Solutions Manual to Accompany

NONLINEAR PROGRAMMING

Theory and Algorithms

Third Edition

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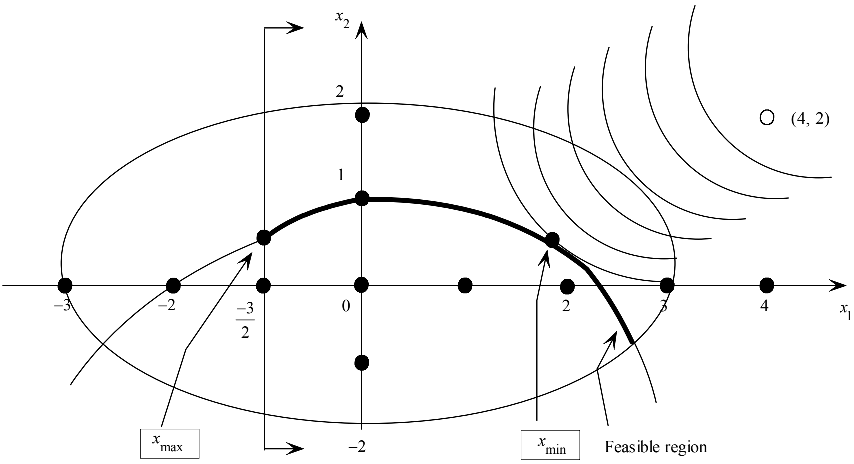
TABLE OF CONTENTS

CHAPTER 1:

INTRODUCTION

1.1 In the figure below, $x_{\min}$ and $x_{\max}$ denote optimal solutions for Part (a) and Part (b), respectively.



1.2 a. The total cost per time unit (day) is to be minimized given the storage limitations, which yields the following model:

Minimize $\quad f(Q_1, Q_2) = k_1 \dfrac{d_1}{Q_1} + h_1 \dfrac{Q_1}{2} + k_2 \dfrac{d_2}{Q_2} + h_2 \dfrac{Q_2}{2} + c_1 d_1 + c_2 d_2$

subject to $\quad s_1 Q_1 + s_2 Q_2 \le S$

$\quad Q_1 > 0, \; Q_2 > 0.$

Note that the last two terms in the objective function are constant and thus can be ignored while solving this problem.

b. Let S_j denote the lost sales (in each cycle) of product j, $j = 1, 2$. In this case, we replace the objective function in Part (a) with $F(Q_1, Q_2, S_1, S_2)$, where $F(Q_1, Q_2, S_1, S_2) = F_1(Q_1, S_1) + F_2(Q_2, S_2)$, and where

$$F_j(Q_j, S_j) = \frac{d_j}{Q_j + S_j}(k_j + c_j Q_j + \ell_j S_j - PQ_j) + h_j \frac{Q_j^2}{2(Q_j + S_j)}, \quad j = 1, 2.$$

This follows since the cycle time is $\frac{Q_j + S_j}{d_j}$, and so over some T days, the number of cycles is $\frac{Td_j}{Q_j + S_j}$. Moreover, for each cycle, the fixed setup cost is k_j, the variable production cost is $c_j Q_j$, the lost sales cost is $\ell_j S_j$, the profit (negative cost) is PQ_j, and the inventory carrying cost is $\frac{h_j}{2} Q_j (\frac{Q_j}{d_j})$. This yields the above total cost function on a daily basis.

1.4 Notation: x_j : production in period j, $j = 1,\ldots,n$

d_j : demand in period j, $j = 1,\ldots,n$

I_j : inventory at the end of period j, $j = 0, 1,\ldots,n$.

The production scheduling problem is to:

$$\text{Minimize } \sum_{j=1}^{n} [f(x_j) + cI_{j-1}]$$

subject to

$$x_j - d_j + I_{j-1} = I_j \quad \text{for } j = 1,\ldots,n$$
$$I_j \le K \quad \text{for } j = 1,\ldots,n-1$$
$$I_n = 0$$
$$x_j \ge 0,\ I_j \ge 0 \quad \text{for } j = 1,\ldots,n-1.$$

1.6 Let X denote the set of feasible portfolios. The task is to find an $x^* \in X$ such that there does not exist an $\bar{x} \in X$ for which $\bar{c}^t \bar{x} \ge \bar{c}^t x^*$ and $\bar{x}^t \mathrm{V} \bar{x} \le x^{*^t} \mathrm{V} x^*$, with at least one inequality strict. One way to find efficient portfolios is to solve:

$$\text{Maximize } \{\mu_1 \bar{c}^t x - \mu_2 x^t \mathrm{V} x : x \in X\}$$

for different values of $(\mu_1, \mu_2) > 0$ such that $\mu_1 + \mu_2 = 1$.

