Ulrich Häussler-Combe

Computational Methods for Reinforced Concrete Structures

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Cover: The photo shows a part of the façade of the Pinakothek der Moderne, Munich. The grid indicates the subdivision of a complex structure into small simple elements or finite elements, respectively.

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Preface

This book grew out of lectures the author gives at the Technische Universität Dresden. These lectures are entitled "Computational Methods for Reinforced Concrete Structures" and "Design of Reinforced Concrete Structures." Reinforced concrete is a composite of concrete and reinforcement connected by bond. Bond is a key item for the behavior of the composite which utilizes compressive strength of concrete and tensile strength of reinforcement while leading to considerable multiple cracking. This makes reinforced concrete unique compared to other construction materials such as steel, wood, glass, masonry, plastic materials, fiber reinforced plastics, geomaterials, etc.

Numerical methods like the finite element method on the other hand disclose a way for a realistic computation of the behavior of structures. But the implementations generally present themselves as black boxes in the view of users. Input is fed in and the output has to be trusted. The assumptions and methods in between are not transparent. This book aims to establish transparency with special attention for the unique properties of reinforced concrete structures. Appropriate approaches will be discussed with their potentials and limitations while integrating them in the larger framework of computational mechancis and connecting aspects of numerical mathematics, mechanics, and reinforced concrete.

This is a wide field and the scope has to be limited. The focus will be on the behavior of whole structural elements and structures and not on local problems like tracking single cracks or mesoscale phenomena. Basics of multiaxial material laws for concrete will be treated but advanced theories for multiaxial concrete behavior are not a major subject of this book. Such theories are still a field of ongoing research which by far seems not to be exhausted up to date.

The book aims at advanced students of civil and mechanical engineering, academic teachers, designing and supervising engineers involved in complex problems of reinforced concrete, and researchers and software developers interested in the broad picture. Chapter 1 describes basics of modeling and discretization with finite element methods and solution methods for nonlinear problems insofar as is required for the particular methods applied to reinforced concrete structures. Chapter 2 treats uniaxial behavior of concrete and its combination with reinforcement while discussing mechanisms of bond and cracking. This leads to the model of the reinforced tension bar which provides the basic understanding of reinforced concrete mechanisms. Uniaxial behavior is also assumed for beams and frames under bending, normal forces and shear which is described in Chapter 3. Aspects of prestressing, dynamics and second-order effects are also treated in this chapter. Chapter 4 deals with strut-and-tie models whereby still a uniaxial material behavior is assumed. This chapter also refers to rigid plasticity and limit theorems.

Modeling of multiaxial material behavior within the framework of macroscopic continuum mechanics is treated in Chapter 5. The concepts of plasticity and damage are described with simple specifications for concrete. Multiaxial cracking is integrated within the model of continuous materials. Aspects of strain softening are treated leading to concepts of regularization to preserve the objectivity of discretizations. A bridge from microscopic behavior to macroscopic material modeling is given with a sketch of the microplane theory. Chapter 6 treats biaxial states of stress and strain as they arise with plates or deep beams. Reinforcement design is described based on linear elastic plate analysis and the lower bound limit theorem. While the former neglects kinematic compatibility, this is involved again with biaxial specifications of multiaxial stress–strain relations including crack modeling.

Slabs are described as the other type of plane surface structures in Chapter 7. But in contrast to plates their behavior is predominantly characterized by internal forces like bending moments. Thus, an adaption of reinforcement design based on linear elastic analysis and the lower bound limit theorem is developed. Kinematic compatibility is again brought into play with nonlinear moment–curvature relations. Shell structures are treated in Chapter 8. A continuum-based approach with kinematic constraints is followed to derive internal forces from multiaxial stress–strain relations suitable for reinforced cracked concrete. The analysis of surface structures is closed in this chapter with the plastic analysis of simple slabs based on the upper bound limit theorem. Chapter 9 gives an overview about uncertainty and in particular about the determination of the failure probability of structures and safety factor concepts. Finally, the appendix adds more details about particular items completing the core of numerical methods for reinforced concrete structures.

Most of the described methods are complemented with examples computed with a software package developed by the author and coworkers using the PYTHON programming language.

- Programs and example data should be available under `www.concrete-fem.com`. More details are given in Appendix F.

These programs exclusively use the methods described in this book. Programs and methods are open for discussion with the disclosure of the source code and should give a stimulation for alternatives and further developments.

Thanks are given to the publisher Ernst & Sohn, Berlin, and in particular to Mrs. Claudia Ozimek for the engagement in supporting this work. My education in civil engineering, and my professional and academic career were guided by my academic teacher Prof. Dr.-Ing. Dr.-Ing. E.h. Dr. techn. h.c. Josef Eibl, former head of the department of Concrete Structures at the Institute of Concrete Structures and Building Materials at the Technische Hochschule Karlsruhe (nowadays KIT – Karlsruhe Institute of Technology), to whom I express my gratitude. Further thanks are given to former or current coworkers Patrik Pröchtel, Jens Hartig, Mirko Kitzig, Tino Kühn, Joachim Finzel and Jörg Weselek for their specific contributions. I appreciate the inspiring and collaborative environment of the Institute of Concrete Structures at the Technische Unversität Dresden. It is my pleasure to teach and research at this institution. And I have to express my deep gratitude to my wife Caroline for her love and patience.

Ulrich Häussler-Combe — Dresden, in spring 2014

Contents

Notations

The same symbols may have different meanings in some cases. But the different meanings are used in different contexts and misunderstandings should not arise.

