by a fixed positive integer. This reviewer used Joseph's work extensively in *AMS Memoir* **519**, where he computed just which quotients by maximal ideals are completely prime in the classical cases. Joseph also computed the Goldie ranks of certain quotients by induced primitive ideals in [56], verifying for such quotients the Gelfand–Kirillov conjecture that their Goldie fields are always Weyl skew fields. He verified the same conjecture for type A in [55].

By 1982 the classification of the primitive spectrum was essentially complete, but still rather mysterious. The appearance of special Weyl group representations can be most satisfactorily explained by identifying them with the representations attached by Springer to certain nilpotent orbits (also called special). In turn the closures of these orbits arise as the associated varieties of primitive ideals. Borho and Brylinski had proved that such associated varieties are irreducible (and thus closures of single nilpotent orbits) in many special cases; Joseph proved it in general in [65], having earlier verified it for type A in [52].

He also gave a more direct construction of the Springer representation π attached to a nilpotent orbit $\mathcal{O}$ in [64], as follows. Intersect $\bar{\mathcal{O}}$ with the nilradical $\mathfrak{n}$ of a fixed Borel subalgebra of the ambient semisimple Lie algebra. One obtains an algebraic variety $\mathcal{V}$ with exactly $\dim \pi$ irreducible components, as was known from early work of Springer and Spaltenstein. Joseph looked at the leading terms of Hilbert–Samuel polynomials of the coordinate rings of these components as a function of the grading coming from the action of a Cartan subgroup. He showed that their numerators may be regarded as polynomials in the symmetric algebra of the dual of the Cartan subalgebra and proved that the span of these polynomials attached to the components of $\mathcal{V}$ is stable under the Weyl group W. He then conjectured that the resulting representation of W is isomorphic to the Springer representation π. Hotta proved this in 1982 by a fairly complicated computation; following work of Rossmann, Joseph later gave a more direct proof by realizing the polynomials in question as integrals which represent classes in a homology ring whose structure is well known [80]. The components of $\mathcal{V}$ are called orbital varieties; they form the basic building blocks of the associated variety of a highest weight module. Joseph also used earlier work of Steinberg on his famous triple variety to deduce in [64] that every orbital variety lying in $\mathfrak{n}$ arises by taking the intersection of $\mathfrak{n}$ with a translate of itself by some element of W, taking the saturation under the action of the Borel subgroup B, and finally passing to the closure. In later work with Hinich [121], Joseph showed that inclusions among orbital variety closures behave quite similarly to inclusions among primitive ideals, though there are also subtle differences between these two kinds of inclusions.

It turns out that simple highest weight modules, unlike primitive ideals, need not have irreducible associated varieties; Joseph gave the first counterexample in [64] and Tanisaki later gave other examples. Following the Kostant–Souriau program of geometric quantization, Joseph formulated a quantization subprogram of his own in [77], which essentially sought to invert the associated variety construction. More precisely, given an orbital variety V, Joseph asked first whether there is a simple highest weight module with associated variety V. If so, he called V weakly quantizable; the terminology comes from [101]. Whenever V is weakly quantizable, he asked more strongly whether the coordinate ring of V agrees as a module over a Cartan subalgebra with some highest weight module M up to a weight shift. If so, he called V strongly quantizable. Joseph's student Melnikov later showed that every orbital variety is weakly quantizable in type A (*C. R. Acad. Sci.* **316** (1993)).

Another student, Benlolo, exhibited two varieties in type A that are strongly quantizable, but only via nonsimple highest weight modules. Joseph himself showed that his quantization program fails in general by exhibiting orbital varieties of the minimal nilpotent orbit that are not even weakly quantizable [101]. Nevertheless both he and his students Benlolo, Melnikov, and Perelman have continued to investigate the quantizability of various orbital varieties and obtained many positive results (see e.g., [117]). In particular, orbital varieties lying in abelian nilradicals of maximal parabolic subalgebras all turn out to be strongly quantizable in an especially nice way, via unitary highest weight modules [84]. The unitary highest weight modules arising in this way are in some sense the most interesting ones; Enright and Joseph constructed all unitary highest weight modules starting from these in [83]. More generally, orbital varieties which are hypersurfaces in nilradicals of parabolic subalgebras have nice quantization properties.

