

Solutions Manual to Accompany

NONLINEAR PROGRAMMING

Theory and Algorithms

Third Edition

MOKHTAR S. BAZARAA

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WILEY

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to Accompany
Nonlinear Programming:
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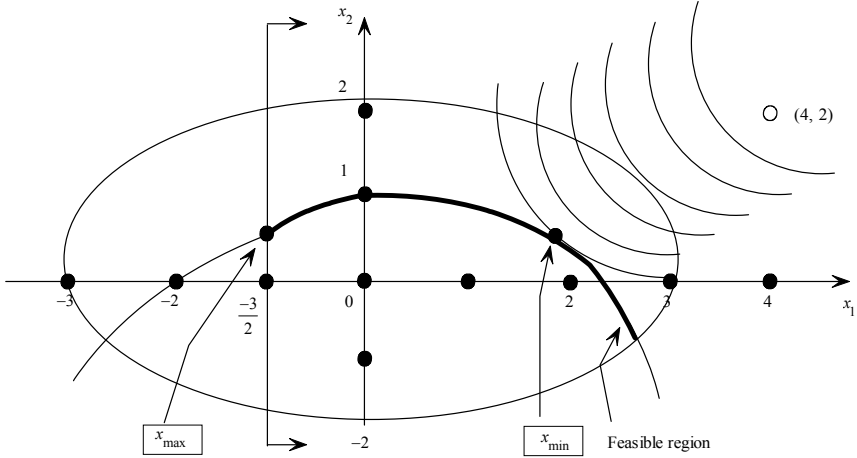
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CHAPTER 1:

INTRODUCTION

- 1.1 In the figure below, x_{\min} and x_{\max} denote optimal solutions for Part (a) and Part (b), respectively.



- 1.2 a. The total cost per time unit (day) is to be minimized given the storage limitations, which yields the following model:

$$\text{Minimize } f(Q_1, Q_2) = k_1 \frac{d_1}{Q_1} + h_1 \frac{Q_1}{2} + k_2 \frac{d_2}{Q_2} + h_2 \frac{Q_2}{2} + c_1 d_1 + c_2 d_2$$

$$\text{subject to } s_1 Q_1 + s_2 Q_2 \leq S$$

$$Q_1 > 0, Q_2 > 0.$$

Note that the last two terms in the objective function are constant and thus can be ignored while solving this problem.

- b. Let S_j denote the lost sales (in each cycle) of product $j, j = 1, 2$. In this case, we replace the objective function in Part (a) with $F(Q_1, Q_2, S_1, S_2)$, where $F(Q_1, Q_2, S_1, S_2) = F_1(Q_1, S_1) + F_2(Q_2, S_2)$, and where

$$F_j(Q_j, S_j) = \frac{d_j}{Q_j + S_j} (k_j + c_j Q_j + \ell_j S_j - P Q_j) + h_j \frac{Q_j^2}{2(Q_j + S_j)}, \quad j = 1, 2.$$

This follows since the cycle time is $\frac{Q_j + S_j}{d_j}$, and so over some T days, the number of cycles is $\frac{Td_j}{Q_j + S_j}$. Moreover, for each cycle, the fixed setup cost is k_j , the variable production cost is $c_j Q_j$, the lost sales cost is $\ell_j S_j$, the profit (negative cost) is PQ_j , and the inventory carrying cost is $\frac{h_j}{2} Q_j \left(\frac{Q_j}{d_j}\right)$. This yields the above total cost function on a daily basis.

- 1.4** Notation: x_j : production in period $j, j = 1, \dots, n$
 d_j : demand in period $j, j = 1, \dots, n$
 I_j : inventory at the end of period $j, j = 0, 1, \dots, n$.

The production scheduling problem is to:

$$\text{Minimize } \sum_{j=1}^n [f(x_j) + cI_{j-1}]$$

subject to

$$x_j - d_j + I_{j-1} = I_j \quad \text{for } j = 1, \dots, n$$

$$I_j \leq K \quad \text{for } j = 1, \dots, n-1$$

$$I_n = 0$$

$$x_j \geq 0, I_j \geq 0 \quad \text{for } j = 1, \dots, n-1.$$

- 1.6** Let X denote the set of feasible portfolios. The task is to find an $x^* \in X$ such that there does not exist an $\bar{x} \in X$ for which $\bar{c}^t \bar{x} \geq \bar{c}^t x^*$ and $\bar{x}^t \forall \bar{x} \leq x^* \forall x^*$, with at least one inequality strict. One way to find efficient portfolios is to solve:

$$\text{Maximize } \{\mu_1 \bar{c}^t x - \mu_2 x^t \forall x : x \in X\}$$

for different values of $(\mu_1, \mu_2) > 0$ such that $\mu_1 + \mu_2 = 1$.

