Solutions Manual to Accompany

COMBINATORIAL REASONING

An Introduction to the Art of Counting

DUANE DETEMPLE
WILLIAM WEBB

WILEY

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PREFACE

This manual provides the statements and complete solutions to all of the odd numbered problems in the textbook *Combinatorial Reasoning: An Introduction to the Art of Counting.* The definitions, theorems, figures, and other problems referenced in the solutions contained in this manual are to that book.

HOW TO USE THIS MANUAL

The most important thing to remember is that you should not turn to any solution in this manual without first attempting to solve the problem on your own. Many of the problems are subtle or complex and therefore require considerable thought—and time!—before you can expect to find a correct method of solution. You will learn best by trying to solve a problem on your own, even if you are unsuccessful. We hope that most often you will consult this manual simply to confirm your own answer. Often your answer will be the same as ours, but sometimes you may have found a different method of solution that is not only correct, but may even be better than ours (if so, please send us your alternative solution at the address below). If the answer to a problem eludes you even after a good effort, then take a look at the solution offered here. Even in this case, it is best only to read the beginning of the solution and see if you can continue to solve the problem on your own.

TIPS FOR SOLVING COMBINATORIAL PROBLEMS

Many students wonder how to go about attacking nonroutine problems. We have listed some suggestions below that may be helpful for solving combinatorial problems and more generally for solving problems in any branch of mathematics.

- Try small cases and look for patterns
- Separate a problem into cases
- Draw a figure
- Make a table of values
- Look for a similar or related problem, one you already know how to solve
- For combinatorial problems, apply one of the strategies explored in the textbook: use the addition and multiplication principles; identify the problem as a permutation, combination, or distribution; find and solve a recurrence relation; use a generating function; use the principle of inclusion/exclusion; restate the problem to relate it to a problem answered by well known numbers such as binomial coefficients, Fibonacci numbers, Stirling numbers, partition numbers, Catalan numbers, and so on.

For a more complete discussion of mathematical problem solving, you are encouraged to consult *How to Solve It,* the classic, but still useful, book of George Pólya.

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PART I
THE BASICS OF ENUMERATIVE COMBINATORICS

CHAPTER 1
INITIAL EnCOUNTers WITH COMBINATORIAL REASONING

PROBLEM SET 1.2

1.2.1 A bag contains 7 blue, 4 red, and 9 green marbles. How many marbles must be drawn from the bag without looking to be sure that we have drawn

a. a pair of red marbles?

b. a pair of marbles of the same color?

c. a pair of marbles with different colors?

d. three marbles of the same color?

e. a red, blue, and green marble?

Answer

(a) 18 **(b)** 4 **(c)** 10 **(d)** 7 **(e)** 17

1.2.3. There are 10 people at a dinner party. Show that at least two people have the same number of acquaintances at the party.

Answer

Each person can know any where from 0 (no one) to 9 (everyone) people. But if someone knows no one, there cannot be someone who knows everyone, and vice versa. Thus, place the 10 people into the 9 boxes that are labeled 1, 2, … , 8, and 0|9. By the pigeonhole principle, some box has at least 2 members. That is, there are at least two people at the party with the same number of acquaintances.

1.2.5 Given any five points in the plane, with no three on the same line, show that there exists a subset of four of the points that form a convex quadrilateral.

[*Hint*: Consider the *convex hull* of the points; that is, consider the convex polygon with vertices at some or all of the given points that encloses all five points. This scenario can be imagined as the figure obtained by bundling the points within a taut rubber band that has been snapped around all five points. There are then three cases to consider, depending on whether the convex hull is a pentagon, a quadrilateral containing the fifth point, or a triangle containing the other two given points.]

Answer

If the convex hull is a pentagon, each set of 4 points are the vertices of a convex quadrilateral. If the convex hull is a quadrilateral, the convex hull itself is the sought quadrilateral. If the convex hull is a triangle, the line formed by the two points within the triangle separates the vertices of the triangle into opposite half planes. By the pigeonhole principle, there are two points of the triangle in the same half plane. These two points, together with the two points within the triangle, can be combined to form the desired convex quadrilateral.

1.2.7 Given five points on a sphere, show that some four of the points lie in a closed hemisphere.

[*Note*: A closed hemisphere includes the points on the bounding great circle.]

Answer

Pick any two of the five points and draw a great circle through them. At least two of the remaining three points belong to the same closed hemisphere determined by the great circle. These two points, and the two starting points, are four points in the same closed hemisphere.

1.2.9. Suppose that 51 numbers are chosen randomly from $[100] = \{1, 2, \ldots, 100\}$. Show that two of the numbers have the sum 101.

Answer

Each of the 51 numbers belongs to one of the 50 sets $\{1, 100\}$, $\{2, 99\}$, … , $\{50, 51\}$. Some set contains two of the chosen numbers, and these sum to 101.

1.2.11. Choose any 51 numbers from $[100] = \{1, 2, \ldots, 100\}$. Show that there are two of the chosen numbers that are relatively prime (i.e., have no common divisor other than 1).

Answer

Place each of the 51 numbers into one of the 50 sets $\{1, 2\}$, $\{3, 4\}$, … , $\{99, 100\}$. One of the sets contains a pair of consecutive integers that are relatively prime.

1.2.13. Choose any 51 numbers from $[100] = \{1, 2, \ldots, 100\}$. Show that there are two of the chosen numbers for which one divides the other.

