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MARC HENRARD

Interest Rate Modelling in the Multi-curve Framework

Foundations, Evolution and Implementation

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Preface

The first lines of this book were written in 2006. At the time the term *multi-curve framework*, which is used for the book's title, had not been coined and my idea was only to write a couple of pages for a note. In the meantime, August 2007 changed the course of writing on interest rate curve modelling for ever.

> Festina lente.
>
> *Latin saying*

> Personal translation: Haste slowly.

Why did it take me so long?

> Chacun sa méthode... Moi, je travaille en dormant et la solution de tous les problèmes, je la trouve en rêvant. (Each his own method . . . Myself, I work sleeping and the solution to all problems, I find it dreaming).
>
> *Drôle de drame (1937) – Marcel Carné*

This means a lot of nights spent working to dream up all these pages.

The starting point of the reflection was my quest to answer the question 'What is the present value of an FRA(Forward Rate Agreement)?' in a convincing way. I could not find a satisfactory answer in the literature. The answers I could find were either 'it is trivial', or a description of a replication argument for which it was not acceptable to discuss the numerous hidden hypotheses. Discussing the hypothesis was not politically correct as a scientist nor as a business executive. For the former, it questioned a foundation of quantitative finance, and thus the developments built on those foundations. For the latter, it was not seen positively by board members and supervisors as it casted doubts on accounting figures – maybe rightly so – and had legal implications.

The reason for my interest in discounting is that

> Gentlemen prefer bonds.
>
> *Andrew Mellon – 1855–1937*

Ironically, the first article to come out of those reflections, titled *Irony in derivatives discounting*, was published just one month before the now famous August 2007. The publication was not a prediction of what would happen in the derivative market just after and should not be seen as a premonition. Neither should it be seen as a cause of the crisis. It was nevertheless an indication of inconsistencies in the practice of derivative pricing that were not answered by a coherent theoretical framework.

The book, which started, unbeknown to its author, seven years ago, is intended to be a description of the current status of the subject, which is now called the *multi-curve framework*. It borrows from the developments of numerous practitioners and academics working on the subject. The length of the bibliography, with most of the references dating from 2009 and after, is a witness of the activity on the subject over the last years.

The book profited from fruitful conversations with and comments on drafts by Marco Bianchetti, Damiano Brigo, Antonio Castagna, Stéthane Crépey, Christophe Laforge, Andrea Macrina, Fabio Mercurio, Massimo Morini, and Chyng Wen Tee.

This book would not have been possible without the support of OpenGamma. The company encouraged me to complete the book and gave me the time to do it. The quality of the book was improved by discussion on its content with OpenGamma colleagues: Casey Clement, Yukinori Iwashita, Elaine McLeod, Soila Patajoki, Joan Puig, Richard White, and Arroub Zine-Eddine.

It is also my pleasure in this book to follow in the footsteps of giants in the art of relevant and irrelevant quotes. I have done my best in that regard, but I can not claim to match them.

Beyond the written literature, another source of inspiration for the development of this book was the participation in numerous practitioner and academic conferences and the perspicacious questions and remarks by their attendants. While preparing the final version of this book, I also had the pleasure of lecturing on the subject in the mathematical finance masters program at University College London. The book profited from the course preparation and from the questions, naive in appearance but fundamentally deep, asked by the participants.

In the book I use the term ‘we’ with the general and vague meaning of the author, the quantitative finance research community and the readers. The terms ‘I’, ‘me’ or ‘my’ are used with the precise meaning of *the author personally with his opinions and bias*. The term ‘I’ should be used as a warning sign that the sentence contains opinions and maybe not only facts. You have been warned!

Hopefully the reader will find as much interest in the subject as I have over the last seven years. Hopefully he will also be intrigued, surprised, amused and maybe amazed at some of the subject facets.

Oh, by the way, my quest is still on! I’m still asking myself ‘What is the present value of an FRA?’, even if the question has changed to ‘What is the collateral quote of an FRA?’. A question that is the source of insomnia …and maybe the starting point of another book in a couple of years.