- 1.10** Let x and p denote the demand and production levels, respectively, and let Z denote a standard normal random variable. Then we need p to be such that $P(p < x - 5) \leq 0.01$, which by the continuity of the normal random variable is equivalent to $P(x \geq p + 5) \leq 0.01$. Therefore, p must satisfy

$$P(Z \geq \frac{p + 5 - 150}{7}) \leq 0.01,$$

where Z is a standard normal random variable. From tables of the standard normal distribution we have $P(Z \geq 2.3267) = 0.01$. Thus, we want

$$\frac{p - 145}{7} \geq 2.3267, \text{ or that the chance constraint is equivalent to } p \geq 161.2869.$$

1.13 We need to find a positive number K that minimizes the expected total cost. The expected total cost is $\alpha(1 - p)P(\bar{x} \leq K | \mu = \mu_2) + \beta pP(\bar{x} > K | \mu = \mu_1)$. Therefore, the mathematical programming problem can be formulated as follows:

$$\begin{aligned} \text{Minimize } & \alpha(1 - p) \int_0^K f(\bar{x} | \mu_2) d\bar{x} + \beta p \int_0^\infty f(\bar{x} | \mu_1) d\bar{x} \\ \text{subject to } & K \geq 0. \end{aligned}$$

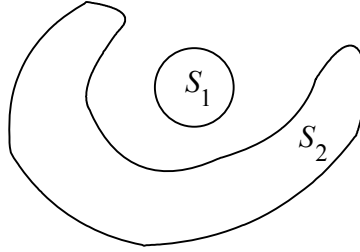
If the conditional distribution functions $F(\bar{x} | \mu_2)$ and $F(\bar{x} | \mu_1)$ are known, then the objective function is simply $\alpha(1 - p)F(K | \mu_2) + \beta p(1 - F(K | \mu_1))$.

CHAPTER 2:

CONVEX SETS

- 2.1** Let $x \in \text{conv}(S_1 \cap S_2)$. Then there exists $\lambda \in [0,1]$ and $x_1, x_2 \in S_1 \cap S_2$ such that $x = \lambda x_1 + (1 - \lambda)x_2$. Since x_1 and x_2 are both in S_1 , x must be in $\text{conv}(S_1)$. Similarly, x must be in $\text{conv}(S_2)$. Therefore, $x \in \text{conv}(S_1) \cap \text{conv}(S_2)$. (Alternatively, since $S_1 \subseteq \text{conv}(S_1)$ and $S_2 \subseteq \text{conv}(S_2)$, we have $S_1 \cap S_2 \subseteq \text{conv}(S_1) \cap \text{conv}(S_2)$ or that $\text{conv}[S_1 \cap S_2] \subseteq \text{conv}(S_1) \cap \text{conv}(S_2)$.)

An example in which $\text{conv}(S_1 \cap S_2) \neq \text{conv}(S_1) \cap \text{conv}(S_2)$ is given below:



Here, $\text{conv}(S_1 \cap S_2) = \emptyset$, while $\text{conv}(S_1) \cap \text{conv}(S_2) = S_1$ in this case.

- 2.2** Let S be of the form $S = \{x : Ax \leq b\}$ in general, where the constraints might include bound restrictions. Since S is a polytope, it is bounded by definition. To show that it is convex, let y and z be any points in S , and let $x = \lambda y + (1 - \lambda)z$, for $0 \leq \lambda \leq 1$. Then we have $Ay \leq b$ and $Az \leq b$, which implies that

$$Ax = \lambda Ay + (1 - \lambda)Az \leq \lambda b + (1 - \lambda)b = b,$$

or that $x \in S$. Hence, S is convex.

Finally, to show that S is closed, consider any sequence $\{x_n\} \rightarrow x$ such that $x_n \in S, \forall n$. Then we have $Ax_n \leq b, \forall n$, or by taking limits as $n \rightarrow \infty$, we get $Ax \leq b$, i.e., $x \in S$ as well. Thus S is closed.

