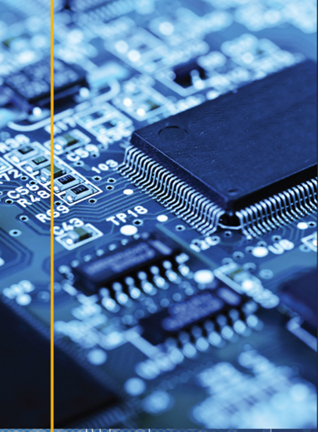
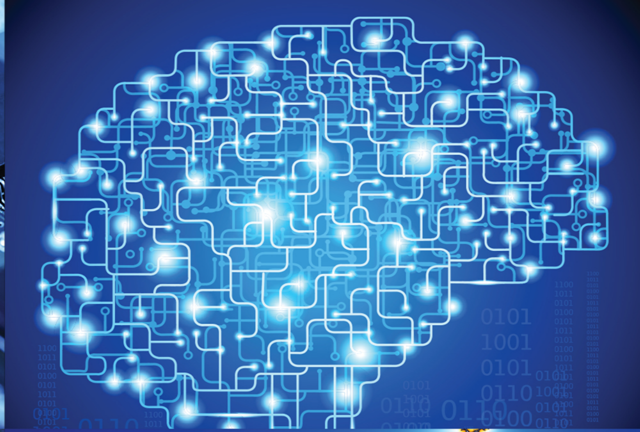
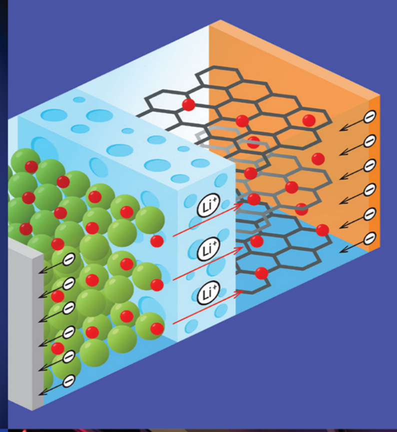
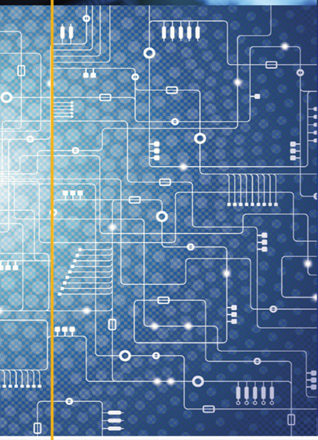
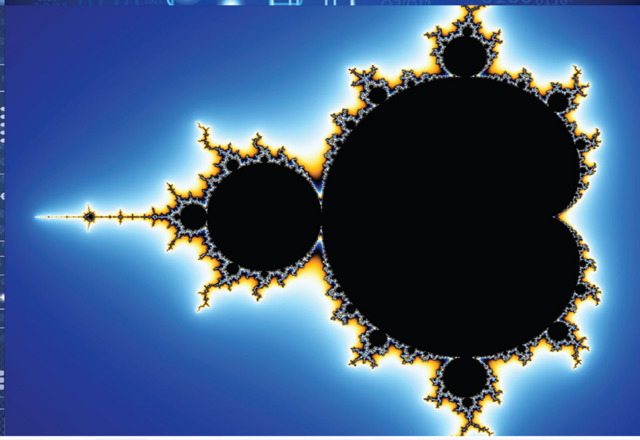
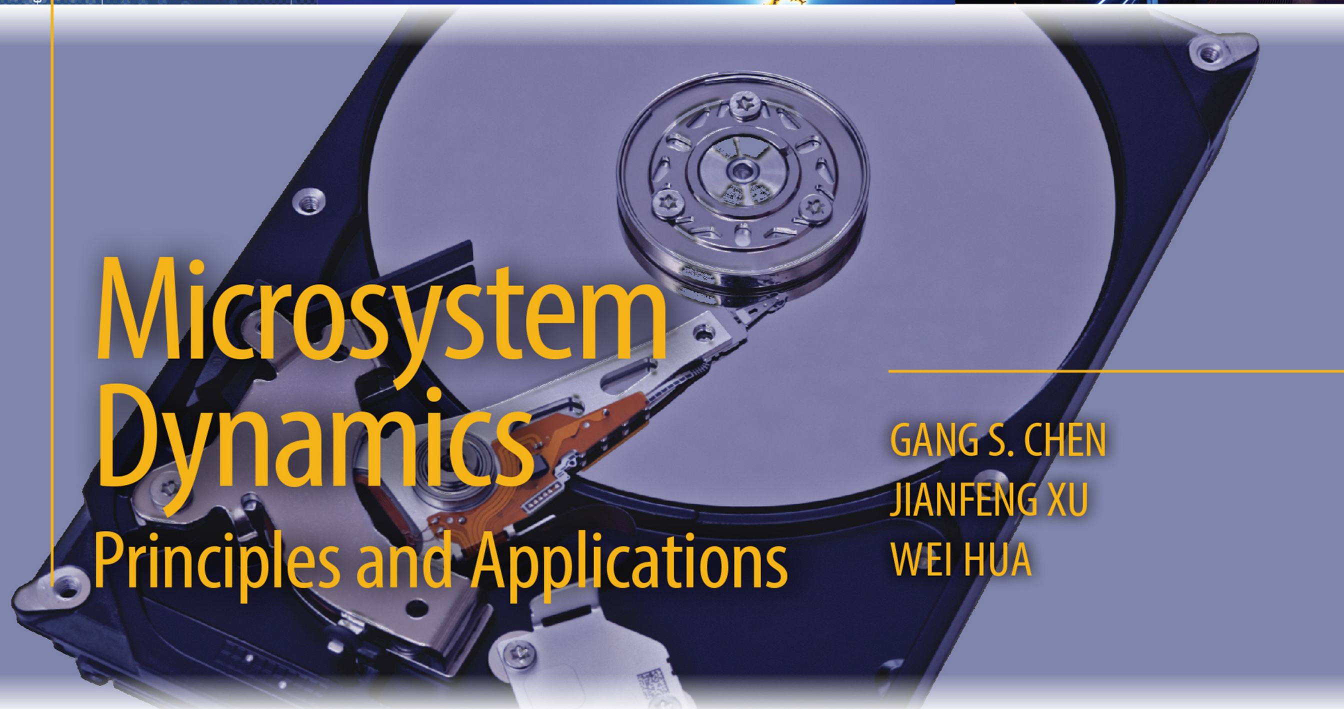