Enjoy!

1 Introduction

1.1 9 August 2007

9 August 2007 is probably the day that the *multi-curve framework* became paramount, even if the term had not yet been coined at that time. What happened on that date can be found in history books. From an interest rate curve modelling perspective, it can be summarised by

> Houston, we have a problem!
>
> *Attributed to the crew of the US's Apollo 13 – 14 April 1970*

From the pre-crisis period to the time of finishing this book, the pricing of Ibor[1] and overnight-related derivatives has changed significantly. Before 2007, the standard textbook formulas,[2] using a unique multipurpose curve in each currency, were generally used in banks and software packages.

Nevertheless not everybody believed this was a perfect approach. Several banks, at least in their front-office, were using different pragmatic approaches incorporating cost of funding, risk-free rates different from Libor and similar issues. I have personally developed and used related techniques for pricing on a trading desk from 2003 onward. I would not, however, call the techniques used at that time a theoretically sound framework. It was more a set of ad hoc adjustments done for specific products with a narrow range of applicability. A theoretical foundation was missing and the coherency between instruments was not guaranteed.

For non-par swaps,[3] noticeable quote differences could be observed between banks with different credit qualities and quantitative development levels. It was a clear sign that adjustments to the textbook formulas were done in some places (and

[1] We use the term Ibor as a generic term for all rates fixing with similar rules. In particular the term covers Libor (originally ten currencies and since May 2013 reduced to six currencies), Euribor (EUR), Tibor (JPY), Cibor (DKK), and so on. See Appendix B for the conventions in the main currencies and Quantitative-Research (2012c) for more currencies and details.

[2] Some of them are described in the first part of Chapter 2.

[3] The interbank swaps are mainly traded at par, with the noticeable exception of the asset swaps.

probably not in others). Since then, several stories have appeared in newspapers and journals describing how some more knowledgeable banks may have used that knowledge to make profits at the detriment of their counterparts and clients. It is difficult to know how much was systematic misinformation and how much was simply taking the risk of using techniques other than the consensus. It should serve the readers as a warning that the current multi-curve framework, as described in this book, is today's equivalent of the 2007 textbook formulas. The framework is still evolving. and the approach is not future proof. As this book is published, there are a lot of people researching and implementing similar subjects in academia and in the industry. In particular, Chapter 8 is a work in progress, both for its theoretical developments and its implementation in practice.

Since the development of the framework became public some people have claimed that the framework was used as early as the mid-1990s in some places. I have not seen any evidence of those claims and all the texts from that period that are now publicly surfacing fall in the *precursor* category, which is discussed in the next section. They described some incoherencies in the single-curve textbook formulas, and proposed some local adjustments, but failed to propose a global, theoretically sound and practically implementable alternative.

Certainly the framework was not developed by one person or in one place. It is the collective work of many people, coming to the problem from a theoretical or practical point of view. It is a child with a lot of fathers and mothers, a child that is still growing fast.

What is certain is that the framework became paramount even before most people knew it existed, and that happened in August 2007.

1.2 Foundations, evolution, and implementation

This book's subtitle is *Foundations, Evolution, and Implementation.* The subtitle acts as a quick summary of the book content.

The *foundations* are paramount. As the framework is different from the previous one-curve approach in fundamental aspects for index-linked financial instruments, one can not rely on a 'copy and paste' style approach. Any claim, even the most basic one, should be backed by clear definitions and clear proofs. Before building the skyscraper that is the pricing of an exotic derivative, one has not only to be sure that the foundations are sound but also indicate explicitly where they are. In some cases, mathematically sophisticated developments are made, only to notice later that their domain of applicability is empty. Their starting hypotheses, which seem acceptable in a one-curve world, lead to contradictions for the multi-curve framework. This is why the book systematically uses an 'axiomatic approach'. The fundamental hypotheses are displayed in the text as quotes with a bold letter in front reminding us of the content of the hypothesis – such as **D** for the discounting hypothesis at the beginning of Section 2.2. These *axioms* or fundamental hypotheses are the foundations of the framework. No development in interest rate modelling can be done without referring to them or to their equivalent in a different framework.