- 2.3** Consider the closed set S shown below along with $\text{conv}(S)$, where $\text{conv}(S)$ is not closed:



Now, suppose that $S \subseteq \mathbb{R}^p$ is closed. Toward this end, consider any sequence $\{x_n\} \rightarrow x$, where $x_n \in \text{conv}(S)$, $\forall n$. We must show that $x \in \text{conv}(S)$. Since $x_n \in \text{conv}(S)$, by definition (using Theorem 2.1.6),

we have that we can write $x_n = \sum_{r=1}^{p+1} \lambda_{nr} x_n^r$, where $x_n^r \in S$ for $r = 1, \dots, p+1$, $\forall n$, and where $\sum_{r=1}^{p+1} \lambda_{nr} = 1$, $\forall n$, with $\lambda_{nr} \geq 0$, $\forall r, n$.

Since the λ_{nr} -values as well as the x_n^r -points belong to compact sets, there exists a subsequence K such that $\{\lambda_{nr}\}_K \rightarrow \lambda_r$, $\forall r = 1, \dots, p+1$, and $\{x_n^r\} \rightarrow x^r$, $\forall r = 1, \dots, p+1$. From above, we have taking limits as $n \rightarrow \infty$, $n \in K$, that

$$x = \sum_{r=1}^{p+1} \lambda_r x^r, \text{ with } \sum_{r=1}^{p+1} \lambda_r = 1, \lambda_r \geq 0, \forall r = 1, \dots, p+1,$$

where $x^r \in S$, $\forall r = 1, \dots, p+1$ since S is closed. Thus by definition, $x \in \text{conv}(S)$ and so $\text{conv}(S)$ is closed. \square

2.7 a. Let y^1 and y^2 belong to AS . Thus, $y^1 = Ax^1$ for some $x^1 \in S$ and $y^2 = Ax^2$ for some $x^2 \in S$. Consider $y = \lambda y^1 + (1 - \lambda)y^2$, for any $0 \leq \lambda \leq 1$. Then $y = A[\lambda x^1 + (1 - \lambda)x^2]$. Thus, letting $x \equiv \lambda x^1 + (1 - \lambda)x^2$, we have that $x \in S$ since S is convex and that $y = Ax$. Thus $y \in AS$, and so, AS is convex.

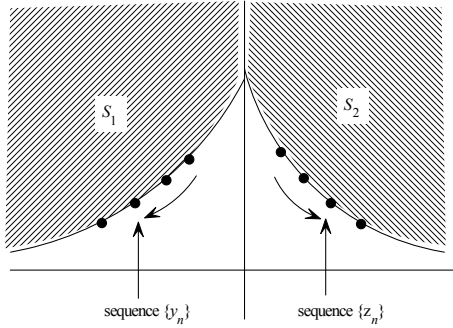
b. If $\alpha \equiv 0$, then $\alpha S \equiv \{0\}$, which is a convex set. Hence, suppose that $\alpha \neq 0$. Let αx^1 and $\alpha x^2 \in \alpha S$, where $x^1 \in S$ and $x^2 \in S$. Consider $\alpha x = \lambda \alpha x^1 + (1 - \lambda)\alpha x^2$ for any $0 \leq \lambda \leq 1$. Then, $\alpha x = \alpha[\lambda x^1 + (1 - \lambda)x^2]$. Since $\alpha \neq 0$, we have that $x = \lambda x^1 + (1 - \lambda)x^2$, or that $x \in S$ since S is convex. Hence $\alpha x \in \alpha S$ for any $0 \leq \lambda \leq 1$, and thus αS is a convex set.

2.8 $S_1 + S_2 = \{(x_1, x_2) : 0 \leq x_1 \leq 1, 2 \leq x_2 \leq 3\}$.

$$S_1 - S_2 = \{(x_1, x_2) : -1 \leq x_1 \leq 0, -2 \leq x_2 \leq -1\}.$$

2.12 Let $S = S_1 + S_2$. Consider any $y, z \in S$, and any $\lambda \in (0,1)$ such that $y = y_1 + y_2$ and $z = z_1 + z_2$, with $\{y_1, z_1\} \subseteq S_1$ and $\{y_2, z_2\} \subseteq S_2$. Then $\lambda y + (1 - \lambda)z = \lambda y_1 + \lambda y_2 + (1 - \lambda)z_1 + (1 - \lambda)z_2$. Since both sets S_1 and S_2 are convex, we have $\lambda y_i + (1 - \lambda)z_i \in S_i, i = 1, 2$. Therefore, $\lambda y + (1 - \lambda)z$ is still a sum of a vector from S_1 and a vector from S_2 , and so it is in S . Thus S is a convex set.