Answer

Any natural number has the form $m = 2^{d_m} k_m$, where $d_m \geq 0$ and k_m is odd. Call k_m the *odd factor* of *m.* For example, the odd factor of $100 = 2^2 \cdot 25$ is $k_{100} = 25$. Thus, the odd factors of the 51 chosen numbers are in the set {1, 3, 5, ... , 99}. Since this is a set with 50 members, two of the 51 chosen numbers have the same odd factor. The smaller is then a divisor of the larger, with a quotient that is a power of two.

1.2.15. Consider a string of $3n$ consecutive natural numbers. Show that any subset of $n + 1$ of the numbers has two members that differ by at most 2.

Answer

Suppose the $3n$ consecutive numbers are $a, a + 1, \ldots, b.$ Each of the $n + 1$ numbers in the given subset belongs to one of the sets $\{a, a + 1, a + 2\}$, $\{a + 3, a + 4, a + 5\}$. ... , $\{b - 2, b - 1, b\}$. By the pigeonhole principle, one of these sets has two members of the subset and these differ by at most 2.

1.2.17. Suppose that the numbering of the squares along the spiral path shown in Example 1.9 is continued. What number k is assigned to the square S whose lower left corner is at the point (9, 5)?

Answer

We want to find a solution to the equations $k = 11i + 9$ and $k = 16j + 5$ for some integers i and j. This gives us $11i + 4 = 16j$. Both 4 and 16 are divisible by 4, so we see that i is divisible by 4. If we let $i = 4$, then $j = 3$ and we obtain the solution $k = 53$. The next multiple of 4 giving a solution is $i = 20$, but then $k = 229$ and we see that the spiral is overlapping itself with repeated squares covered a second time.

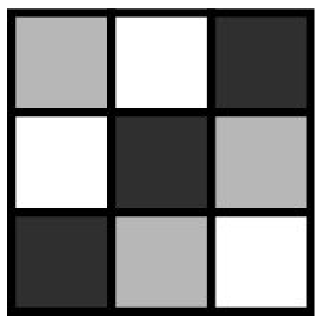
1.2.19. Generalize the results of Problem 1.2.18.

a. How many spiral paths exist on the torus if $m = n$?

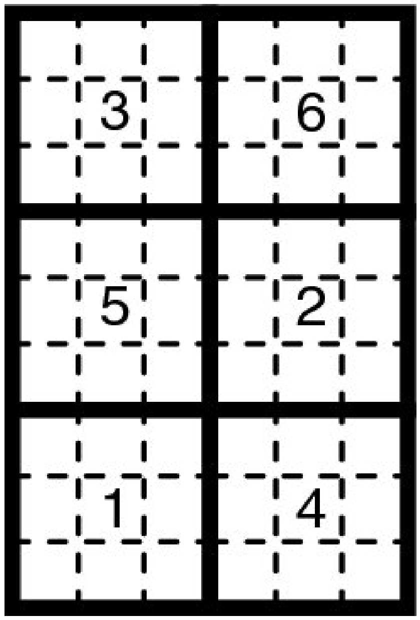
b. Suppose $d \geq 2$ is the largest common divisor of m and n. How many distinct spiral paths exist on the torus?

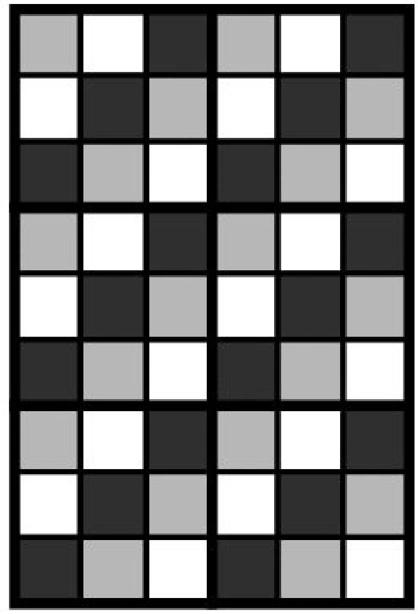
Answer

a. Any path returns to its starting position in m steps, so there are m spirals each covering m squares. For example, there are three paths when $m = n = 3$, as shown here.



b. Since m/d and n/d are relatively prime, there is a unique spiral with mn/d^2 steps that covers a d by d square at each step. For example, if $m = 6$ and $n = 9$, then $d = 3$, and there is a unique spiral of length $\frac{mn}{d^2} = \frac{6 \cdot 9}{3^2} = 6$ of 3×3 squares that covers the torus. This is seen at the left in the figure below. By part (a) we see there are d nonintersecting spirals on the torus, each of length $\frac{mn}{d}$. The case $m = 6$, $n = 9$, $d = 3$, is shown at the right below, with the $d = 3$ paths each of length $\frac{mn}{d} = \frac{6 \cdot 9}{3} = 18$ shown in black, white, and gray.





PROBLEM SET 1.3

1.3.1. Consider an $m \times n$ chessboard, where m is even and n is odd. Prove that if two opposite corners of the board are removed, the trimmed board can be tiled with dominoes.

Answer

The left and right hand columns of height $n - 1$ of the trimmed board can each be tiled with vertical dominoes. The remaining board is has all of its rows of even length $m - 2$, so it can be tiled with horizontal dominoes.

1.3.3. Suppose that the lower left $j \times k$ rectangle is removed from an $m \times n$ chessboard, leaving an angle-shaped chessboard. Prove that that angular board can be tiled with dominoes if it contains an even number of squares.

Answer

Since $mn - jk = (m - k)n + (n - j)k$ is even, $(m - k)n$ and $(n - j)k$ have the same parity. If both are even, we can tile the resulting $(m - k) \times n$ and $(n - j) \times k$ rectangles. If both are odd, then n and k are odd thus m and j must be even. We can then tile the $m \times (n - j)n$ and $(m - k) \times j$ rectangles.