The *evolution* is important to understand how this new framework started and where it is coming from. Several choices made in the new framework are recycling previous approaches with appropriate and justified twists. Why and where those twists were made is an important piece of information. Looking at the final product, it may seem to have been produced by black magic. It is not any more when seen through the eyes of Darwin, as the result of evolution.

In chronological order, the evolution came before the foundations. The theory is more an explanation of the practice obtained by tinkering with different approaches than a new, ground-breaking theory that started to be used from scratch. We refer the reader to the book *Antifragile* by Taleb (2012) for more on how practice often precedes theory. In this book, we reverse the chronological order and start with the foundations.

> Si, à l'égard de plusieurs questions traitées dans cette étude, j'ai comparé les résultats de l'observation à ceux de la théorie, ce n'est pas pour vérifier des formules établies par des méthodes mathématiques, mais pour montrer seulement que le marché, à son insu, obéit à une loi qui le domine : la loi de la probabilité.
>
> *Louis Bachelier, Théorie de la spéculation, 1900*
>
> Personal translation: If, regarding several questions analysed in this study, I compared the observed results to those of the theory, it is not to verify the formulas obtained by mathematical methods, but only to show that the market, unwittingly, complies to a law that dominates it: the law of probability.

This quote, which is the final sentence of Bachelier's 1900 thesis, can be applied in some ways to this book. We do not claim that the theory described appeared in 2007 and started to be used by the market. It first started to be used, and then its use was codified. It was first used in a particular context, on interest rate derivatives desks, and then coherency requirements, analysed by the theory, lead to its general use in finance. The market, unwittingly, complies to a law and the framework tries to axiomatise it.

The *implementation* is the 'raison d'être' of the framework and of this book. The foundations make sense only if they are used in practice. Every piece of theory, every example in this book, was created because it is useful in practice and was implemented for a practical reason to serve a practitioner request. It is possible to implement all of it in a unique and coherent framework. Moreover, the framework is not a goal by itself but is the foundation and an initial building block of interest rate modelling, including forex – as interest rates in several currencies – and inflation.

When you have read and fully implemented all the details of this book you will have completed only the first stage of your journey! Thankfully, you need not set out upon the project on your own. You can join an open community that has worked on it, through its foundations, evolution and implementation.

1.3 Standard textbook framework

The book by Hull (2006), one of the most popular introductory textbooks to derivative pricing, is used as an example of the way earlier literature treats the curves question. In the section on the *type of rates* in the *Interest rates* chapter (p. 76) the existence of both Libor and Libid is acknowledged. The latter is described in the above book as the rate at which a cash rich investor can invest in the interbank market. In 2006, those rates were relatively close to the OIS (Overnight Indexed Swap) rates.

When it comes to valuing the first derivatives (Section 4.7: Forward Rate Agreements) the explanation is 'the assumption underlying the contract is that the borrowing or lending would normally be done at Libor'. This is first a misleading statement on the instrument itself. The reality is that the contract *settlement amount* is computed by discounting with a *Libor fixing* rate between the end of the accrual period and its start. There is *no actual borrowing or lending* and there is *no assumption* in the FRA[4] contract, only a clear (contingent) settlement formula. For the valuation of the instrument before its fixing date, the approach described in the above book is to use the same Libor rate to discount the resulting quantity to the valuation date. But there is no justification in the text for choosing that particular rate from the different ones described in the previous sections. The choice of Libor for that purpose is a modelling choice and not a contractual obligation. This vagueness is certainly a witness of the consensus at that time: if the instrument is related to an Ibor rate, use similar rates for everything, even if there is no modelling or legal reason to do so. The hidden explanation is that it is easier and everybody is doing it.