Consider the following example, where S_1 and S_2 are closed, and convex.



Let $x_n = y_n + z_n$, for the sequences $\{y_n\}$ and $\{z_n\}$ shown in the figure, where $\{y_n\} \subseteq S_1$, and $\{z_n\} \subseteq S_2$. Then $\{x_n\} \rightarrow 0$ where $x_n \in S, \forall n$, but $0 \notin S$. Thus S is not closed.

Next, we show that if S_1 is compact and S_2 is closed, then S is closed. Consider a convergent sequence $\{x_n\}$ of points from S , and let x denote its limit. By definition, $x_n = y_n + z_n$, where for each $n, y_n \in S_1$ and $z_n \in S_2$. Since $\{y_n\}$ is a sequence of points from a compact set, it must be bounded, and hence it has a convergent subsequence. For notational simplicity and without loss of generality, assume that the sequence $\{y_n\}$ itself is convergent, and let y denote its limit. Hence, $y \in S_1$. This result taken together with the convergence of the sequence $\{x_n\}$ implies that $\{z_n\}$ is convergent to z , say. The limit, z , of $\{z_n\}$ must be in S_2 , since S_2 is a closed set. Thus, $x = y + z$, where $y \in S_1$ and $z \in S_2$, and therefore, $x \in S$. This completes the proof. \square

2.15 a. First, we show that $\text{conv}(S) \subseteq \hat{S}$. For this purpose, let us begin by showing that S_1 and S_2 both belong to \hat{S} . Consider the case of S_1 (the case of S_2 is similar). If $x \in S_1$, then $A_1x \leq b_1$, and so, $x \in \hat{S}$ with $y = x$, $z = 0$, $\lambda_1 = 1$, and $\lambda_2 = 0$. Thus $S_1 \cup S_2 \subseteq \hat{S}$, and since \hat{S} is convex, we have that $\text{conv}[S_1 \cup S_2] \subseteq \hat{S}$.

Next, we show that $\hat{S} \subseteq \text{conv}(S)$. Let $x \in \hat{S}$. Then, there exist vectors y and z such that $x = y + z$, and $A_1y \leq b_1\lambda_1$, $A_2z \leq b_2\lambda_2$ for some $(\lambda_1, \lambda_2) \geq 0$ such that $\lambda_1 + \lambda_2 = 1$. If $\lambda_1 = 0$ or $\lambda_2 = 0$, then we readily obtain $y = 0$ or $z = 0$, respectively (by the boundedness of S_1 and S_2), with $x = z \in S_2$ or $x = y \in S_1$, respectively, which yields $x \in S$, and so $x \in \text{conv}(S)$. If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $x = \lambda_1 y_1 + \lambda_2 z_2$, where $y_1 = \frac{1}{\lambda_1}y$ and $z_2 = \frac{1}{\lambda_2}z$. It can be easily verified in this case that $y_1 \in S_1$ and $z_2 \in S_2$, which implies that both vectors y_1 and z_2 are in S . Therefore, x is a convex combination of points in S , and so $x \in \text{conv}(S)$. This completes the proof \square

b. Now, suppose that S_1 and S_2 are not necessarily bounded. As above, it follows that $\text{conv}(S) \subseteq \hat{S}$, and since \hat{S} is closed, we have that $\text{clconv}(S) \subseteq \hat{S}$. To complete the proof, we need to show that $\hat{S} \subseteq \text{clconv}(S)$. Let $x \in \hat{S}$, where $x = y + z$ with $A_1y \leq b_1\lambda_1$, $A_2z \leq b_2\lambda_2$, for some $(\lambda_1, \lambda_2) \geq 0$ such that $\lambda_1 + \lambda_2 = 1$. If $(\lambda_1, \lambda_2) > 0$, then as above we have that $x \in \text{conv}(S)$, so that $x \in \text{clconv}(S)$. Thus suppose that $\lambda_1 = 0$ so that $\lambda_2 = 1$ (the case of $\lambda_1 = 1$ and $\lambda_2 = 0$ is similar). Hence, we have $A_1y \leq 0$ and $A_2z \leq b_2$, which implies that y is a recession direction of S_1 and $z \in S_2$ (if S_1 is bounded, then $y \equiv 0$ and then $x = z \in S_2$ yields $x \in \text{clconv}(S)$). Let $\bar{y} \in S_1$ and consider the sequence

$$x_n = \lambda_n [\bar{y} + \frac{1}{\lambda_n}y] + (1 - \lambda_n)z, \text{ where } 0 < \lambda_n \leq 1 \text{ for all } n.$$