		firstly used
General		
$\bullet^T$	transpose of vector or matrix $\bullet$	Eq. (1.5)
$\bullet^{-1}$	inverse of quadratic matrix $\bullet$	Eq. (1.13)
$\delta\bullet$	virtual variation of $\bullet$, test function	Eq. (1.5)
$\delta\bullet$	solution increment of $\bullet$ within an iteration of nonlinear equation solving	Eq. (1.70)
$\tilde{\bullet}$	$\bullet$ transformed in (local) coordinate system	Eq. (5.15)
$\dot{\bullet}$	time derivative of $\bullet$	Eq. (1.4)
Normal lowercase italics		
a_s	reinforcement cross section per unit width	Eq. (7.70)
b	cross-section width	Section 3.1.2
b_w	crack-band width	Section 2.1
d	structural height	Section 7.6.2
e	element index	Section 1.3
f	strength condition	Eq. (5.42)
f_c	uniaxial compressive strength of concrete (unsigned)	Section 2.1
f_{ct}	uniaxial tensile strength of concrete	Section 2.1
f_t	uniaxial failure stress – reinforcement	Section 2.3
f_{yk}	uniaxial yield stress – reinforcement	Section 2.3
f_E	probability density function of random variable E	Eq. (9.2)
g_f	specific crack energy per volume	Section 2.1
h	cross-section height	Section 3.1.2
m_x, m_y, m_{xy}	moments per unit width	Eq. (7.8)
n	total number of degrees of freedom in a discretized system	Section 1.2
n_E	total number of elements	Section 3.3.1
n_i	order of Gauss integration	Section 1.6
n_N	total number of nodes	Section 3.3.1
n_x, n_y, n_{xy}	normal forces per unit width	Eq. (7.8)
p	pressure	Eq. (5.8)
p_F	failure probability	Eq. (9.18)
$\bar{p}_x, \bar{p}_z$	distributed beam loads	Eq. (3.58)
r	local coordinate	Section 1.3
s	local coordinate	Section 1.3

s_{bf}	slip at residual bond strength	Section 2.4
$s_{b\,\max}$	slip at bond strength	Section 2.4
t	local coordinate	Section 1.3
t	time	Section 1.2
t_x, t_y, t_{xy}	couple force resultants per unit width	Eq. (7.67)
u	specific internal energy	Eq. (5.12)
v_x, v_y	shear forces per unit width	Eq. (7.8)
w	deflection	Eq. (1.56)
w	fictitious crack width	Eq. (2.4)
w_{cr}	critical crack width	Section 5.7.1
z	internal lever arm	Section 3.5.4

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$\mathbf{b}$	body forces	Section 1.2
$\mathbf{f}$	internal nodal forces	Section 1.2
$\mathbf{p}$	external nodal forces	Section 1.2
$\mathbf{n}$	normal vector	Eq. (5.5)
$\mathbf{t}$	surface traction	Section 1.2
$\mathbf{t}_c$	crack traction	Eq. (5.123)
$\mathbf{u}$	displacement field	Section 1.2
$\boldsymbol{v}$	nodal displacements	Section 1.2
$\mathbf{w}_c$	fictitious crack width vector	Eq. (5.122)

Normal uppercase italics

A	surface	Section 1.2, Eq. (1.5)
A	cross-sectional area of a bar or beam	Eq. (1.54)
A_s	cross-sectional area reinforcement	Example 2.4
A_t	surface with prescribed tractions	Section 1.2, Eq. (1.5)
A_u	surface with prescribed displacements	Eq. (1.53)
C	material stiffness coefficient	Eq. (2.32)
C_T	tangential material stiffness coefficient	Eq. (2.34)
D	scalar damage variable	Eq. (5.106)
D_T	tangential material compliance coefficient	Eq. (5.160)
D_{cT}	tangential compliance coefficient of cracked element	Eq. (5.132)
D_{cLT}	tangential compliance coefficient of crack band	Eq. (5.132)
E	Young's modulus	Eq. (1.43)
E_0	initial value of Young's modulus	Eq. (2.13)
E_c	initial value of Young's modulus of concrete	Section 2.1
E_s	initial Young's modulus of steel	Section 2.3
E_T	tangential modulus	Eq. (2.2)
F	yield function	Eq. (5.64)
F_E	distribution function of random variable E	Eq. (9.1)
G	shear modulus	Eq. (3.8)

G	flow function	Eq. (5.63)
G_f	specific crack energy per surface	Eq. (2.7)
I_1	first invariant of stress	Eq. (5.20)
J	determinant of Jacobian	Eq. (1.67)
J_2, J_3	second, third invariant of stress deviator	Eq. (5.20)
L_c	characteristic length of an element	Eq. (6.32)
L_e	length of bar or beam element	Section 1.3
M	bending moment	Section 3.1.2
N	normal force	Section 3.1.2
P	probability	Eq. (9.1)
T	natural period	Eq. (3.211)
V	shear force	Section 3.1.2
V	volume	Section 1.2, Eq. (1.5)

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$\mathbf{B}$	matrix of spatial derivatives of shape functions	Section 1.2, Eq. (1.2)
$\mathbf{C}$	material stiffness matrix	Eq. (1.47)
$\mathbf{C}_T$	tangential material stiffness matrix	Eq. (1.50)
$\mathbf{D}$	material compliance matrix	Eq. (1.51)
$\mathbf{D}_T$	tangential material compliance matrix	Eq. (1.51)
$\mathbf{E}$	coordinate independent strain tensor	Eq. (8.15)
$\mathbf{G}_1, \mathbf{G}_2, \mathbf{G}_3$	unit vectors of covariant system	Eq. (8.16)
$\mathbf{G}^1, \mathbf{G}^2, \mathbf{G}^3$	unit vectors of contravariant system	Eq. (8.17)
$\mathbf{I}$	unit matrix	Eq. (1.85)
$\mathbf{J}$	Jacobian	Eq. (1.20)
$\mathbf{K}$	stiffness matrix	Eq. (1.11)
$\mathbf{K}_e$	element stiffness matrix	Eq. (1.61)
$\mathbf{K}_T$	tangential stiffness matrix	Eq. (1.66)
$\mathbf{K}_{Te}$	tangential element stiffness matrix	Eq. (1.65)
$\mathbf{M}$	mass matrix	Eq. (1.60)
$\mathbf{M}_e$	element mass matrix	Eq. (1.58)
$\mathbf{N}$	matrix of shape functions	Section 1.2, Eq. (1.1)
$\mathbf{Q}$	vector/tensor rotation matrix	Eq. (5.15)
$\mathbf{S}$	coordinate independent stress tensor	Eq. (8.24)
$\mathbf{T}$	element rotation matrix	Eq. (3.109)
$\mathbf{V}_n$	shell director	Section 8.1
$\mathbf{V}_\alpha, \mathbf{V}_\beta$	unit vectors of local shell system	Eq. (8.2)

Normal lowercase Greek

α	tie inclination	Eq. (3.157)
α_E, α_R	sensitivity parameters	Eq. (9.13)
α	coefficient for several other purposes	
β	shear retention factor	Eq. (5.137)
β	reliability index	Eq. (9.12)