Another standard textbook on swaps and curve construction is Sadr (2009). In the description of the pricing of a swap, the terms used to describe the rates linked to the Ibor leg are 'hypothetical loan', 'deposits', 'funding' and 'risk-free rates'. The text indicates that to obtain the standard one-curve formula, those different rates have to be the same. Using the equivalence between those rates the book proceeds through standard arbitrage-free arguments to obtain the standard swap pricing formula.

[4] The FRA contract details are described in Appendix B.3.

The approach is described in the chapter called 'Swaps: It's still About Discounting', which for a 2009 book is maybe an unfortunate title. There is no analysis of what happens if the rates are different. The book also describes a method that separates the forward and discount curves, in a way similar to Fruchard et al. (1995) described below. This can be viewed as an example of pre-crisis consensus where Libor, risk-free and funding were somehow considered equivalent.

In his book on model risk, Morini (2011) has a good description of the multiple hidden hypotheses that were used to price simple instruments in that framework.

It is a valuable exercise to read the past approaches with the hindsight of the recent developments and to ask why it was done in that way. Every reader interested in model risk or model validation should do the exercise as a warning to any model, including the ones presented here.

More details on the standard textbook framework are provided at the start of Chapter 2. We use this introduction as a starting point for our *evolution* analysis and as a justification on the choice of fundamental hypotheses and definitions for the rest of the book.

1.4 The precursors

We now review some of the *precursors* literature. We describe why the questions asked and the features described in those articles were important, but also why we do not incorporate them in the *multi-curve framework* literature stricto sensu.

These precursors articles include Fruchard et al. (1995), Tuckman and Porfirio (2003) and Boenkost and Schmidt (2004).

Fruchard et al. (1995) describes the existence of a cross-currency basis and proposes an approach to the pricing of cross-currency swaps. One of the hypotheses (Hypothesis 4 of the paper) is that floating references are independent of their payment frequency, and implicitly that they can be obtained from the same pseudo-discount factor curve. The justification is: 'This is a generally accepted consequence of the risk neutral probability approach.' This approach is based on a market reference for liquidity, the USD Libor (Hypothesis 5). In that main currency, there is a unique curve for discounting and forward estimation. In other currencies there is one discounting and one forward curve. The discounting curves in the other currencies are deduced from the (interest rate swaps) IRS fixed rates and cross-currency basis spread. The paper explicitly excludes the possibility of basis between Ibor rates of different tenors. This basis is one of the features of the post August 2007 market. The approach is able to take into account one cross-currency basis between the base currency and each other currency. The explanation of the difference between currencies is based on an hypothetical difference of liquidity. It potentially explains a difference between currencies but not within a given currency. The formulas to obtain the pricing are quite involved and use several layers of derivatives and integrals.

Tuckman and Porfirio (2003) also focuses on cross-currency swaps. The authors clearly point to incoherences in the then approach for the pricing of those instruments. The solution proposed uses two swaps of the same tenor, one of them not available in the market – a zero-coupon swap – but does not propose a theoretically satisfactory solution. It is an important hint, as it uses two different types of asset to build the curves, like the current multi-curve framework uses two types of instrument. Unfortunately, one of the two assets proposed by the article does not exist in the market and the solution can not be implemented in practice.

Boenkost and Schmidt (2004) also analyses cross-currency swaps starting from the visible incoherencies. They propose two different approaches. One of them is based on two curves in each currency. The two curves are not used for different financial purposes but one is used for each leg of a swap. The result is a different set of incoherencies, like a swap in its last period with zero netted cash-flows having a non-zero present value.

All the above articles asked very good questions, and...

> In mathematics, the art of posing a question is more important than the art of solving one.
>
> *Georg Cantor, 1867*

1.5 Early multi-curve framework literature

To my knowledge, the first article to propose a coherent valuation framework where the discounting is explicitly differentiated from the index forward estimation was *The irony in the derivatives discounting* (Henrard 2007). It was published in July 2007, just one month before the crisis started. To my knowledge there is no link between the article publication and the crisis.