In addition to his research papers on primitive ideals, Joseph has also written a number of expository papers on these and related topics: see [62, 63, 68, 78, 85, 94, 96]. He has also written on a variety of other topics in classical representation theory. His first papers in pure mathematics dealt with Weyl algebras, which he became interested in through his previous work in mathematical physics [25, 27, 44]. He also studied homomorphic images of subalgebras of the first Weyl algebra in [43]. He proved the

Gelfand–Kirillov conjecture mentioned above for solvable Lie algebras in [29], building on his work in [30], and generalized it in [34]. He studied double commutants in enveloping algebras in [36]. In [67, 69] he ventured into finite-dimensional representation theory, proving the Demazure character formula for $\mathfrak{b}$-submodules generated by extremal weight vectors in finite-dimensional irreducible $\mathfrak{g}$-modules whose highest weight is sufficiently far from the walls ($\mathfrak{b}$ a Borel subalgebra); later Andersen and Kumar proved this formula in general and Kumar and Mathieu deduced the PRV conjecture from it.

The Demazure operators, which appeared for the first time in Joseph's work in these papers, reappeared beautifully and unexpectedly in [101]. They appear yet again in a very recent paper on crystal bases [118], which established the existence of Demazure flags for certain tensor products; see also [122]. Joseph continued to study finite-dimensional modules in [110]; here he, Letzter, and Zelikson gave a proof valid for all integral weights that the jump polynomials attached to Brylinski's filtration of a finite-dimensional module can be computed from Lusztig's q-polynomials. In [66] Joseph and Stafford studied the homological properties of primitive quotients, showing that such quotients behave well at regular central character but can be quite pathological otherwise.

In [72] Joseph revisited the induced ideals he had studied in [56] from a very different point of view, studying the conditions under which they are generated by minimal primitive ideals together with one copy of the adjoint representation. (He was led to study this question by previous unpublished work of R. Brylinski on it.) Along the way he derived an interesting and surprisingly difficult formula for the multiplicity of the adjoint representation in any quotient of the enveloping algebra by a maximal ideal [73]. He studied modules in category $\mathcal{O}$ with a ring structure in [74], finding that such modules have a remarkably rigid structure. These modules arose again in [77], where he studied very general rings of differential operators and developed a criterion for such a ring to be a quotient of the enveloping algebra. This paper was motivated by work of Levasseur and Stafford and recovered their results as a special case. Such rings of differential operators are not in general Noetherian; Joseph showed that they are nevertheless Goldie in [81].

In [79] Joseph studied unipotent representations of complex groups, which had been defined (purely algebraically) by Barbasch and Vogan five years earlier. He gave a new proof of the Barbasch–Vogan character formulas for them, using some fundamental calculations of Lusztig. In [88] he, Perets, and Polo gave an elementary ring-theoretic proof of the Beilinson–Bernstein equivalence of categories between modules over an enveloping algebra and D-modules, using just the Hilbert Nullstellensatz and the translation principle. Their construction carries over to the quantum case [90], which Lunts and Rosenberg interpreted as a Beilinson–Bernstein equivalence of categories in a nonabelian framework.

Following Polo, Joseph studied Galois theory in the setting of an enveloping algebra in [99]. Given a finite group G of automorphisms of the quotient U/I of the enveloping algebra by a minimal primitive ideal of regular integral central character, he asked whether the ring $(U/I)^G$ of G-invariants could be another quotient U'/I' of the same type. He was able to show under these conditions that the Lie algebras corresponding to the enveloping algebras U, U' must have the same Coxeter diagram, but he could not quite show that their Dynkin diagrams are the same. In 1997 he returned to the Harish-Chandra homomorphism, which he had used in one of his first papers on primitive ideals ([39]). This time he worked with the extended Harish-Chandra homomorphism, defined on the space of invariant differential operators on a semisimple Lie algebra, and showed that it is graded surjective [102]. This gave an affirmative answer to a question of Wallach, a weaker version of which had been answered affirmatively by Levasseur and Stafford.