β_t	tension stiffening coefficient	Section 2.7
ϵ	uniaxial strain	Section 1.4, Eq. (1.43)
ϵ	strain of a beam reference axis	Section 3.1.1, Eq. (3.4)
$\epsilon_1, \epsilon_2, \epsilon_3$	principal strains	Section 5.2.3
ϵ_{ct}	concrete strain at uniaxial tensile strength	Section 2.1
ϵ_{cu}	concrete failure strain at uniaxial tension	Eq. (5.152)
ϵ_{c1}	concrete strain at uniaxial compressive strength (signed)	Section 2.1
ϵ_{cu1}	concrete failure strain at uniaxial compression (signed)	Section 2.1
ϵ_I	imposed uniaxial strain	Section 2.2
ϵ_V	volumetric strain	Eq. (5.102)
ϕ	cross-section rotation	Eq. (3.1)
ϕ	angle of external friction	Eq. (5.91)
φ	angle of orientation	Section 6.1, Eq. (6.5)
φ	creep coefficient	Eq. (2.26)
φ_c	creep coefficient of concrete	Eq. (3.119)
γ	shear angle	Eq. (3.1)
γ_E, γ_R	partial safety factors	Eq. (9.44)
κ	curvature of a beam reference axis	Section 3.1.1, Eq. (3.4)
κ_p	state variable for plasticity	Section 5.5.1
κ_d	state variable for damage	Section 5.6
μ_E	mean of random variable E	Section 9.1
ν	Poisson's ratio	Eq. (1.44)
ν	coefficient of variation	Eq. (9.46)
θ	strut inclination	Eq. (3.148)
θ	deviatoric angle	Eq. (5.46)
ϑ	angle of internal friction	Eq. (5.89)
ρ	deviatoric length	Eq. (5.45)
ρ_s	reinforcement ratio	Eq. (6.8)
ϱ_s	specific mass	Eq. (1.52)
σ	uniaxial stress	Section 1.4, Eq. (1.43)
$\sigma_1, \sigma_2, \sigma_3$	principal stresses	Section 5.2.3
σ_E	standard deviation of random variable E	Section 9.1
τ	bond stress	Section 2.4, Eq. (2.44)
τ	time variable in time history	Section 2.2
τ_{bf}	residual bond strength	Section 2.4
$\tau_{b\max}$	bond strength	Section 2.4
ω	circular natural frequency	Eq. (3.211)
ξ	hydrostatic length	Eq. (5.44)

Bold lowercase Greek

$\boldsymbol{\epsilon}$	small strain	Section 1.2
$\boldsymbol{\epsilon}$	generalized strain	Eq. (1.33)
$\boldsymbol{\epsilon}_p$	plastic small strain	Eq. (5.61)

$\boldsymbol{\kappa}$	vector of internal state variables	Eq. (5.39)
$\boldsymbol{\sigma}$	Cauchy stress	Section 1.2
$\boldsymbol{\sigma}$	generalized stress	Eq. (1.34)
$\boldsymbol{\sigma}'$	deviatoric part of Cauchy stress	Section 5.2.2

Normal uppercase Greek

Φ	standardized normal distribution function	Eq. (9.19)

Bold uppercase Greek

$\boldsymbol{\Sigma}$	viscous stress surplus	Eq. (1.76)

Chapter 1

Finite Elements Overview

1.1 Modeling Basics

"There are no exact answers. Just bad ones, good ones and better ones. Engineering is the art of approximation." Approximation is performed with models. We consider a reality of interest, e.g., a concrete beam. In a first view, it has *properties* such as dimensions, color, surface texture. From a view of structural analysis the latter ones are irrelevant. A more detailed inspection reveals a lot of more properties: composition, weight, strength, stiffness, temperatures, conductivities, capacities, and so on. From a structural point of view some of them are essential. We combine those essential properties to form a *conceptual model*. Whether a property is essential is obvious for some, but the valuation of others might be doubtful. We have to choose. By choosing properties our model becomes approximate compared to reality. Approximations are more or less accurate.

On one hand, we should reduce the number of properties of a model. Any reduction of properties will make a model less accurate. Nevertheless, it might remain a good model. On the other hand, an over-reduction of properties will make a model inaccurate and therefore useless. Maybe also properties are introduced which have no counterparts in the reality of interest. Conceptual modeling is the art of choosing properties. As all other arts it cannot be performed guided by strict rules.

The chosen properties have to be related to each other in quantitative manner. This leads to a *mathematical model*. In many cases, we have systems of differential equations relating variable properties or simply *variables*. After prescribing appropriate boundary and initial conditions an exact, unique solution should exist for variables depending on spatial coordinates and time. Thus, a particular variable forms a field. Such fields of variables are infinite as space and time are infinite.

As analytical solutions are not available in many cases, a discretization is performed to obtain approximate numerical solutions. *Discretization* reduces underlying infinite space and time into a finite number of supporting points in space and time and maps differential equations into algebraic equations relating a finite number of variables. This leads to a *numerical model*.

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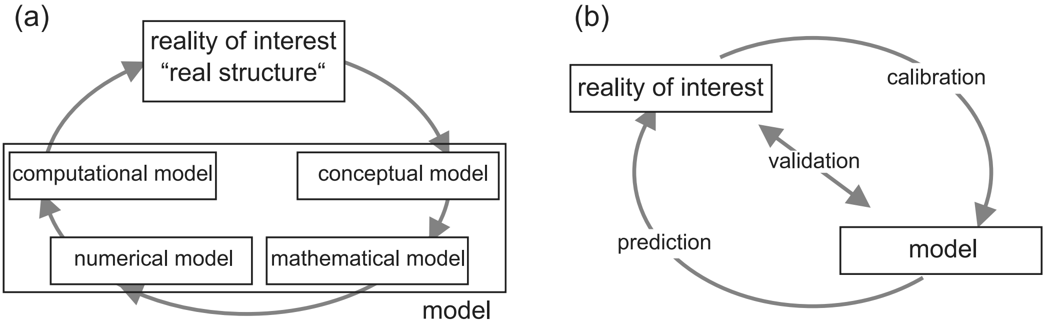

Figure 1.1: Modeling (a) Type of models following [83]. (b) Relations between model and reality.

A numerical model needs some completion as it has to be described by means of programming to form a *computational model*. Finally, programs yield solutions through processing by computers. The whole cycle is shown in Fig. 1.1. Sometimes it is appropriate to merge the sophisticated sequence of models into the *model*.