Note that $\bar{y} + \frac{1}{\lambda_n}y \in S_1$, $z \in S_2$, and so $x_n \in \text{conv}(S)$, $\forall n$.

Moreover, letting $\{\lambda_n\} \rightarrow 0^+$, we get that $\{x_n\} \rightarrow y + z \equiv x$, and so $x \in \text{clconv}(S)$ by definition. This completes the proof. \square

- 2.21 a.** The extreme points of S are defined by the intersection of the two defining constraints, which yield upon solving for x_1 and x_2 in terms of x_3 that

$$x_1 = -1 \pm \sqrt{5 - 2x_3}, \quad x_2 = \frac{3 - x_3 \mp \sqrt{5 - 2x_3}}{2}, \quad \text{where } x_3 \leq \frac{5}{2}.$$

For characterizing the extreme directions of S , first note that for any fixed x_3 , we have that S is bounded. Thus, any extreme direction must have $d_3 \neq 0$. Moreover, the maximum value of x_3 over S is readily verified to be bounded. Thus, we can set $d_3 = -1$. Furthermore, if $\bar{x} \equiv (0, 0, 0)$ and $d = (d_1, d_2, -1)$, then $\bar{x} + \lambda d \in S$, $\forall \lambda > 0$, implies that

$$d_1 + 2d_2 \leq 1 \tag{1}$$

and that $4\lambda d_2 \geq \lambda^2 d_1^2$, i.e., $4d_2 \geq \lambda^2 d_1^2$, $\forall \lambda > 0$. Hence, if $d_1 \neq 0$, then we will have $d_2 \rightarrow \infty$, and so (for bounded direction components) we must have $d_1 = 0$ and $d_2 \geq 0$. Thus together with (1), for extreme directions, we can take $d_2 = 0$ or $d_2 = 1/2$, yielding $(0, 0, -1)$ and $(0, \frac{1}{2}, -1)$ as the extreme directions of S .

- b. Since S is a polyhedron in R^3 , its extreme points are feasible solutions defined by the intersection of three linearly independent defining hyperplanes, of which one must be the equality restriction $x_1 + x_2 = 1$. Of the six possible choices of selecting two from the remaining four defining constraints, we get extreme points defined by four such choices (easily verified), which yields $(0, 1, \frac{3}{2})$, $(1, 0, \frac{3}{2})$, $(0, 1, 0)$, and $(1, 0, 0)$ as the four extreme points of S . The extreme directions of S are given by extreme points of $D \equiv \{(d_1, d_2, d_3) : d_1 + d_2 + 2d_3 \leq 0, d_1 + d_2 = 0, d_1 + d_2 + d_3 = 1, d \geq 0\}$, which is empty. Thus, there are no extreme directions of S (i.e., S is bounded).

- c. From a plot of S , it is readily seen that the extreme points of S are given by $(0, 0)$, plus all point on the circle boundary $x_1^2 + x_2^2 = 2$ that lie between the points $(-\sqrt{2/5}, 2\sqrt{2/5})$ and $(\sqrt{2/5}, 2\sqrt{2/5})$, including the two end-points. Furthermore, since S is bounded, it has no extreme direction.

2.24 By plotting (or examining pairs of linearly independent active constraints), we have that the extreme points of S are given by $(0, 0)$, $(3, 0)$, and $(0, 2)$. Furthermore, the extreme directions of S are given by extreme points of $D = \{(d_1, d_2) : -d_1 + 2d_2 \leq 0, d_1 - 3d_2 \leq 0, d_1 + d_2 = 1, d \geq 0\}$,

which are readily obtained as $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{3}{4}, \frac{1}{4})$. Now, let

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \lambda \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}, \text{ where } \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \mu \begin{bmatrix} 3 \\ 0 \end{bmatrix} + (1 - \mu) \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

for $(\mu, \lambda) > 0$. Solving, we get $\mu = 7/9$ and $\lambda = 20/9$, which yields