The article focuses on discounting in interest rate derivatives. The starting observation is that different curves are used to value different instruments (OIS, IRS, cross-currency swaps), which is clearly incoherent and creates portfolio level arbitrages.

The approach proposed described only one Ibor curve; it is not an important feature of the approach but a description of the then reality. The approach can be used with several Ibor related curves without changes. The spread between the curves is described in a multiplicative way and is equivalent to the β we define in Chapter 2; our notation is adapted from that used in the paper. Only the case of a constant spread is discussed in the original paper; again, it is not an important feature of the approach but more a description of the then reality.

The first instrument for which the price is discussed is the FRA and the paper provides a pricing formula, including all the hypotheses required to obtain the result. The valuation of swaps in the framework and the impact on the value and risk for not-at-par instruments is discussed. The approach is extended in a coherent way to (Short Term Interest Rate) STIR futures, caps/floors and swaptions. The Ibor fixings are clearly dissociated from the discounting rate through the introduction of an exogenous spread. The spread can be obtained from the market value of different instruments; this is equivalent to the curve building process we propose in Chapter 5.

For practical examples, the article uses an OIS-based discounting curve (called Libid in the paper). The article was probably overly simplistic, as it used only a constant spread between the curves and did not use explicitly different curves for the different Ibor tenors. Nevertheless it was an early proposal leading to today's most commonly used approach.

The importance of the multi-curve framework, with each curve having a specific purpose, is attested by the numerous related literature that appeared in the following years. Throughout the book we have tried to cite as many sources as possible in the relevant sections. We want to give readers the opportunity to go back to the original sources. In this section we comment on a couple of the early papers which describe some particular aspects of the framework. In the literature the framework has received many names. Some of those names are: *two curves*, *multi-curve framework*, *derivative tenor curves*, *funding-Ibor*, *discounting-estimation*, *discounting-forecast*, *discounting-forward*, and *multi-curve market*. Through the book, we will use the name used in the title: *multi-curve framework*.

Ametrano and Bianchetti (2009) is the earliest paper describing the impact of the multi-curve framework on curve construction. Kijima et al. (2009) is important as the first paper describing the impact of collateral. Our approach to collateral, described in Chapter 8, is different but leads to the same fundamental result. Morini (2009) links the framework to the credit risk of the Libor basket obligors. Chibane et al. (2009) describes a multi-curve framework similar to the one described here and a bootstrapping approach to curve calibration. Bianchetti (2010) proposes a description of a multi-curve approach. Piterbarg (2010) is an early description of collateral when collateral rates and risk-free rates are equal. Moreni and Pallavicini (2010) propose a parsimonious simultaneous modelling of both discounting and forward curves. Mercurio (2009), Mercurio (2010a) and Mercurio (2010c) propose a comprehensive Libor Market Model approach for discounting and forward curves. Traven (2010) presents a simplified analysis of the framework based on a restricted set of instruments and on spread around a dominant curve. Bianchetti and Carlicchi (2011) analyses the impact of the framework on optional instruments like swaptions and cap/floor.

In Figure 1.1 the spreads between EUR swaps with different floating leg indexes are displayed in relation to the above literature for the period 2004–13. Note the