A final solution provided after computer processing is approximate compared to the exact solution of the underlying mathematical model. This is caused by discretization and round-off errors. Let us assume that we can minimize this mathematical approximation error in some sense and consider the final solution as a *model solution*. Nevertheless, the relation between the model solution and the underlying reality of interest is basically an issue. Both – model and reality of interest – share the same properties by definition or conceptual modeling, respectively. Let us also assume that the real data of properties can be objectively determined, e.g., by measurements.

Thus, real data of properties should be properly approximated by their computed model counterparts for a problem under consideration. The difference between model solution data and real data yields a *modeling error*. In order to distinguish between bad (inaccurate), good (accurate), and better model solutions, we have to choose a reference for the modeling error. This choice has to be done within a larger context, allows for discretion and again is not guided by strict rules like other arts. Furthermore, the reference may shift while getting better model solutions during testing.

A bad model solution may be caused by a bad model – bad choice of properties, poor relations of properties, insufficient discretization, programming errors – or by incorrect model parameters. *Parameters* are those properties which are assumed to be known in advance for a particular problem and are not object to a computation. Under the assumption of a good model, the model parameters can be corrected by a *calibration*. This is based upon appropriate problems from the reality of interest with the known real data. On one hand calibration minimizes the modeling error by adjusting of parameters. On the other hand, *validation* chooses other problems with known real data and assesses the modeling error without adjusting of parameters. Hopefully model solutions are still good.

Regarding reinforced concrete structures, calibrations usually involve the adaption of material parameters like strength and stiffness as part of *material models*. These parameters are chosen such that the behavior of material specimen observed in experiments is reproduced. A validation is usually performed with structural elements such as bars, beams, plates, and slabs. Computational results of *structural models* are compared with the corresponding experimental data.

This leads to basic peculiarities. Reproducible experiments performed with structural elements are of a small simplified format compared with complex unique buildings. Furthermore, repeated experimental tests with the same nominal parameters exhibit scattering results. Standardized *benchmark tests* carving out different aspects of reinforced concrete behavior are required. Actually a common agreement about such benchmark tests exists only in the first attempts. Regarding a particular problem a corresponding model has to be validated on a case-by-case strategy using adequate experimental investigations. Their choice again has no strict rules as the preceding arts.

Complex proceedings have been sketched hitherto outlining a model of modeling. Some benefit is desirable finally. Thus, a model which passed validations is usable for *predictions*. Structures created along such predictions hopefully prove their worth in the reality of interest.

This textbook covers the range of conceptual models, mathematical models, and numerical models with special attention to reinforced concrete structures. Notes regarding the computational model including available programs and example data are given in Appendix F. A major aspect of the following is modeling of *ultimate limit states*: states with maximum bearable loading or acceptable deformations and displacements in relation to failure. Another aspect is given with *serviceability*: Deformations and in some cases oscillations of structures have to be limited to allow their proper usage and fulfillment of intended services. *Durability* is a third important aspect for building structures: deterioration of materials through, e.g., corrosion, has to be controlled. This is strongly connected to cracking and crack width in the case of reinforced concrete structures. Both topics are also treated in the following.

1.2 Discretization Outline

The finite element method (FEM) is a predominant method to derive numerical models from mathematical models. Its basic theory is described in the remaining sections of this chapter insofar as it is needed for its application to different types of structures with reinforced concrete in the following chapters.

The underlying mathematical model is defined in one-, two-, or three-dimensional fields of space related to a *body* and one-dimensional space of time. A body undergoes deformations during time due to loading. We consider a simple example with a plate defined in 2D space, see Fig. 1.2. Loading is generally defined depending on time whereby time may be replaced by a loading factor in the case of quasistatic problems. Field variables depending on spatial coordinates and time are, e.g., given by the displacements.

- Such fields are discretized by dividing space into *elements* which are connected by *nodes*, see Fig. 1.3a. Elements adjoin but do not overlap and fill out the space of the body under consideration.

- Discretization basically means *interpolation*,, i.e., displacements within an element are interpolated using the values at nodes belonging to the particular element.

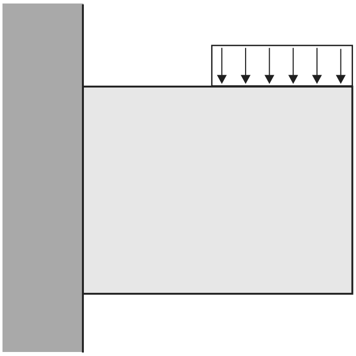

Figure 1.2: Model of a plate.

In the following this will be written as

$$\mathbf{u} = \mathbf{N} \cdot \boldsymbol{\upsilon} \tag{1.1}$$

with the displacements $\mathbf{u}$ depending on spatial coordinates and time, a matrix $\mathbf{N}$ of *shape functions* depending on spatial coordinates and a vector $\boldsymbol{\upsilon}$ depending on time and collecting all displacements at nodes. The number of components of $\boldsymbol{\upsilon}$ is n. It is two times the number of nodes in the case of the plate as the displacement $\mathbf{u}$ has components u_x, u_y. Generally some values of $\boldsymbol{\upsilon}$ may be chosen such that the *essential or displacement boundary conditions* of the problem under consideration is fulfilled by the displacements interpolated by Eq. (1.1). This is assumed for the following.

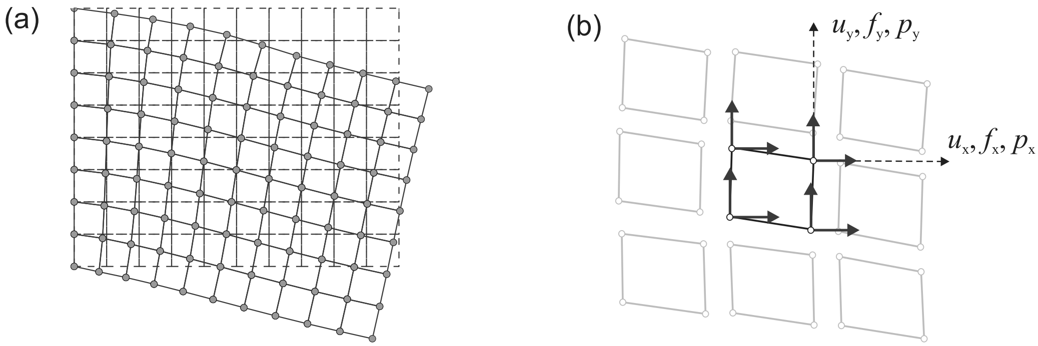

Figure 1.3: (a) Elements and nodes (deformed). (b) Nodal quantities.