1.10 Let x and p denote the demand and production levels, respectively, and let Z denote a standard normal random variable. Then we need p to be such that $P(p < x - 5) \le 0.01$, which by the continuity of the normal random variable is equivalent to $P(x \ge p + 5) \le 0.01$. Therefore, p must satisfy

$$P(Z \geq \frac{p + 5 - 150}{7}) \leq 0.01,$$

where Z is a standard normal random variable. From tables of the standard normal distribution we have $P(Z \geq 2.3267) = 0.01$. Thus, we want $\frac{p - 145}{7} \geq 2.3267$, or that the chance constraint is equivalent to $p \geq 161.2869$.

1.13 We need to find a positive number K that minimizes the expected total cost. The expected total cost is $\alpha(1 - p)P(\bar{x} \leq K \,|\, \mu = \mu_2) + \beta p P(\bar{x} > K \,|\, \mu = \mu_1)$. Therefore, the mathematical programming problem can be formulated as follows:

$$\text{Minimize} \quad \alpha(1 - p)\int_0^K f(\bar{x}\,|\,\mu_2)d\bar{x} + \beta p \int_0^\infty f(\bar{x}\,|\,\mu_1)d\bar{x}$$

subject to $K \geq 0$.

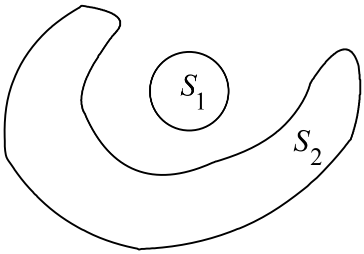
If the conditional distribution functions $F(\bar{x}\,|\,\mu_2)$ and $F(\bar{x}\,|\,\mu_1)$ are known, then the objective function is simply $\alpha(1 - p)F(K\,|\,\mu_2) + \beta p(1 - F(K\,|\,\mu_1))$.

CHAPTER 2:

CONVEX SETS

2.1 Let $x \in conv(S_1 \cap S_2)$. Then there exists $\lambda \in [0,1]$ and x_1, $x_2 \in S_1 \cap S_2$ such that $x = \lambda x_1 + (1-\lambda)x_2$. Since x_1 and x_2 are both in S_1, x must be in $conv(S_1)$. Similarly, x must be in $conv(S_2)$. Therefore, $x \in conv(S_1) \cap conv(S_2)$. (Alternatively, since $S_1 \subseteq conv(S_1)$ and $S_2 \subseteq conv(S_2)$, we have $S_1 \cap S_2 \subseteq conv(S_1) \cap conv(S_2)$ or that $conv[S_1 \cap S_2] \subseteq conv(S_1) \cap conv(S_2)$.)

An example in which $conv(S_1 \cap S_2) \neq conv(S_1) \cap conv(S_2)$ is given below:



Here, $conv(S_1 \cap S_2) = \varnothing$, while $conv(S_1) \cap conv(S_2) = S_1$ in this case.

2.2 Let S be of the form $S = \{x : Ax \leq b\}$ in general, where the constraints might include bound restrictions. Since S is a polytope, it is bounded by definition. To show that it is convex, let y and z be any points in S, and let $x = \lambda y + (1-\lambda)z$, for $0 \leq \lambda \leq 1$. Then we have $Ay \leq b$ and $Az \leq b$, which implies that

$$Ax = \lambda Ay + (1-\lambda)Az \leq \lambda b + (1-\lambda)b = b,$$

or that $x \in S$. Hence, S is convex.

Finally, to show that S is closed, consider any sequence $\{x_n\} \to x$ such that $x_n \in S$, $\forall n$. Then we have $Ax_n \leq b$, $\forall n$, or by taking limits as $n \to \infty$, we get $Ax \leq b$, i.e., $x \in S$ as well. Thus S is closed.