Alternate answer

View the angular region as the union of rectangles *A, B,* and *C,* where the corner rectangle *B* shares an edge with each of *A* and *C.* If all three rectangles have even area, the angle can be tiled since *A, B,* and *C* can each be tiled individually. If *A* and *B,* or *B* and *C,* each have odd area, then combining the odd rectangles shows that the angle is a union of two even area rectangles and therefore can be tiled. If *A* and *C* are odd, their edges are all of odd length and therefore rectangle *B* is also odd; the angular board therefore is not of even area.

1.3.5. Consider a rectangular solid of size $l \times m \times n$, where l, m, and n are all odd positive integers. Imagine that the unit cubes forming the solid are alternately colored gray and black, with a black cube at the corner in the first column, first row, and first layer.

a. What is the color of each of the remaining corner cubes of the solid?

b. How can the color of the cube in column j, row k, and layer h of the solid be determined?

c. Prove that removing any black cube leaves a trimmed solid that can be filled with solid $1 \times 1 \times 2$ dominoes.

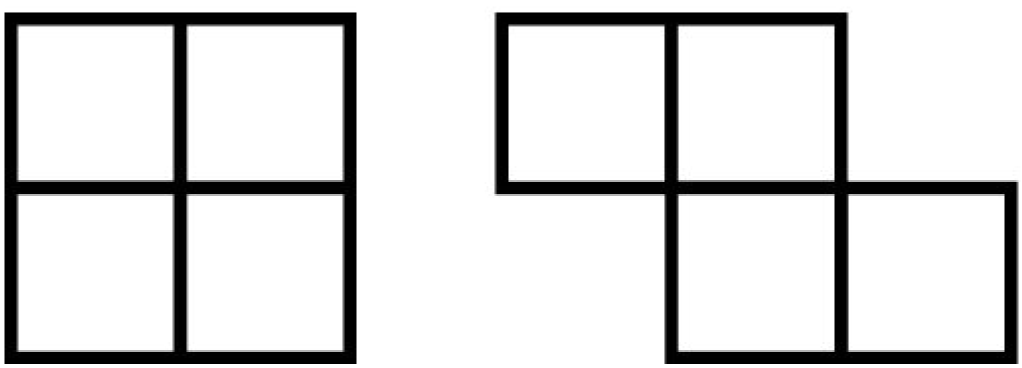
Answer

a. Since the colors alternate, all eight corners of the solid are black.

b. The cube is black if and only if the sum $j + k + h$ is odd. For example, the cube in column 1, row 1, and layer 1 is black since $1 + 1 + 1 = 3$, an odd number. That is, j, k, and h must all be odd, or one must be odd and the other two even.

c. If the cube that is removed is black, and is in column j, row k, and layer h, then j, k, and h are all odd or two are even and one is odd. With no loss in generality assume that $j + k$ is even and h is odd. Theorem 1.21 tells us that layer h can be tiled with dominoes confined to that layer. When layer h is removed it leaves two (possibly one if $h = 1$ or n) rectangular solids with an even dimension and so it can be tiled with solid dominoes.

1.3.7. A *tetromino* is formed with four squares joined along common edges. For example, the *O* and the *Z* tetromino are shown here.



a. Find the three other tetrominoes, called the *I*, *J*, and *T* tetrominoes.

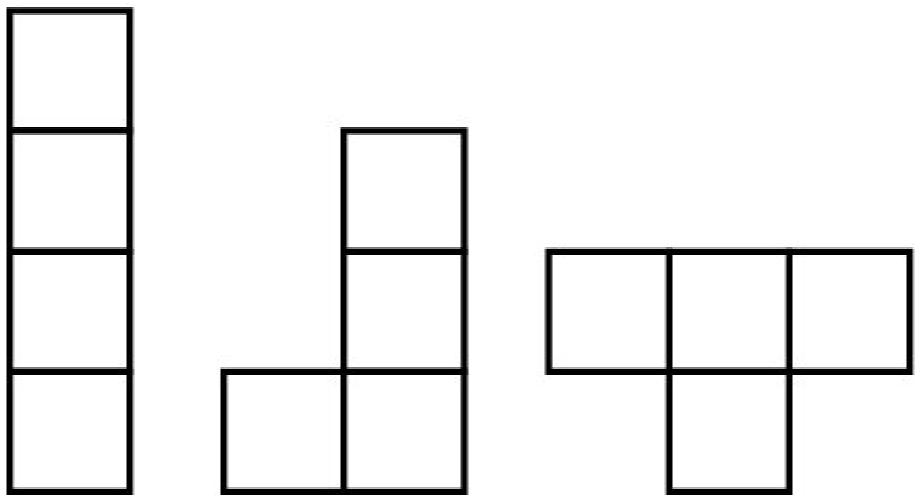
b. The set of five tetrominoes has a total area of 20 square units. Explain why it is not possible to tile a 4×5 rectangle with a set of tetrominoes.

c. Show that a 4×10 rectangle can be tiled with two sets of tetrominoes.

d. Show that a 5×8 rectangle can be tiled with two sets of tetrominoes.