Strains are derived from displacements by differentiation with respect to spatial coordinates. In the following, this will be written as

$$\boldsymbol{\epsilon} = \mathbf{B} \cdot \boldsymbol{\upsilon} \tag{1.2}$$

with the strains $\boldsymbol{\epsilon}$ depending on spatial coordinates and time, a matrix $\mathbf{B}$ of spatial derivatives of shape functions depending on spatial coordinates and the vector $\boldsymbol{\upsilon}$ as has been used in Eq. (1.1). The first examples for Eqs. (1.1, 1.2) will be given in Section 1.3.

- A field variable $\mathbf{u}$ is discretized with Eqs. (1.1, 1.2), i.e., the infinite field in space is reduced into a finite number n of variables in supporting spatial points or nodes which are collected in $\boldsymbol{\upsilon}$.

Thereby *kinematic compatibility* should be assured regarding interpolated displacements, i.e., generally spoken a coherence of displacements and deformations should be given.

Strains $\boldsymbol{\epsilon}$ lead to stresses $\boldsymbol{\sigma}$. A *material law* connects both. Material laws for solids are a science in itself. This textbook mainly covers their flavors for reinforced concrete structures. To begin with, such laws are abbreviated with

$$\boldsymbol{\sigma} = f(\boldsymbol{\epsilon}) \tag{1.3}$$

Beyond total values of stress and strain their small changes in time t have to be considered. They are measured with time derivatives

$$\dot{\boldsymbol{\epsilon}} = \frac{\partial \boldsymbol{\epsilon}}{\partial t}, \quad \dot{\boldsymbol{\sigma}} = \frac{\partial \boldsymbol{\sigma}}{\partial t} \tag{1.4}$$

Nonlinear material behavior is mainly formulated as a relation between $\dot{\boldsymbol{\epsilon}}$ and $\dot{\boldsymbol{\sigma}}$. The first concepts about material laws are given in Section 1.4.

An *equilibrium condition* is the third basic element of structural analysis beneath kinematic compatibility and material laws. It is advantageously formulated as principle of virtual work leading to

$$\int_V \delta\boldsymbol{\epsilon}^T \cdot \boldsymbol{\sigma}\, \mathrm{d}V = \int_V \delta\mathbf{u}^T \cdot \mathbf{b}\, \mathrm{d}V + \int_{A_t} \delta\mathbf{u}^T \cdot \mathbf{t}\, \mathrm{d}A \tag{1.5}$$

for quasistatic cases with the volume V of the solid body of interest, its body forces $\mathbf{b}$, its surface A, and its surface tractions $\mathbf{t}$ which are prescribed at a part A_t of the whole boundary A. Furthermore, virtual displacements $\delta\mathbf{u}$ and the corresponding virtual strains $\delta\boldsymbol{\epsilon}$ are introduced. They are arranged as vectors and $\delta\mathbf{u}^T, \delta\boldsymbol{\epsilon}^T$ indicate their transposition into row vectors to have a proper scalar product with $\boldsymbol{\sigma}, \mathbf{b}, \mathbf{t}$ which are also arranged as vectors. Fields of $\mathbf{b}$ and $\mathbf{t}$ are generally prescribed for a problem under consideration while the field of stresses $\boldsymbol{\sigma}$ remains to be determined. Surface tractions $\mathbf{t}$ constitute the *natural* or *force boundary conditions.*

- Stresses $\boldsymbol{\sigma}$ and loadings $\mathbf{b}$, $\mathbf{t}$ are in static equilibrium for the problem under consideration if Eq. (1.5) is fulfilled for arbitrary virtual displacements $\delta\mathbf{u}$ and the corresponding virtual strains $\delta\boldsymbol{\epsilon}$.

Thereby, $\delta\mathbf{u}$ is zero at the part A_u of the whole boundary A with prescribed displacement boundary conditions. Applying the displacement interpolation equation (1.1) to virtual displacements leads to

$$\delta\mathbf{u} = \mathbf{N} \cdot \delta\boldsymbol{\upsilon}, \qquad \delta\boldsymbol{\epsilon} = \mathbf{B} \cdot \delta\boldsymbol{\upsilon} \tag{1.6}$$

and using this with Eq. (1.5) to

$$\delta\boldsymbol{\upsilon}^T \cdot \left[\int_V \mathbf{B}^T \cdot \boldsymbol{\sigma}\, \mathrm{d}V\right] = \delta\boldsymbol{\upsilon}^T \cdot \left[\int_V \mathbf{N}^T \cdot \mathbf{b}\, \mathrm{d}V + \int_{A_t} \mathbf{N}^T \cdot \mathbf{t}\, \mathrm{d}A\right] \tag{1.7}$$

with transpositions $\delta\boldsymbol{v}^T, \mathbf{B}^T, \mathbf{N}^T$ of the vector $\delta\boldsymbol{v}$ and the matrices $\mathbf{B}, \mathbf{N}$. As $\delta\boldsymbol{v}$ is arbitrary a *discretized condition of static equilibrium* is derived in the form

$$\mathbf{f} = \mathbf{p} \tag{1.8}$$

with the vector $\mathbf{f}$ of *internal nodal forces* and the vector $\mathbf{p}$ of *external nodal forces*

$$\begin{aligned} \mathbf{f} &= \int_V \mathbf{B}^T \cdot \boldsymbol{\sigma} \, \mathrm{d}V \\ \mathbf{p} &= \int_V \mathbf{N}^T \cdot \mathbf{b} \, \mathrm{d}V + \int_{A_t} \mathbf{N}^T \cdot \mathbf{t} \, \mathrm{d}A \end{aligned} \tag{1.9}$$

Corresponding to the length of the vector $\boldsymbol{v}$ the vectors $\mathbf{f}$, $\mathbf{p}$ have n components.

- By means of $\boldsymbol{\sigma} = \mathbf{f}(\boldsymbol{\epsilon})$ and $\boldsymbol{\epsilon} = \mathbf{B} \cdot \boldsymbol{v}$, Eq. (1.8) constitutes a system of n nonlinear algebraic equations whereby the nodal displacements $\boldsymbol{v}$ have to be determined such that – under the constraint of displacement boundary conditions – internal nodal forces $\mathbf{f}$ are equal to prescribed external nodal forces $\mathbf{p}$.