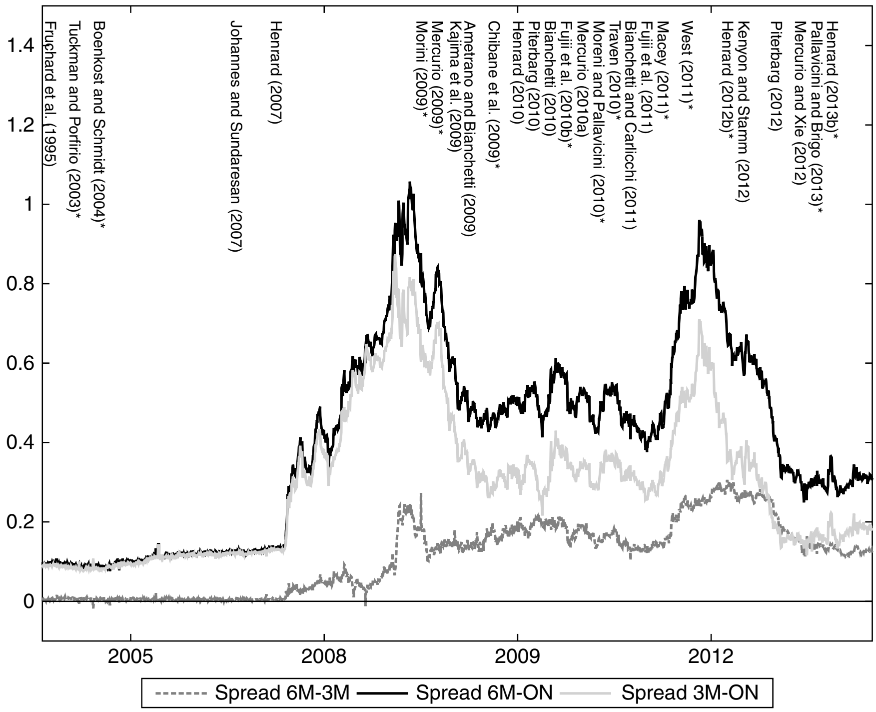


Figure 1.1 Relationship between swap spreads and multi-curves related literature. The spreads are between six months Euribor and three months Euribor and between six months Euribor and OIS, both for swaps with two years tenor.

burst of activity a couple of years after the start of the period with larger spreads. The dates on the graph are approximative in order to avoid text overlap. Literature with a * indicates working papers that have not been published in peer reviewed journals. Obviously the dates proposed are biased, as the publication in peer reviewed journals is strongly impacted by the referee process and the publication back-log. The chronology nevertheless clearly indicates a burst of activity soon after the actual widening of the different spreads, a period where the impact of the new approach was larger.

In the years following the spread widening, a large effort has been made by most banks and software developers to adjust their systems to the new reality. Several financial institutions had started these developments in one form or another before the actual crisis. Nevertheless, robust and general implementations as described here take time and they were not available immediately.

The foundations of the multi-curve framework presented in this book are mainly based on the approach described in Henrard (2010) and Henrard (2013a), the first version of which dates back to mid-2008. It is to my knowledge the first theoretically sound publicly available description of what is now called the *multi-curve framework*. We changed some of the original definition to introduce a more general framework as described in Henrard (2012c).

In that new multi-curve framework context, all of the previous textbook approaches to derivative pricing, and in particular the valuation of vanilla products such as IRS, FRA and futures need to be reviewed carefully. Even for some of the vanilla instruments, strong modelling assumptions are required to obtain simple formulas.

The notions of *yield curves* and *discount factors* need to be handled carefully. The framework presented here is based on a unique curve used to discount all the cash-flows related to the derivatives, whatever the tenors they relate to, and other curves for index estimation. Even if those estimation curves are different from the discounting and old curves, we will in some circumstances use the same names of discount factor, yield curve and forward rate.

The choice of the discounting curve is by itself an open question. Different people will choose different curves. One possibility is to relate it to the rate that can be achieved for the residual cash; the name used then is the *funding* approach. Another possible starting point is that most of the interbank transactions are done on a collateral basis, and in many contracts the collateral renumeration is set to the same rate as the one used in OIS. In particular, Kijima et al. (2009) showed that in the presence of collateral, the derivative pricing is related to discounting at collateral rate. The exact meaning of the claim needs to be made precise; this will be done in Chapter 8.