2.3 Consider the closed set S shown below along with $conv(S)$, where $conv(S)$ is not closed:

Now, suppose that $S \subseteq \mathbb{R}^p$ is closed. Toward this end, consider any sequence $\{x_n\} \to x$, where $x_n \in conv(S)$, $\forall n$. We must show that $x \in conv(S)$. Since $x_n \in conv(S)$, by definition (using Theorem 2.1.6), we have that we can write $x_n = \sum_{r=1}^{p+1} \lambda_{nr} x_n^r$, where $x_n^r \in S$ for $r = 1,\ldots,p+1$, $\forall n$, and where $\sum_{r=1}^{p+1} \lambda_{nr} = 1$, $\forall n$, with $\lambda_{nr} \geq 0$, $\forall r,n$. Since the λ_{nr}-values as well as the x_n^r-points belong to compact sets, there exists a subsequence K such that $\{\lambda_{nr}\}_K \to \lambda_r$, $\forall r = 1,\ldots,p+1$, and $\{x_n^r\} \to x^r$, $\forall r = 1,\ldots,p+1$. From above, we have taking limits as $n \to \infty$, $n \in K$, that

$$x = \sum_{r=1}^{p+1} \lambda_r x^r \text{, with } \sum_{r=1}^{p+1} \lambda_r = 1,\ \lambda_r \geq 0,\ \forall r = 1,\ldots,p+1,$$

where $x^r \in S$, $\forall r = 1,\ldots,p+1$ since S is closed. Thus by definition, $x \in conv(S)$ and so $conv(S)$ is closed. □

2.7 a. Let y^1 and y^2 belong to *AS*. Thus, $y^1 = Ax^1$ for some $x^1 \in S$ and $y^2 = Ax^2$ for some $x^2 \in S$. Consider $y = \lambda y^1 + (1-\lambda)y^2$, for any $0 \leq \lambda \leq 1$. Then $y = A[\lambda x^1 + (1-\lambda)x^2]$. Thus, letting $x \equiv \lambda x^1 + (1-\lambda)x^2$, we have that $x \in S$ since S is convex and that $y = Ax$. Thus $y \in AS$, and so, AS is convex.

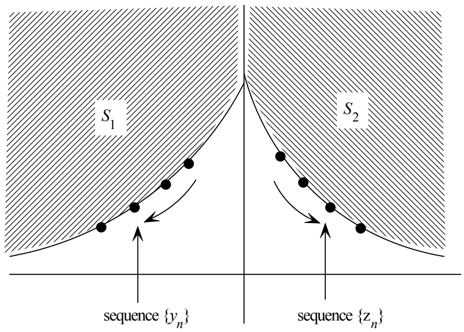
b. If $\alpha \equiv 0$, then $\alpha S \equiv \{0\}$, which is a convex set. Hence, suppose that $\alpha \neq 0$. Let αx^1 and $\alpha x^2 \in \alpha S$, where $x^1 \in S$ and $x^2 \in S$. Consider $\alpha x = \lambda \alpha x^1 + (1-\lambda)\alpha x^2$ for any $0 \leq \lambda \leq 1$. Then, $\alpha x = \alpha[\lambda x^1 + (1-\lambda)x^2]$. Since $\alpha \neq 0$, we have that $x = \lambda x^1 + (1-\lambda)x^2$, or that $x \in S$ since S is convex. Hence $\alpha x \in \alpha S$ for any $0 \leq \lambda \leq 1$, and thus αS is a convex set.

2.8 $S_1 + S_2 = \{(x_1, x_2) : 0 \leq x_1 \leq 1,\ 2 \leq x_2 \leq 3\}$.

$S_1 - S_2 = \{(x_1, x_2) : -1 \le x_1 \le 0,\ -2 \le x_2 \le -1\}$.

2.12 Let $S = S_1 + S_2$. Consider any $y,\ z \in S$, and any $\lambda \in (0,1)$ such that $y = y_1 + y_2$ and $z = z_1 + z_2$, with $\{y_1, z_1\} \subseteq S_1$ and $\{y_2, z_2\} \subseteq S_2$. Then $\lambda y + (1-\lambda)z = \lambda y_1 + \lambda y_2 + (1-\lambda)z_1 + (1-\lambda)z_2$. Since both sets S_1 and S_2 are convex, we have $\lambda y_i + (1-\lambda)z_i \in S_i$, $i = 1, 2$. Therefore, $\lambda y + (1-\lambda)z$ is still a sum of a vector from S_1 and a vector from S_2, and so it is in S. Thus S is a convex set.