Nonlinear stress–strain relations, i.e., *physical nonlinearities*, are always an issue for reinforced concrete structures. It is a good practice in nonlinear simulation to start with a linearization to have a reference for the refinements of a conceptual model. *Physical linearity* is described with

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\epsilon} \tag{1.10}$$

with a constant material matrix $\mathbf{C}$. Thus, using Eq. (1.2) internal forces $\mathbf{f}$ (Eq. (1.9)) can be formulated as

$$\mathbf{f} = \mathbf{K} \cdot \boldsymbol{v}, \qquad \mathbf{K} = \int_V \mathbf{B}^T \cdot \mathbf{C} \cdot \mathbf{B} \, \mathrm{d}V \tag{1.11}$$

with a constant stiffness matrix $\mathbf{K}$ leading to

$$\mathbf{K} \cdot \boldsymbol{v} = \mathbf{p} \tag{1.12}$$

This allows for a direct determination of nodal displacements which is symbolically written as

$$\boldsymbol{v} = \mathbf{K}^{-1} \cdot \mathbf{p} \tag{1.13}$$

Actually the solution is not determined with a matrix inversion but with more efficient techniques, e.g., Gauss triangularization. Stresses $\boldsymbol{\sigma}$ and strains $\boldsymbol{\epsilon}$ follow with a solution $\boldsymbol{v}$ given. A counterpart of physical linearity is *geometric linearity*:

- Small displacements and geometric linearity are assumed throughout the following if not otherwise stated.

This was a fast track for the finite element method. The rough outline will be filled out in the following. Comprehensive descriptions covering all aspects are given in, e.g., [98], [99], [9], [3]. The special aspects of reinforced concrete structures are treated in [16], [44], [81].

1.3 Elements

Interpolation performed with finite elements will be described with more details in the following. We consider the mechanical behavior of *material points* within a body. A material point is identified by its spatial coordinates. It is convenient to use a different coordinate system simultaneously. First of all, the global *Cartesian coordinate system*, see Appendix C, which is shared by all material points of a body. Thus, a material point is identified by global Cartesian coordinates

$$\mathbf{x} = \begin{pmatrix} x & y & z \end{pmatrix}^T \tag{1.14}$$

in 3D space. In the following, we assume that the space occupied by the body has been divided into finite elements. Thus, a material point may alternatively be identified by the label I of the element it belongs to and its local coordinates

$$\mathbf{r} = \begin{pmatrix} r & s & t \end{pmatrix}^T \tag{1.15}$$

related to a particular *local coordinate system* belonging to the element e. A material point undergoes *displacements*. In the case of translations they are measured in the global Cartesian system by

$$\mathbf{u} = \begin{pmatrix} u & v & w \end{pmatrix}^T \tag{1.16}$$

Displacements in a general sense may also be measured by means of rotations

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_x & \varphi_y & \varphi_z \end{pmatrix}^T \tag{1.17}$$

if we consider a material point embedded in some neighborhood of surrounding points. The indices indicate the respective reference axes of rotation.

Isoparametric interpolation will be used in the following. The general interpolation form (Eq. (1.1)) is particularized as

$$\mathbf{u} = \mathbf{N}(\mathbf{r}) \cdot \boldsymbol{v}_e \tag{1.18}$$

whereby the global coordinates of the corresponding material point are given by

$$\mathbf{x} = \mathbf{N}(\mathbf{r}) \cdot \mathbf{x}_e \tag{1.19}$$

The vector $\boldsymbol{v}_e$ collects all nodal displacements of all nodes belonging to the element e and the vector $\mathbf{x}_e$ all global nodal coordinates of that element. Isoparametric interpolation is characterized by the same interpolation for geometry and displacements with the same shape functions $\mathbf{N}(\mathbf{r})$. Global and local coordinates are related by the *Jacobian*

$$\mathbf{J} = \frac{\partial \mathbf{x}}{\partial \mathbf{r}} \tag{1.20}$$

which may be up to a 3×3 matrix for 3D cases. Strains may be derived with displacements related to global coordinates through isoparametric interpolation. Their definition depends on the type of the structural problem. A general formulation

$$\boldsymbol{\epsilon} = \mathbf{B}(\mathbf{r}) \cdot \boldsymbol{v}_e \tag{1.21}$$

is used. Strains $\boldsymbol{\epsilon}$ finally lead to stresses $\boldsymbol{\sigma}$. Examples are given in the following.

– Two-node bar element along a line.

The line is measured by a coordinate x. Each coordinate has a cross section with a cross-sectional area. The *kinematic assumption* of a bar is that every material point in the cross section has the same displacement in the line direction.

A bar element e has nodes I, J with coordinates x_I, x_J. The nodes have the displacements u_I, u_J along the line. The origin of the local coordinate r is placed in the center between the two nodes. Regarding Eqs. (1.18, 1.19) we have

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \end{pmatrix} \\ \mathbf{N} &= \begin{bmatrix} \frac{1}{2}(1-r) & \frac{1}{2}(1+r) \end{bmatrix} \\ \mathbf{x}_e &= \begin{pmatrix} x_I \\ x_J \end{pmatrix}, \quad \boldsymbol{\upsilon}_e = \begin{pmatrix} u_I \\ u_J \end{pmatrix} \end{aligned} \tag{1.22}$$

This leads to a scalar Jacobian

$$J = \frac{\partial x}{\partial r} = \frac{L_e}{2} \tag{1.23}$$

Strains are uniaxial and defined by

$$\epsilon = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} \tag{1.24}$$

leading to

$$\mathbf{B} = \frac{2}{L_e} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{1.25}$$

with a bar length $L_e = x_J - x_I$ and finally, regarding Eq. (1.3), to uniaxial strains and stresses

$$\boldsymbol{\epsilon} = (\ \epsilon\ ), \quad \boldsymbol{\sigma} = (\ \sigma\ ) \tag{1.26}$$

which are constant along the element.

– Two-node bar element in a plane

The plane is measured by coordinates x, y. The center axis of a bar is a line in this plane. Each point of the center axis again has a cross-sectional area and again the *kinematic assumption* of this bar is that every material point in the cross section has the same displacement in the direction of the center axis.