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{7}{9} \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \frac{2}{9} \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \frac{20}{9} \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}.$$

2.31 The following result from linear algebra is very useful in this proof:

(*) An $(m + 1) \times (m + 1)$ matrix G with a row of ones is invertible if and only if the remaining m rows of G are linearly independent. In other words,

if $G = \begin{bmatrix} B & a \\ e^t & 1 \end{bmatrix}$, where B is an $m \times m$ matrix, a is an $m \times 1$ vector, and e

is an $m \times 1$ vector of ones, then G is invertible if and only if B is invertible. Moreover, if G is invertible, then

$$G^{-1} = \begin{bmatrix} M & g \\ h^t & f \end{bmatrix}, \text{ where } M = B^{-1}(I + \frac{1}{\alpha} a e^t B^{-1}), g = -\frac{1}{\alpha} B^{-1} a,$$

$$h^t = -\frac{1}{\alpha} e^t B^{-1}, \text{ and } f = \frac{1}{\alpha}, \text{ and where } \alpha = 1 - e^t B^{-1} a.$$

By Theorem 2.6.4, an n -dimensional vector d is an extreme point of D if and only if the matrix $\begin{bmatrix} A \\ e^t \end{bmatrix}$ can be decomposed into $[B_D \ N_D]$ such that

$$\begin{bmatrix} d_B \\ d_N \end{bmatrix}, \text{ where } d_N = 0 \text{ and } d_B = B_D^{-1} b_D \geq 0, \text{ where } b_D = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}.$$

Property (*) above, the matrix $\begin{bmatrix} A \\ e^t \end{bmatrix}$ can be decomposed into $[B_D \ N_D]$,

where B_D is a nonsingular matrix, if and only if A can be decomposed into $[B \ N]$, where B is an $m \times m$ invertible matrix. Thus, the matrix B_D must

necessarily be of the form $\begin{bmatrix} B & a_j \\ e^t & 1 \end{bmatrix}$, where B is an $m \times m$ invertible submatrix of A . By applying the above equation for the inverse of G , we obtain

$$d_B = B_D^{-1}b_D = \begin{bmatrix} -\frac{1}{\alpha}B^{-1}a_j \\ \frac{1}{\alpha} \end{bmatrix} = \frac{1}{\alpha} \begin{bmatrix} -B^{-1}a_j \\ 1 \end{bmatrix},$$

where $\alpha = 1 - e^t B^{-1}a_j$. Notice that $d_B \geq 0$ if and only if $\alpha > 0$ and $B^{-1}a_j \leq 0$. This result, together with Theorem 2.6.6, leads to the conclusion that d is an extreme point of D if and only if d is an extreme direction of S .

Thus, for characterizing the extreme points of D , we can examine bases of $\begin{bmatrix} A \\ e^t \end{bmatrix}$, which are limited by the number of ways we can select $(m+1)$ columns out of n , i.e.,

$$\binom{n}{m+1} = \frac{n!}{(m+1)!(n-m-1)!},$$

which is fewer by a factor of $\frac{1}{(m+1)}$ than that of the Corollary to Theorem 2.6.6.

2.42 Problem P : Minimize $\{c^t x : Ax = b, x \geq 0\}$.

(Homogeneous) Problem D : Maximize $\{b^t y : A^t y \leq 0\}$.

Problem P has no feasible solution if and only if the system $Ax = b, x \geq 0$, is inconsistent. That is, by Farkas' Theorem (Theorem 2.4.5), this occurs if and only if the system $A^t y \leq 0, b^t y > 0$ has a solution, i.e., if and only if the homogeneous version of the dual problem is unbounded. \square

2.45 Consider the following pair of primal and dual LPs, where e is a vector of ones in \mathbb{R}^m :

$$\begin{array}{ll} \mathbf{P:} & \text{Max} \quad e^t p \\ & \text{subject to} \quad A^t p = 0 \\ & \quad \quad \quad p \geq 0. \\ \mathbf{D:} & \text{Min} \quad 0^t x \\ & \quad \quad \quad Ax \geq e \\ & \quad \quad \quad x \text{ unres.} \end{array}$$

Then, System 2 has a solution $\Leftrightarrow P$ is unbounded (take any feasible solution to System 2, multiply it by a scalar λ , and take $\lambda \rightarrow \infty$) $\Leftrightarrow D$