Consider the following example, where S_1 and S_2 are closed, and convex.



Let $x_n = y_n + z_n$, for the sequences $\{y_n\}$ and $\{z_n\}$ shown in the figure, where $\{y_n\} \subseteq S_1$, and $\{z_n\} \subseteq S_2$. Then $\{x_n\} \to 0$ where $x_n \in S$, $\forall n$, but $0 \notin S$. Thus S is not closed.

Next, we show that if S_1 is compact and S_2 is closed, then S is closed. Consider a convergent sequence $\{x_n\}$ of points from S, and let x denote its limit. By definition, $x_n = y_n + z_n$, where for each n, $y_n \in S_1$ and $z_n \in S_2$. Since $\{y_n\}$ is a sequence of points from a compact set, it must be bounded, and hence it has a convergent subsequence. For notational simplicity and without loss of generality, assume that the sequence $\{y_n\}$ itself is convergent, and let y denote its limit. Hence, $y \in S_1$. This result taken together with the convergence of the sequence $\{x_n\}$ implies that $\{z_n\}$ is convergent to z, say. The limit, z, of $\{z_n\}$ must be in S_2, since S_2 is a closed set. Thus, $x = y + z$, where $y \in S_1$ and $z \in S_2$, and therefore, $x \in S$. This completes the proof. □

2.15 a. First, we show that $conv(S) \subseteq \hat{S}$. For this purpose, let us begin by showing that S_1 and S_2 both belong to $\hat{S}$. Consider the case of S_1 (the case of S_2 is similar). If $x \in S_1$, then $A_1 x \le b_1$, and so, $x \in \hat{S}$ with $y = x$, $z = 0$, $\lambda_1 = 1$, and $\lambda_2 = 0$. Thus $S_1 \cup S_2 \subseteq \hat{S}$, and since $\hat{S}$ is convex, we have that $conv[S_1 \cup S_2] \subseteq \hat{S}$.

Next, we show that $\hat{S} \subseteq conv(S)$. Let $x \in \hat{S}$. Then, there exist vectors y and z such that $x = y + z$, and $A_1 y \le b_1\lambda_1$, $A_2 z \le b_2\lambda_2$ for some $(\lambda_1, \lambda_2) \ge 0$ such that $\lambda_1 + \lambda_2 = 1$. If $\lambda_1 = 0$ or $\lambda_2 = 0$, then we readily obtain $y = 0$ or $z = 0$, respectively (by the boundedness of S_1 and S_2), with $x = z \in S_2$ or $x = y \in S_1$, respectively, which yields $x \in S$, and so $x \in conv(S)$. If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $x = \lambda_1 y_1 + \lambda_2 z_2$, where $y_1 = \frac{1}{\lambda_1} y$ and $z_2 = \frac{1}{\lambda_2} z$. It can be easily verified in this case that $y_1 \in S_1$ and $z_2 \in S_2$, which implies that both vectors y_1 and z_2 are in S. Therefore, x is a convex combination of points in S, and so $x \in conv(S)$. This completes the proof □

b. Now, suppose that S_1 and S_2 are not necessarily bounded. As above, it follows that $conv(S) \subseteq \hat{S}$, and since $\hat{S}$ is closed, we have that $c\ell conv(S) \subseteq \hat{S}$. To complete the proof, we need to show that $\hat{S} \subseteq c\ell conv(S)$. Let $x \in \hat{S}$, where $x = y + z$ with $A_1 y \le b_1\lambda_1$, $A_2 z \le b_2\lambda_2$, for some $(\lambda_1, \lambda_2) \ge 0$ such that $\lambda_1 + \lambda_2 = 1$. If $(\lambda_1, \lambda_2) > 0$, then as above we have that $x \in conv(S)$, so that $x \in c\ell conv(S)$. Thus suppose that $\lambda_1 = 0$ so that $\lambda_2 = 1$ (the case of $\lambda_1 = 1$ and $\lambda_2 = 0$ is similar). Hence, we have $A_1 y \le 0$ and $A_2 z \le b_2$, which implies that y is a recession direction of S_1 and $z \in S_2$ (if S_1 is bounded, then $y \equiv 0$ and then $x = z \in S_2$ yields $x \in c\ell conv(S)$). Let $\bar{y} \in S_1$ and consider the sequence

$$x_n = \lambda_n[\bar{y} + \frac{1}{\lambda_n} y] + (1 - \lambda_n)z, \text{ where } 0 < \lambda_n \le 1 \text{ for all } n.$$