A bar element e has nodes I, J with coordinates x_I, y_I, x_J, y_J. The nodes have the displacements u_I, v_J, u_I, v_J in a plane. The origin of the local coordinate r is placed in the center between the two nodes. Regarding Eqs. (1.18) and (1.19) we have

$$\begin{aligned} \mathbf{x} &= \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \\ \mathbf{N} &= \begin{bmatrix} \frac{1}{2}(1-r) & 0 & \frac{1}{2}(1+r) & 0 \\ 0 & \frac{1}{2}(1-r) & 0 & \frac{1}{2}(1+r) \end{bmatrix} \\ \mathbf{x}_e &= \begin{pmatrix} x_I \\ y_I \\ x_J \\ y_J \end{pmatrix}, \quad \boldsymbol{\upsilon}_e = \begin{pmatrix} u_I \\ v_I \\ u_J \\ v_J \end{pmatrix} \end{aligned} \tag{1.27}$$

Uniaxial strain is measured in the direction of the bar's center axis, i.e., in a rotated coordinate system x', y' with x' being aligned to the center axis. The rotation angle α (counterclockwise positive) and the transformation matrix $\mathbf{T}$ for global coordinates and displacements are given by

$$\mathbf{T} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}, \quad \cos\alpha = \frac{x_J - x_I}{L_e}, \quad \sin\alpha = \frac{y_J - y_I}{L_e} \tag{1.28}$$

with a bar length $L_e = \sqrt{(y_J - y_I)^2 + (x_J - x_I)^2}$. The scalar Jacobian is similar as before

$$J = \frac{\partial x'}{\partial r} = \frac{L_e}{2} \tag{1.29}$$

Strains are again uniaxial and defined by

$$\epsilon = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial r}\frac{\partial r}{\partial x'} \tag{1.30}$$

leading to

$$\mathbf{B} = \frac{2}{L_I}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \tag{1.31}$$

regarding Eqs. (1.22_2, 1.28). Uniaxial strains and stresses have a form as given by Eq. (1.26).

- Two-node spring element along a line.

  The line is measured by a coordinate x. A spring element e has nodes I, J with coordinates x_I, x_J. The nodes may coincide and have the same coordinates. A *kinematic assumption* for springs may be stated as follows: only the displacement difference of two nodes is relevant irrespective of their original distance.

  Springs are an abstract concept and do not occupy a space. They miss material points, local coordinates, and a Jacobian. Thus, regarding Eq. (1.21) it is

$$\boldsymbol{\epsilon} = \begin{pmatrix} \Delta u \end{pmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \quad \boldsymbol{v}_e = \begin{pmatrix} u_I \\ u_J \end{pmatrix} \tag{1.32}$$

  whereby this particular strain $\boldsymbol{\epsilon} = \begin{pmatrix} \Delta u \end{pmatrix}$ corresponds to a difference in displacements of nodes and leads to a force $\boldsymbol{\sigma} = \begin{pmatrix} F \end{pmatrix}$. The relation between Δu and F or spring characteristics may be linear or nonlinear.

- Two-node spring element in a plane.

  The plane is measured with coordinates x, y. A spring element e has nodes I, J with coordinates x_I, y_I, x_J, y_J which may again coincide. In analogy to Eq. (1.32)

$$\boldsymbol{\epsilon} = \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \boldsymbol{v}_e = \begin{pmatrix} u_I \\ v_I \\ u_J \\ v_J \end{pmatrix} \tag{1.33}$$

Generalized strain $\boldsymbol{\epsilon}$ leads to a generalized stress

$$\boldsymbol{\sigma} = \begin{pmatrix} F_x \\ F_y \end{pmatrix} \tag{1.34}$$

The relation between $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ may again be linear or nonlinear. It may be appropriate to transform $\boldsymbol{\epsilon}$ to a rotated coordinate system before evaluating $\boldsymbol{\sigma}$ using a transformation matrix as given by $\mathbf{T}$ in Eq. (1.28). This requires back transformation of $\boldsymbol{\sigma}$ to the original coordinate system with the transposed $\mathbf{T}^T$.

– Four-node continuum element in a plane or *quad element*

The plane is measured with coordinates x, y. A continuum element has nodes I, J, K, L with coordinates $x_i, y_i, i = I, \ldots, L$. They span a quad and are ordered counterclockwise. The following local coordinates are assigned: $I : r_I = -1, s_I = -1$; $J : r_J = 1, s_J = -1$; $K : r_K = 1, s_K = 1$; $L : r_L = -1, s_L = 1$. The *kinematic assumption* of a continuum is that displacements are continuous, i.e., no gaps or overlapping occur. Regarding Eqs. (1.18, 1.19), we have

$$\begin{gathered} \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix} \\ \mathbf{N}_i(r,s) = \frac{1}{4}\begin{bmatrix} (1+r_i r)(1+s_i s) & 0 \\ 0 & (1+r_i r)(1+s_i s) \end{bmatrix} \\ \mathbf{x}_{e,i} = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad \boldsymbol{v}_{e,i} = \begin{pmatrix} u_i \\ v_i \end{pmatrix} \end{gathered} \tag{1.35}$$

with $i = I, \ldots, L$ and

$$\mathbf{x}(r,s) = \sum_i \mathbf{N}_i(r,s) \cdot \mathbf{x}_{e,i}, \quad \mathbf{u}(r,s) = \sum_i \mathbf{N}_i(r,s) \cdot \boldsymbol{v}_{e,i} \tag{1.36}$$

This leads to a Jacobian matrix

$$\mathbf{J}(r,s) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{bmatrix}, \quad J = \det \mathbf{J} \tag{1.37}$$