In Chapter 2, we suppose that the curves are given. The actual curve calibration process, that is how the market information can be used to obtain those theoretical objects, is proposed in Chapter 5. The curves related to indexes are often called *estimation*, *projection*, *forecast* or *forward* curves. Here the plural is used as there is one curve for each index. In practice, the most popular tenors are one, three, six and twelve months and each currency has its favourite family of indexes – Libor, Euribor, CDOR and so on. In theory one should have one curve for each tenor (two, four, and so on months) and for each index family for each currency. For those curves we will also use the terms yield, discount factor and forward rate at the risk of slight abuse of terminology. The forward curves will never be used for discounting.[5] In the next section, we set the main hypothesis used in the framework. Then we price the simple derivatives (IRS, FRA, STIR futures). The term 'simple' has to be taken with a pinch of salt. In this framework, OISs and FRAs are really contingent claims and a curve hypothesis is needed to price them. Of course STIR futures require a so-called convexity adjustment and a model must be specified. In the subsequent chapter, we describe curve construction in detail. We not only

[5] Except for FRAs where the discounting using the Ibor fixing rate is part of the contract specification.

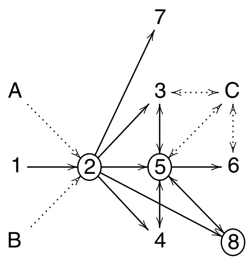
describe the basic requirements for such a process but also describe some nice-to-haves that are useful tools to have in practice. To our knowledge this is the most generic curve calibration process described in the literature, from both a theoretical and an implementation point of view.

1.6 Collateral and funding

The description in the first chapters of the multi-curve framework is in a credit risk-free world without the market reality of the *collateral* and *funding* issues. Even if the central theme of the book is the multi-curve framework and not the collateral and funding issues, a book on the curve framework resulting from the crisis would be missing an important part of the reality if the crisis impact on collateral and funding was not discussed at all. Chapter 8 describes some of the issues related to those features. It provides results that were initially presented in Kijima et al. (2009), Macey (2011), Pallavicini et al. (2012) and Piterbarg (2012). It also presents extensions of the above results. In particular we generalise the definition of collateral and propose results about collateral with assets themselves collateralised – called collateral square. Those results are original and were first published in Henrard (2013b). Our approach to proofs borrows heavily on Macey (2011) and Piterbarg (2012) even if we add the missing self-financing property to the original sketches.

1.7 How to read this book

The easiest way to read this book is to start at Chapter 1 and go to Chapter 8, with a stop in the Appendices when required – you are allowed to eat and sleep between chapters. This is certainly not the only way to read it. In the graph below I have represented the *dependency graph* between the different chapters of the book. The dotted lines for the Appendices indicate where they fit best, but some readers may want to skip them entirely. The links with bidirectional arrows indicate that in some way the two chapters depend on each other and are best read in parallel if the reader has that capacity.



From the graph, the reader can see that, in the mind of the author, the most important chapters are Chapter 2 (Foundations), Chapter 5 (Curve calibration), and Chapter 8 (Collateral).

I tried to write this book in the way I would like to read it. The different *axioms* are clearly evidenced and for each result I have tried to be clear on which hypotheses are required. I have added as many references as possible. The reader should have the opportunity to go to the original literature and compare different approaches. I have also added a lot of cross references between sections, to allow the reader to move between sections with related subjects.

1.8 What is not in this book

A large part of the collateral and funding issues are not discussed in this book. For more in-depth analysis of these issues, we refer to the recent book Brigo et al. (2013). In particular we do not analyse non-perfect collateral. In general the credit risk issues, including Credit Value Adjustment (CVA), are not part of this book. We again refer to the above recent book for more on that subject.

The relation between the Ibor index level and the credit risk embedded in the basket is not discussed. A early account of it can be found in Morini (2009) and a more recent explanation in Crépey (2013) and Filipovic and Trolle (2013). In this book we do not try to explain from first principles the level of the spreads; we take them as an input to the framework.

The inflation derivatives can be viewed as a particular case of Ibor related derivatives in the sense that an index, different from the value of risk-free bonds, is used to reset floating payments. But the inflation market is sufficiently different, with its own idiosyncrasies, and would require an extra chapter or two, not a simple paragraph. We have not tried to cover the subject here.