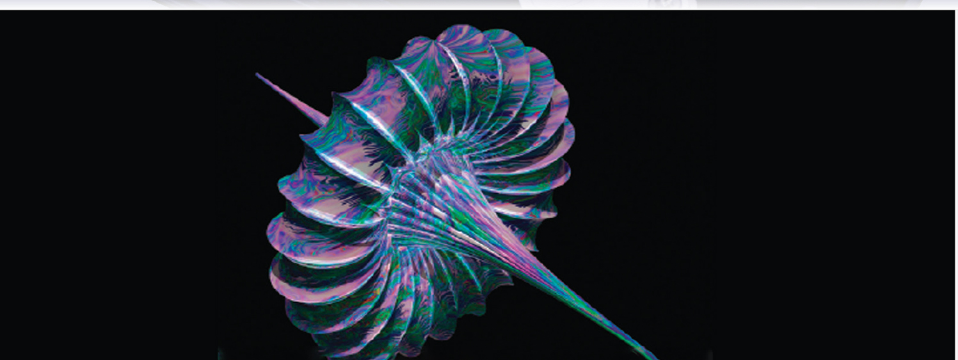




Microsystem Dynamics

Principles and Applications

GANG S. CHEN
JIANFENG XU
WEI HUA



WILEY

Microsystem Dynamics

Principles and Applications

Gang S. Chen
Jianfeng Xu
Wei Hua

WILEY

This edition first published 2025

Registered Offices
John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, USA
John Wiley & Sons Ltd, New Era House, 8 Oldlands Way, Bognor Regis, West Sussex, PO22 9NQ, UK

For details of our global editorial offices, customer services, and more information about Wiley products visit us at www.wiley.com.

The manufacturer's authorized representative according to the EU General Product Safety Regulation is Wiley-VCH GmbH, Boschstr. 12, 69469 Weinheim, Germany, e-mail: Product_Safety@wiley.com.

Wiley also publishes its books in a variety of electronic formats and by print-on-demand. Some content that appears in standard print versions of this book may not be available in other formats.