Note that $\bar{y} + \frac{1}{\lambda_n} y \in S_1$, $z \in S_2$, and so $x_n \in conv(S)$, $\forall n$. Moreover, letting $\{\lambda_n\} \to 0^+$, we get that $\{x_n\} \to y + z \equiv x$, and so $x \in c\ell conv(S)$ by definition. This completes the proof. □

2.21 a. The extreme points of *S* are defined by the intersection of the two defining constraints, which yield upon solving for x_1 and x_2 in terms of x_3 that

$$x_1 = -1 \pm \sqrt{5 - 2x_3}\,,\; x_2 = \frac{3 - x_3 \mp \sqrt{5 - 2x_3}}{2}, \text{ where } x_3 \le \frac{5}{2}.$$

For characterizing the extreme directions of *S*, first note that for any fixed x_3, we have that *S* is bounded. Thus, any extreme direction must have $d_3 \neq 0$. Moreover, the maximum value of x_3 over *S* is readily verified to be bounded. Thus, we can set $d_3 = -1$. Furthermore, if $\bar{x} \equiv (0,0,0)$ and $d = (d_1, d_2, -1)$, then $\bar{x} + \lambda d \in S$, $\forall \lambda > 0$, implies that

$$d_1 + 2d_2 \le 1 \qquad (1)$$

and that $4\lambda d_2 \ge \lambda^2 d_1^2$, i.e., $4d_2 \ge \lambda^2 d_1^2$, $\forall \lambda > 0$. Hence, if $d_1 \neq 0$, then we will have $d_2 \to \infty$, and so (for bounded direction components) we must have $d_1 = 0$ and $d_2 \ge 0$. Thus together with (1), for extreme directions, we can take $d_2 = 0$ or $d_2 = 1/2$, yielding $(0,0,-1)$ and $(0, \frac{1}{2}, -1)$ as the extreme directions of *S*.

b. Since *S* is a polyhedron in R^3, its extreme points are feasible solutions defined by the intersection of three linearly independent defining hyperplanes, of which one must be the equality restriction $x_1 + x_2 = 1$. Of the six possible choices of selecting two from the remaining four defining constraints, we get extreme points defined by four such choices (easily verified), which yields $(0,1,\frac{3}{2})$, $(1,0,\frac{3}{2})$, $(0,1,0)$, and $(1,0,0)$ as the four extreme points of *S*. The extreme directions of *S* are given by extreme points of $D \equiv \{(d_1, d_2, d_3) : d_1 + d_2 + 2d_3 \le 0,\; d_1 + d_2 = 0,\; d_1 + d_2 + d_3 = 1,\; d \ge 0\}$, which is empty. Thus, there are no extreme directions of *S* (i.e., *S* is bounded).

c. From a plot of S, it is readily seen that the extreme points of S are given by (0, 0), plus all point on the circle boundary $x_1^2 + x_2^2 = 2$ that lie between the points $(-\sqrt{2/5},\ 2\sqrt{2/5})$ and $(\sqrt{2/5},\ 2\sqrt{2/5})$, including the two end-points. Furthermore, since S is bounded, it has no extreme direction.