This book discuss only derivatives, and not assets such as bonds or equities. The bond market would be a mixture of credit, discounting and maybe Ibor (for Floating Rate Notes). This is not covered in this book.

2 The Multi-curve Framework Foundations

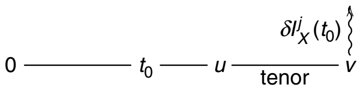
2.1 One-curve world

We start this chapter with a description of some of the one-curve framework pricing formulas. They are important from an historical point of view but also because they explain the evolution of the multi-curve framework, and where its formulas and definitions come from. Chapter 3 proposes different implementations of the framework described here.

Below, we start by describing briefly what an Ibor coupon is. This is the basic financial instrument from which the multi-curve framework is built. More details regarding the conventions related to different coupons and swaps are available in Appendix B. Here we only recall the details important for the understanding of the theory. Note also that we will refrain from using the term *Forward Rate Agreement* (FRA) for one period swaps or single payment Ibor floating coupons. The term FRA has clear market terms and conditions, which are described in Section B.3, including a special settlement procedure which makes them unsuitable as a starting instrument. Some documents in the literature use the term FRA in a loose way as a starting instrument to simplify the description. In my opinion this creates more confusion than simplification.

The general idea of the one-curve world is that all interest rate derivatives depend on only one curve, which is supposed to be at the same time the risk-free curve and the curve relevant for Ibor. The fixed future cash-flows are discounted with that curve. The discount factor at time t for a maturity u is denoted $P(t, u)$. The theoretical deposits underlying the Ibor indexes are priced using the same curve. At any fixing date, the deposit that pays the notional at the settlement date and receives the notional plus the Ibor interest at maturity is supposed to be a fair deposit; this is a deposit for which the total present value is zero. The total value includes the initial settlement of notional and the final payment of notional plus interest. For the purpose of computing the present value, the unique curve is used.

Let j designate an index, such as 'USD LIBOR three months'. We define a j-Ibor floating coupon as a financial instrument which pays at the end of the underlying period the Ibor rate set at the start of the period. The details of the instrument are as follows. The rate is set or fixed at a date t_0 for the period $[u,v]\,(t_0 \leq u < v)$ of length j; at the end date v the amount paid in currency X is the Ibor fixing $I^j_X(t_0)$ multiplied by the conventional accrual factor δ for the period. The lag between t_0 and u is called the *spot lag* or *settlement lag*. The difference between u and v is the tenor of the index j. All periods and accrual factors should be calculated according to the day count, business day convention, calendar and end-of-month rule appropriate to the relevant Ibor or overnight rate. We describe these conventions for the main currencies in Appendix B. The graphical representation of the coupon, with 0 being today, is given below. The vertical undulating curves represent floating payments and the vertical plain lines represent fixed payments.

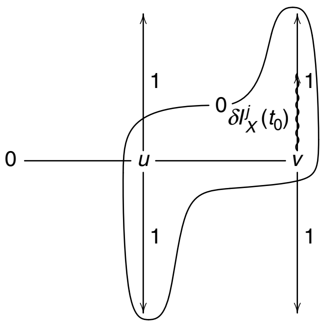


We would like to price this instrument using simple no arbitrage arguments. At this stage we suppose that we are in a credit risk-free world and that the Ibor fixing is equal to the risk-free rate to be paid over the period $[u,v]$ if a deposit is agreed in t_0.

There's no such thing as a free lunch.

Milton Friedman, 1975

To obtain the price we add synthetic payments and reception of the notional both at the start date and at the end date. Ignoring the settlement risk, paying and receiving the same amount at the same date has a zero value. Adding these cash-flows has not changed the total present value of our instrument. This is represented in the next graph.



We link three of the five payments together: the payment of the initial notional, and the receipt of the final notional and of the interest $\delta I^j_X(t_0)$. The value of this bundle of three cash-flows is very easy to compute. The interest received is the fair