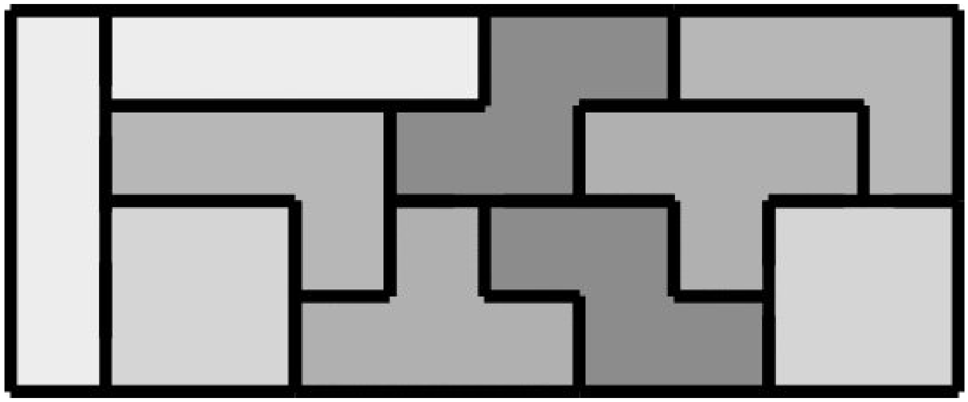
Answer

a.

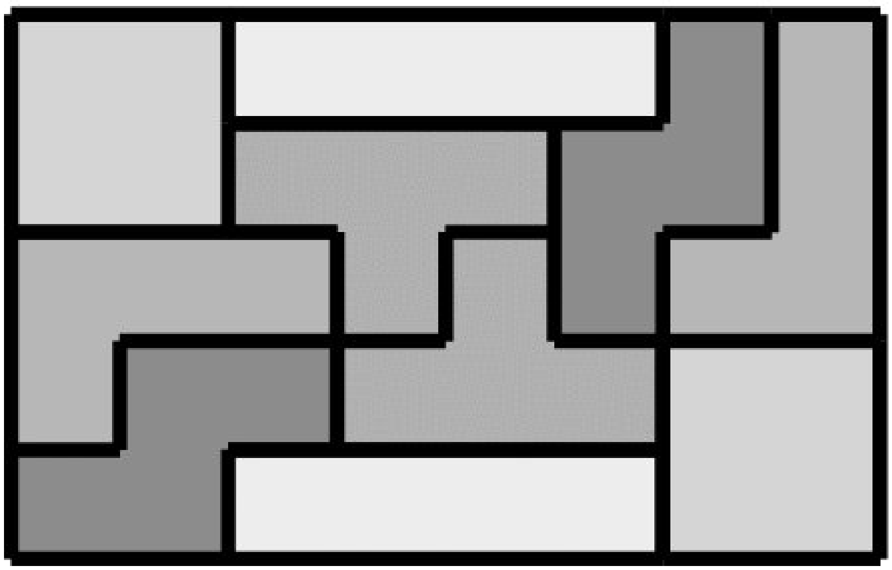


b. A 4×5 chessboard has 10 unit squares of each color. The *O, Z, I,* and *J* tetrominoes each cover 2 unit squares of each color, but the *T* tetromino covers 3 squares of one color and one of the other color. Therefore the 4×5 square cannot be tiled with a set of tetrominoes.

c.

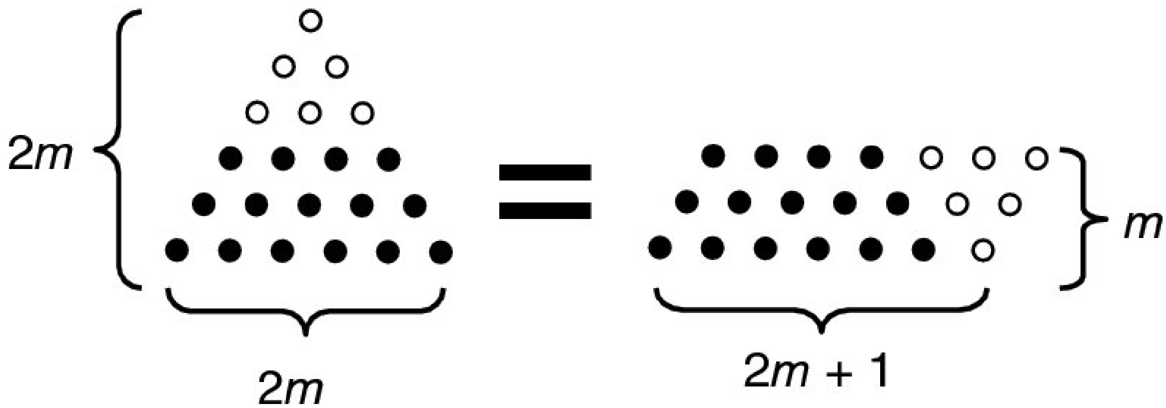


d.



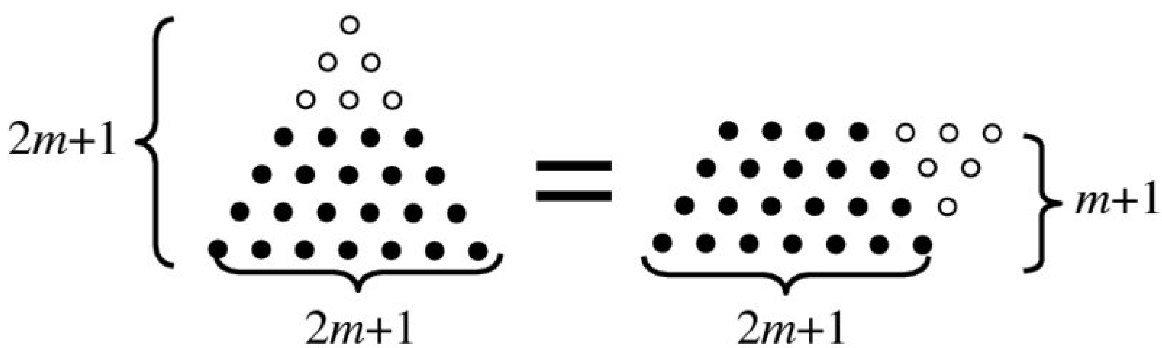
PROBLEM SET 1.4

1.4.1. The following diagram illustrates that $t_{2m} = m(2m + 1)$



Create a similar diagram that illustrates the formula $t_{2m+1} = (2m + 1)(m + 1)$.