The Jacobian relates the partial derivatives of a function $\bullet$ with respect to local and global coordinates

$$\begin{pmatrix} \frac{\partial \bullet}{\partial r} \\ \frac{\partial \bullet}{\partial s} \end{pmatrix} = \mathbf{J} \cdot \begin{pmatrix} \frac{\partial \bullet}{\partial x} \\ \frac{\partial \bullet}{\partial y} \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} \frac{\partial \bullet}{\partial x} \\ \frac{\partial \bullet}{\partial y} \end{pmatrix} = \mathbf{J}^{-1} \cdot \begin{pmatrix} \frac{\partial \bullet}{\partial r} \\ \frac{\partial \bullet}{\partial s} \end{pmatrix} \tag{1.38}$$

with the inverse $\mathbf{J}^{-1}$ of $\mathbf{J}$. Small strains are defined by

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} \\ \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial y} \\ \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial s}\frac{\partial s}{\partial x} \end{pmatrix} \tag{1.39}$$

leading to

$$\boldsymbol{\epsilon}(r,s) = \sum_i \mathbf{B}_i(r,s) \cdot \boldsymbol{v}_{e,i} \tag{1.40}$$

with $i = I \dots J$ and

$$\mathbf{B}_i(r,s) = \frac{1}{4}\begin{bmatrix} r_i(1+s_i s)\frac{\partial r}{\partial x} + s_i(1+r_i r)\frac{\partial s}{\partial x} & 0 \\ 0 & r_i(1+s_i s)\frac{\partial r}{\partial y} + s_i(1+r_i r)\frac{\partial s}{\partial y} \\ r_i(1+s_i s)\frac{\partial r}{\partial y} + s_i(1+r_i r)\frac{\partial s}{\partial y} & r_i(1+s_i s)\frac{\partial r}{\partial x} + s_i(1+r_i r)\frac{\partial s}{\partial x} \end{bmatrix} \quad (1.41)$$

The partial derivatives $\partial r/\partial x \dots$ are given the components of the inverse Jacobian $\mathbf{J}^{-1}$. Matrices $\mathbf{N}_i, \mathbf{B}_i$ related to single nodes are assembled in larger matrices to yield $\mathbf{N}, \mathbf{B}$. Finally, Cauchy stresses

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} \quad (1.42)$$

correspond to strains in a plane. Lateral strains ϵ_z or stresses σ_z come into play with the distinction of *plane stress*, that is $\sigma_z = 0$, which may lead to a lateral strain $\epsilon_z \neq 0$, or *plane strain*, that is $\epsilon_z = 0$ which may lead to a lateral stress $\sigma_z \neq 0$. The particular values in the z-direction have to be determined indirectly with a material law, see Section 1.4.

All mentioned stresses and the corresponding strains are conjugate with respect to energy, i.e., the product $\boldsymbol{\sigma} \cdot \dot{\boldsymbol{\epsilon}}$ corresponds to a rate of *internal energy* or a rate of specific internal energy. The concept of *stresses* may be *generalized*:

- Depending on the type of structural element $\boldsymbol{\sigma}$ may stand for components of Cauchy stresses or for components of forces or for components of internal forces in a beam cross section, see Section 3.1.1. *Strains* $\boldsymbol{\epsilon}$ are *generalized* correspondingly in order to lead to internal energy, e.g., including displacements in the case forces or curvature in the case of moments.

A basic property of the aforementioned elements is that they approximate coordinates and displacements by *interpolation*: nodal values and interpolated values are identical at nodes. For instance, for the four-node continuum element we have $\mathbf{u} = \boldsymbol{\upsilon}_{e,i}$ for $r = r_i, s = s_i\, i = I, \dots, L$. This property is shared by all types of finite elements.

Another issue concerns *continuity*: For the four-node continuum element the interpolation is continuous between adjacent elements along their common boundary. One sided first derivatives of interpolation exist for each element along the boundary but are different for each element. Thus, the four-node continuum element has C^0-continuity with these properties. Furthermore, the integrals for internal and external nodal forces (Eq. (1.9)) are evaluable. Other elements may require higher orders of continuity for nodal forces to be integrable.

Finally, the issue of *element locking* has to be mentioned. The four-node continuum element, e.g., does not allow us to model the behavior of incompressible solids. Constraining Eqs. (1.41) with the condition of incompressibility $\epsilon_x + \epsilon_y + \epsilon_z = 0$ makes the element much to stiff if internal nodal forces are exactly integrated [9, 8.4]. First basic hints to treat locking are given in Section 1.7. The locking problem is exemplary treated for shells in Section 8.6.

Only a few element types were touched up to now. Further elements often used are 3D-continuum elements, 2D- and 3D-beam elements, shell elements and slab elements as a special case of shell elements. Furthermore, elements imposing constraints like contact conditions have become common in practice. For details see, e.g., [3]. Regarding the properties of reinforced concrete more details about 2D-beam elements including Bernoulli beams and Timoshenko beams are given in Section 3.3, about slabs in Section 7.4 and about shells in Chapter 8.

1.4 Material Behavior

From a mechanical point of view, material behavior is primarily focused on strains and stresses. The formal definitions of strains and stresses assume a homogeneous area of matter [64]. Regarding the virgin state of solids their behavior initially can be assumed as linear elastic in nearly all relevant cases. Furthermore, the behavior can be initially assumed as isotropic in many cases, i.e., the reaction of a material is the same in all directions. The concepts of *isotropy* and *anisotropy* are discussed in Section 5.3 with more details.

The following types of elasticity are listed exemplary:

– Uniaxial elasticity

$$\sigma = E \ \epsilon \tag{1.43}$$

with uniaxial stress σ, Young's modulus E, and uniaxial strain ϵ.

– Isotropic plane strain

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1 & \frac{\nu}{1-\nu} & 0 \\ \frac{\nu}{1-\nu} & 1 & 0 \\ 0 & 0 & \frac{1-2\nu}{2(1-\nu)} \end{bmatrix} \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{1.44}$$

with stress components $\sigma_x, \sigma_y, \sigma_{xy}$, Young's modulus E, Poisson's ratio ν, and strain components $\epsilon_x, \epsilon_y, \gamma_{xy}$. This is a subset of the triaxial isotropic linear elastic law as is described in Section 5.3.

– Isotropic plane stress

$$\begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \cdot \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} \tag{1.45}$$

ensuring $\sigma_z = 0$ for every combination $\epsilon_x, \epsilon_y, \gamma_{xy}$

– Plane bending

$$M = EJ\,\kappa \tag{1.46}$$

with the moment M, curvature κ, Young's modulus E, and cross-sectional moment of inertia J. This is covered by the concept of generalized stresses with $\boldsymbol{\sigma} = \big(M\big)$ and generalized strains $\boldsymbol{\epsilon} = \big(\kappa\big)$.

Equations (1.43)–(1.45) are a special case of

$$\boldsymbol{\sigma} = \mathbf{C} \cdot \boldsymbol{\epsilon} \tag{1.47}$$