Joseph's most recent work in classical representation theory is largely motivated by his work in quantized enveloping algebras, described elsewhere in this volume. He gave a new proof of an old result of Duflo that the annihilator of a Verma module is generated by its intersection with the center of the enveloping algebra in [103]; his approach here generalizes to the super and quantum cases. He studied simple weight modules over an affine Kac–Moody algebra in [105], classifying all such modules whose weights have bounded norm and showing in particular that any such module has finite-dimensional weight spaces. In an important series of papers [107, 108, 109] he studied analogues of the classical PRV determinants (studied also by Kostant) for generalized Verma modules. He gave formulas for their characters in [107], worked with G. Letzter to interpret their zeros and give formulas for the degrees in which these zeros occur in [108], and finally gave an explicit formula for the determinants themselves in [109], working with Letzter and Todoric. Working with his student Greenstein, he modified an old construction of Kostant and Chevalley in the semisimple case to show that any integrable highest weight module for affine $\mathfrak{sl}_2$ admits a realization as a subalgebra of a suitable Clifford algebra [113]. This has no analog for $\mathfrak{g}$ semisimple and does not extend to any other affine Lie algebra.

Although Joseph's research interests have largely shifted to quantized enveloping algebras in the last few years, much of what he does with them is motivated by and has ramifications for classical representation theory (as indicated above). Representation theory has benefited immensely from his many contributions. I wish him the best in the years to come.

Quantized representation theory following Joseph

Daniel R. Farkas and Gail Letzter

Department of Mathematics
Virginia Tech
Blacksburg, VA 24061
USA

A quantum group ... is at present a purely mythical being ... A. Joseph

We are ignorant of the meaning of the dragon ... but there is something in the dragon's image that fits man's imagination and this accounts for the dragon's appearance in different places ... L. Borges

At the beginning of his lecture for the Microprogram on Noncommutative Rings held at the Mathematical Sciences Research Institute in July of 1989, Tony Joseph remarked that he had finally seen an application of quantum groups — he had passed a Volkswagen Quantum GL5 on his way to the talks. Not all successful marriages begin with love at first sight.

We quickly set notation. Assume that $\mathfrak{g}$ is a finite-dimensional semisimple Lie algebra of rank n. Fix a root system with simple roots $\alpha_1, \dots, \alpha_n$. Let Q (and let P) denote its root lattice (resp. weight lattice). Assume that q is always an indeterminate. The quantized enveloping algebra $U_q(\mathfrak{g})$, *which we shall always shorten to* U, is the $\mathbf{C}(q)$-algebra generated by symbols x_i, y_i, and $t_i^{\pm 1}$ for $1 \leq i \leq n$, subject to the associative algebra and Hopf algebra relations found in [Jbook]. (Other practitioners use E_i instead of x_i, F_i instead of y_i, and K_i instead of t_i.) The group generated by all t_i is denoted T.

Write τ for the isomorphism that sends the additive group Q to the multiplicative group T via $\alpha_i \mapsto t_i$. Enlarge T to a group $\check{T}$ isomorphic to P by extending τ in the obvious way. The "simply connected" quantized enveloping algebra $\check{U}$ is generated by U and $\check{T}$. For many purposes, $\check{U}$ is a more suitable algebra than U and the scalar field should be replaced with its algebraic closure. We will be blithely sloppy in this review and use all such algebras interchangeably. The reader, concerned whether the dropped ornamentation of U is required for precise descriptions of Joseph's contributions, is welcome to suffer through the subtleties and complications on his or her own.

The moral dual to U is the quantized function algebra (or coordinate ring) $R_q[G]$ where G is the Lie group associated to $\mathfrak{g}$. Again, we abbreviate notation and simply