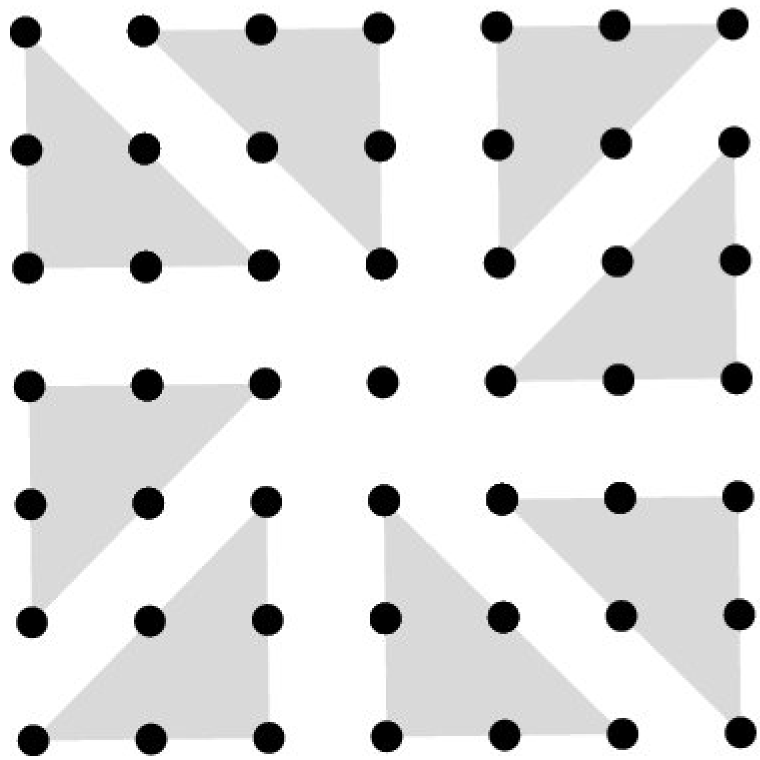
Answer



1.4.3. Use both algebra and dot patterns to show that the square of an odd integer is congruent to 1 modulo 8. That is, show that $s_{2n+1} = 8u_n + 1$ for some integer u_n. Be sure to identify the integer u_n by its well-known name.

Answer

The answer is $s_{2n+1} = 8t_n + 1$, since $(2n+1)^2 = 4n^2 + 4n + 1 = 8\frac{n(n+1)}{2} + 1 = 8t_n + 1$. See the following diagram:



1.4.5. The *centered square numbers* are obtained much like the centered triangle numbers of Problem 1.4.4, except that squares with an increasing number of dots per side surround a center dot.

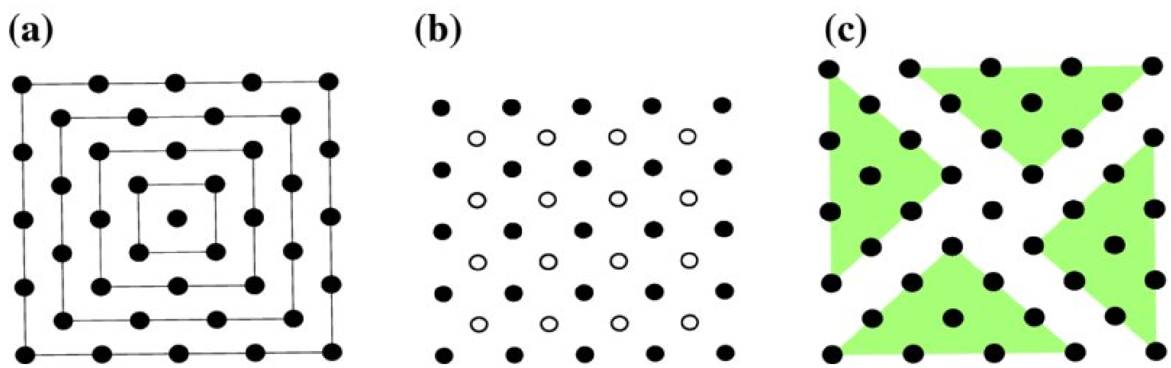
a. Create a diagram that shows the sequence of centered square numbers beginning with 1, 5, 13, 25, and 41.

b. Color the dots in the diagram from part (a) to show that the n^{th} centered square number is given by $(n + 1)^2 + n^2$.

c. Shade your diagram from part (a) to shows that every centered square number is congruent to 1 modulo 4.