2.24 By plotting (or examining pairs of linearly independent active constraints), we have that the extreme points of S are given by (0, 0), (3, 0), and (0, 2). Furthermore, the extreme directions of S are given by extreme points of $D = \{(d_1, d_2): \ -d_1 + 2d_2 \le 0 \quad d_1 - 3d_2 \le 0, \quad d_1 + d_2 = 1, \quad d \ge 0\}$, which are readily obtained as $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{3}{4}, \frac{1}{4})$. Now, let

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \lambda \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}, \text{ where } \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \mu \begin{bmatrix} 3 \\ 0 \end{bmatrix} + (1-\mu) \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

for $(\mu, \lambda) > 0$. Solving, we get $\mu = 7/9$ and $\lambda = 20/9$, which yields

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{7}{9} \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \frac{2}{9} \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \frac{20}{9} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}.$$

2.31 The following result from linear algebra is very useful in this proof:
(*) An $(m+1) \times (m+1)$ matrix G with a row of ones is invertible if and only if the remaining m rows of G are linearly independent. In other words, if $G = \begin{bmatrix} B & a \\ e^t & 1 \end{bmatrix}$, where B is an $m \times m$ matrix, a is an $m \times 1$ vector, and e is an $m \times 1$ vector of ones, then G is invertible if and only if B is invertible. Moreover, if G is invertible, then

$G^{-1} = \begin{bmatrix} M & g \\ h^t & f \end{bmatrix}$, where $M = B^{-1}(I + \frac{1}{\alpha} a e^t B^{-1})$, $g = -\frac{1}{\alpha} B^{-1} a$, $h^t = -\frac{1}{\alpha} e^t B^{-1}$, and $f = \frac{1}{\alpha}$, and where $\alpha = 1 - e^t B^{-1} a$.

By Theorem 2.6.4, an n-dimensional vector d is an extreme point of D if and only if the matrix $\begin{bmatrix} A \\ e^t \end{bmatrix}$ can be decomposed into $[B_D \ N_D]$ such that $\begin{bmatrix} d_B \\ d_N \end{bmatrix}$, where $d_N = 0$ and $d_B = B_D^{-1} b_D \ge 0$, where $b_D = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}$. From Property (*) above, the matrix $\begin{bmatrix} A \\ e^t \end{bmatrix}$ can be decomposed into $[B_D \ N_D]$, where B_D is a nonsingular matrix, if and only if A can be decomposed into $[B \ N]$, where B is an $m \times m$ invertible matrix. Thus, the matrix B_D must necessarily be of the form $\begin{bmatrix} B & a_j \\ e^t & 1 \end{bmatrix}$, where B is an $m \times m$ invertible submatrix of A. By applying the above equation for the inverse of G, we obtain

$$d_B = B_D^{-1} b_D = \begin{bmatrix} -\frac{1}{\alpha} B^{-1} a_j \\ \frac{1}{\alpha} \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} -B^{-1} a_j \\ 1 \end{bmatrix},$$

where $\alpha = 1 - e^t B^{-1} a_j$. Notice that $d_B \geq 0$ if and only if $\alpha > 0$ and $B^{-1} a_j \leq 0$. This result, together with Theorem 2.6.6, leads to the conclusion that d is an extreme point of D if and only if d is an extreme direction of S.

Thus, for characterizing the extreme points of D, we can examine bases of $\begin{bmatrix} A \\ e^t \end{bmatrix}$, which are limited by the number of ways we can select $(m+1)$ columns out of n, i.e.,

$$\binom{n}{m+1} = \frac{n!}{(m+1)!(n-m-1)!},$$

which is fewer by a factor of $\frac{1}{(m+1)}$ than that of the Corollary to Theorem 2.6.6.

2.42 Problem P: Minimize $\{c^t x : Ax = b,\ x \geq 0\}$.

(Homogeneous) Problem D: Maximize $\{b^t y : A^t y \leq 0\}$.

Problem P has no feasible solution if and only if the system $Ax = b$, $x \geq 0$, is inconsistent. That is, by Farkas' Theorem (Theorem 2.4.5), this occurs if and only if the system $A^t y \leq 0$, $b^t y > 0$ has a solution, i.e., if and only if the homogeneous version of the dual problem is unbounded. □

2.45 Consider the following pair of primal and dual LPs, where e is a vector of ones in $\mathbb{R}^m$:

P:	Max	$e^t p$	**D**:	Min	$0^t x$
	subject to	$A^t p = 0$			$Ax \geq e$
		$p \geq 0.$			x unres.

Then, System 2 has a solution $\Leftrightarrow$ P is unbounded (take any feasible solution to System 2, multiply it by a scalar λ, and take $\lambda \to \infty$) $\Leftrightarrow$ D