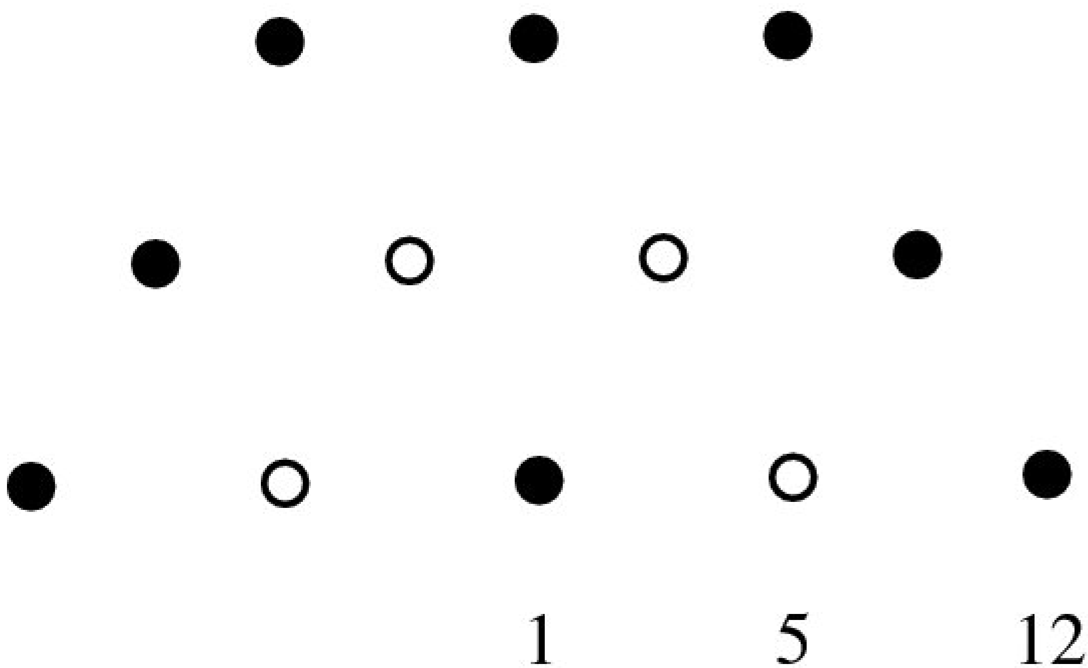
d. Verify part (c) with algebra.

Answer



a. $(n+1)^2 + n^2 = 2n^2 + 2n + 1 = 1 + 2n(n+1) = 1 + 4\dfrac{n(n+1)}{2} = 1 + 4t_n$

1.4.7. The first three *trapezoidal numbers* are 1, 5, and 12, as shown by the dot pattern here.



a. Continue the trapezoidal pattern to find the next three trapezoidal numbers.

b. Draw some lines on your diagram from part (a) to explain why the trapezoidal numbers are simply an alternative pattern for the pentagonal numbers $p_1 = 1$, $p_2 = 5$, $p_3 = 12$, ….

c. Use the trapezoidal diagram to show why each pentagonal number is the sum of a triangular number and a square number. Give an explicit formula for p_n in terms of the triangular and square numbers.

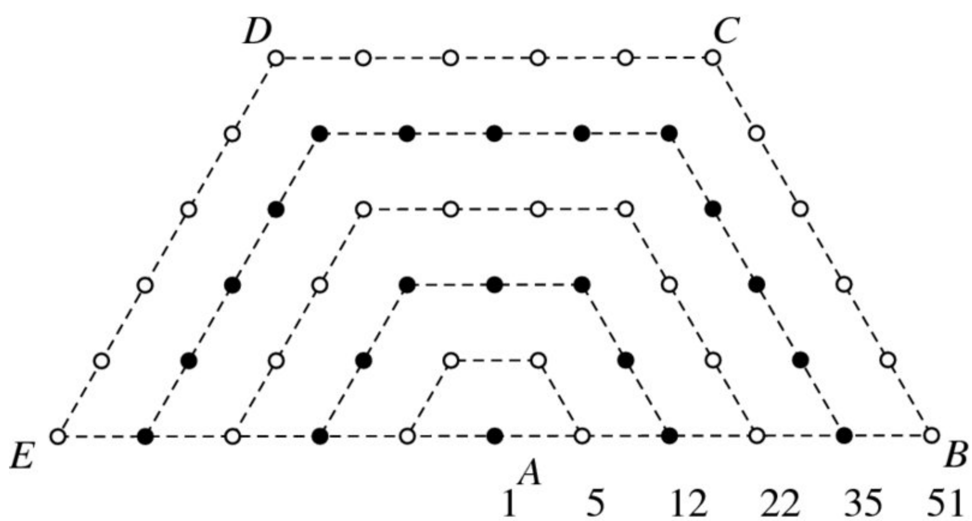
d. The trapezoidal diagram shows that each pentagonal number is the difference of two triangular numbers. Determine the two triangular numbers corresponding to p_n and express this result in a formula.

e. Construct a diagram showing that each pentagonal number is one-third of a triangular number. Give an explicit formula of this property.

Answer

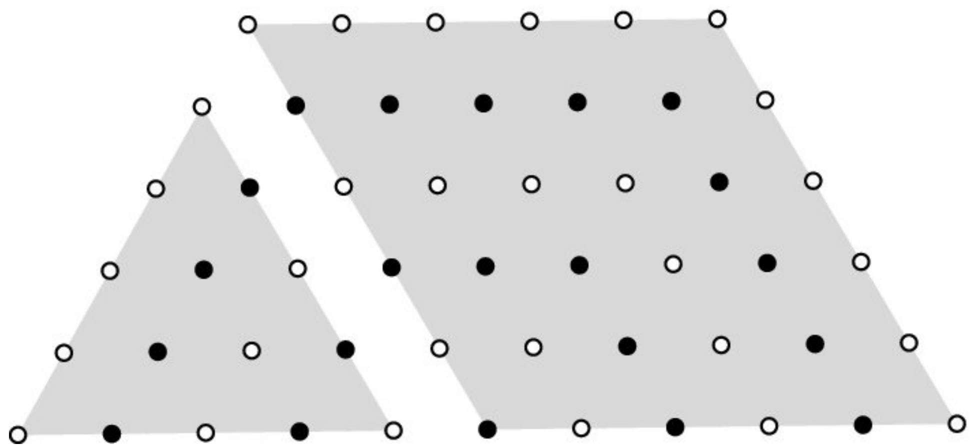
a. The next three trapezoidal numbers are 22, 35, and 51.

b. View a trapezoidal number as a distorted pentagon *ABCDE,* with sides *EA* and *AB* along the same line.



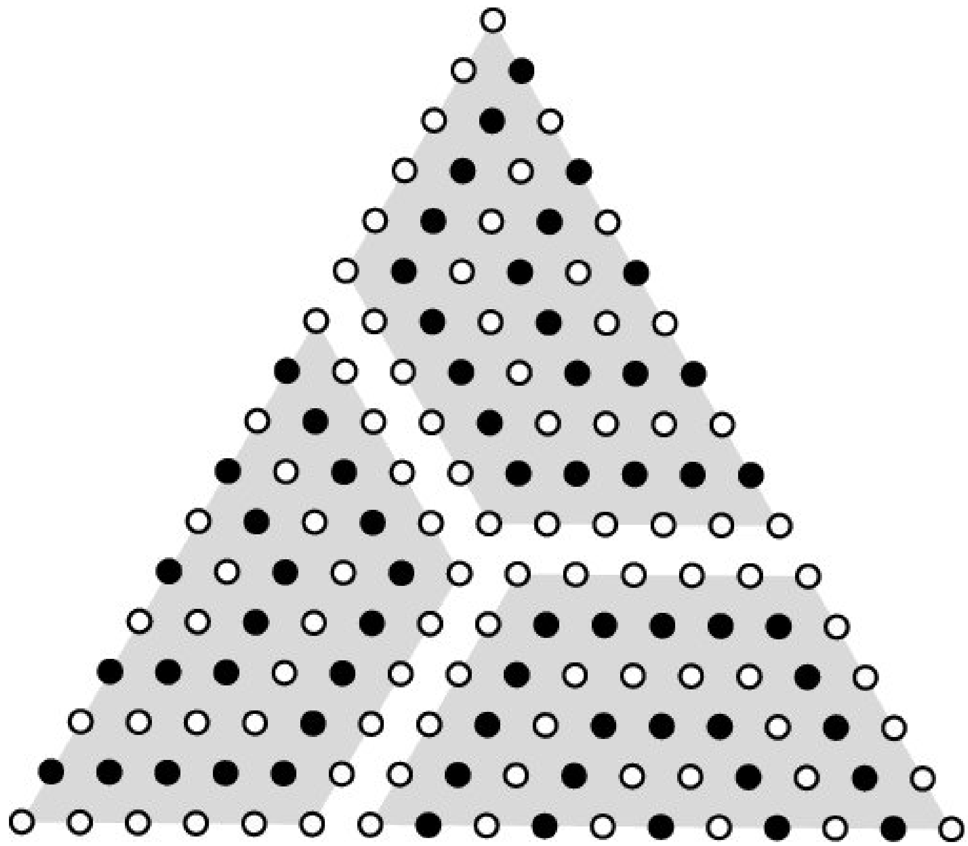
c. $p_n = t_n + n^2$. For example,

$$p_6 = 51 = t_{6-1} + 6^2 = \frac{5 \cdot 6}{2} + 36 = 15 + 36$$



d. Extend the trapezoid to a triangle of side $2n - 1$. We then see that $p_n = t_{2n-1} - t_{n-1}$. For example, $p_4 = 22 = t_7 - t_3 = \frac{7 \cdot 8}{2} - \frac{3 \cdot 4}{2} = 28 - 6.$

e. The shading in the diagram below shows that $p_n = \frac{1}{3} t_{3n-1}$.



For example,

$$p_6 = 51 = \frac{1}{3}t_{3\cdot 6-1} = \frac{1}{3}t_{17} = \frac{1}{3}\left(\frac{17\cdot 18}{2}\right) = \frac{17\cdot 18}{6} = 17.3$$
.

1.4.9. Dominoes, as described in Problem 1.4.8 also come in double-9, double-12, double-15, and even double-18 sets. Consider, more generally, a double-*n* set, so each half-domino is imprinted with 0 to *n* pips.

a. Derive a formula for the number of dominoes in a double-*n* set. Use the formula to determine the number of dominoes in a double-*n* set for n = 6, 9, 12, 15, and 18.

b. Derive a formula for the total number of pips in a double-*n* set. Use the formula to determine the total number of pips in a double-*n* set for n = 6, 9, 12, 15, and 18.

Answer

a. Consider the array of dots in the *x-,y-* coordinate plane with a dot at (*p, q*) that represents the *p* – *q* domino, with $0 \le q \le p \le n$. This array is a triangle with $n + 1$ dots per side, so there are $t_{n+1} = \frac{1}{2}(n+1)(n+2)$ dominoes in a double-*n* set.

For *n* = 3, 6, 9, 12, 15, and 18, the number of dominoes are the triangular numbers 10, 28, 55, 91, 136, and 190.

b. Imagine that you have two double-*n* sets, so that each *p* – *q* domino from one set can be paired with the (*n* – *p*) – (*n* – *q*) complementary domino from the second set. For example, in a double-15 set, pair the 11 – 6 domino from one set with the complementary 4 – 9 domino from the second set. Each pair of complementary dominoes has a total of 2*n* pips, so by part (a) there are $(2n)t_{n+1} = (2n)\frac{1}{2}(n+1)(n+2) = n(n+1)(n+2)$ pips in the two double-*n* sets. Therefore, a single double-*n* set has a total of $\frac{1}{2}n(n+1)(n+2)$ pips.

Alternate answer

Each half-domino with *k* pips, $0 \le k \le n$, occurs $n + 2$ times in a double-*n* set, so the total number of pips is given by

$$(n+2)(0+1+2+\cdots+n) = (n+2)t_n = (n+2)\frac{n(n+1)}{2} = \frac{1}{2}n(n+1)(n+2).$$

PROBLEM SET 1.5

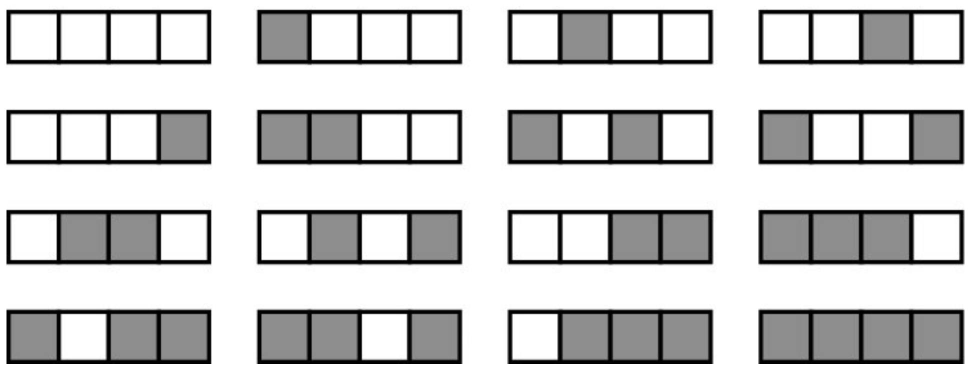
1.5.1.

a. Extend Figure 1.9 to depict the set of 16 tilings of a board of length 4, where each tile is either gray or white.

b. Explain how it is easy to use the 8 tilings of boards of length 3 to draw all of the tiled boards of length 4.

Answer

a.



b. Add a white tile at the right end of each of the 8 tiled boards of length 3, and then add a gray tile at the right end of each of the 8 tiled boards of length 3. Altogether, this forms all 16 of the tilings of boards of length 4.

1.5.3. Use formulas (1.20) and (1.21) to prove Pascal's identity (1.24).

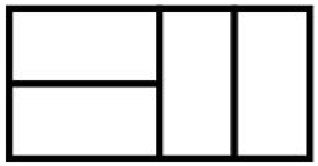
Answer

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!}$$

$$= \frac{(n-1)!}{k!\,(n-k)!}\,[k+n-k] = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

1.5.5.

a. Find all of the ways that a 2 × 4 rectangular board can be tiled with 1 × 2 dominoes. Here is one way to tile the board.

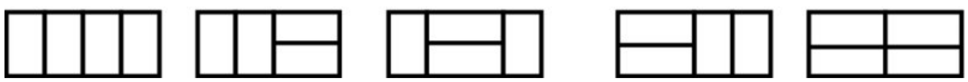


b. Draw all of the ways to tile a 2 × 4 board with dominoes.

c. How many ways can a 2 × n board be tiled with dominoes?

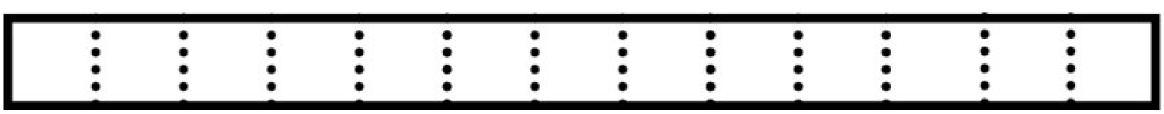
Answer

a. There are five tilings:



b. Draw a horizontal midline through each tiling by dominoes. The lower 1 × n row board is equivalent to a tiling by squares and dominoes, showing there are f_n tilings of a 2 × n board with dominoes.

1.5.7. The following train (see Problem 1.5.6 for the definition of a train) has just one car of length 13.



a. How many ways can a train of length 13 be formed with 2 cars?

b. Why are there $\binom{12}{4}$ trains of length 13 that can be formed with 5 cars?

c. Generalize your answer to part (b) to give a binomial coefficient that expresses the number of trains of length *n* with *r* cars.

Answer

a. Any one of the 12 vertical dashed lines can be chosen to end a car and start a new one, so there are $\binom{12}{1} = 12$ ways to form a train of length 13 with 2 cars.

b. Any choice of 4 of the 12 vertical dashed lines forms a train with 5 cars. Thus, there are $\binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{4 \cdot 3 \cdot 2 \cdot 1} = 495$ trains of length 13 with 5 cars.

c. There are $\binom{n-1}{r-1}$ trains of length *n* with *r* cars.

1.5.9. There are 4 ways to express 5 as a sum of two ordered summands, namely 4 + 1, 3 + 2, 2 + 3, and 1 + 4.

a. How many ways can 5 be expressed as a sum of three ordered summands? (see Problem 1.5.8)

b. How many ways can a positive integer *n* be expressed as a sum of *k* summands?

Answer

a. Six ways

b. $\binom{n-1}{k-1}$ since the summands can be viewed as the lengths of *k* cars that form a train of length