Library of Congress Cataloging-in-Publication Data

Names: Chen, Gang S. (Mechanical engineer) author | Xu, Jianfeng author | Hua, Wei (Scientist) author
Title: Microsystem dynamics : principles and applications / Gang S. Chen, Jianfeng Xu, Wei Hua.
Description: Hoboken, NJ : Wiley, 2025. | Includes bibliographical references and index.
Identifiers: LCCN 2025009176 | ISBN 9781118848906 hardback | ISBN 9781118848876 pdf | ISBN 9781118848883 epub | ISBN 9781118848890
Subjects: LCSH: Microelectromechanical systems | Dynamics
Classification: LCC TK7875 .C48 2025 | DDC 621.381–dc23/eng/20250404
LC record available at https://lccn.loc.gov/2025009176

Cover Images: © PaulPaladin/Getty Images, © VLADGRIN/Getty Images, © sivvector/stock.adobe.com, © mareandmare/Getty Images, © Andreas Nilsson/Shutterstock, © baranozdemir/Getty Images, © Alexey Bragin/Getty Images, © Laguna Design/Getty Images

Set in 9.5/12.5pt STIXTwoText by Lumina Datamatics

Contents

Preface

Microsystems in engineering or Micro Electromechanical Systems are miniaturized devices with components in micrometers, which perform micrometer to nanometer scale complex functions within a tiny space, such as sensing and actuation. Typical microsystems have both mechanical parts and electrical parts, like read-write head in computer storage devices, or bending cantilevers in atomic force microscopes. Dynamics studies the movement of systems of interconnected bodies under the action of external forces. Bodies could be treated with rigid bodies or flexible bodies or both subjects to applications. The dynamics of a rigid body system are defined by its equations of motion, which are derived using either Newton–Euler equations or Lagrangian equations. The dynamics of a flexible body system or structural dynamics have general dynamical equations of motions, including stress and strain relations. Vibration investigates the oscillatory motion of an object about an equilibrium point and the forces associated with it. Microsystem dynamics studies the mechanical behavior and motion of microsystems, analyzes how these microsystems move and respond to external forces, considering multiscales, multiphysics factors.

Unlike most conventional engineering systems, in microsystems, surface and micro level related forces play significant roles and are not ignorable compared with body forces. A microsystem with moving parts functionally operates with varied movements and thus involves vibrations and dynamics. Numerous models have been developed for varied microsystems under individual conditions. Microsystem and dynamics/vibrations used to be two distinct fields. However, with the recent rapid developments in dynamical microsystems – especially the extensive applications of dynamical microsystem in the IT hardware, battery-powered road vehicle and aircraft systems, telecommunication systems, biomedical devices, manufacturing and robotic systems, engineers and scientists are turning to combine microsystem and dynamics/vibrations for integrated and efficient methods to handle and analyze the vast amounts of practical cases.

The need for information storage systems is tremendously high and ever increasing. There are a variety of information storage systems with varying degrees of development and commercialization. To date, magnetic information storage technology, particularly hard disk drive, is most widely used for long-term data storage, whereas memory stick or USB flash drives is typically used for short-term data storage. Microsystem dynamics was once the most challenging and critical problem in the development of hard disk drives, and remains one of the most important techniques to advance hard disk drive technology.

Rechargeable lithium-ion batteries have been used for a wide variety of applications from small-scale portable electronics to massive-scale energy storage systems. Particularly, electric vehicle battery building has been booming worldwide for the last several years. The limitations of current battery technology include underutilization, capacity fade, thermal runaway, stress-induced fracture, and microscale material damage. To overcome these challenges, understanding the complex multiphysics and multiscale microsystem dynamics of lithium-ion batteries is indispensable.

Microactuators or MEMS actuators are the devices that convert electrical energy to mechanical motion. Microactuators are widely used in science and engineering. Examples include variable capacitors, microrelays for low-power VLSI, optical phase shifters, next-generation displays, microgrippers for robotic surgery, and focusing mechanisms for cameras in mobile devices. There are various microactuators using different dynamical systems, which are characterized by microsystem dynamics.

Understanding the nature of microsystem dynamics and solving the technological problems associated with microsystem dynamics are the essence of these fields. The importance of microsystem dynamics cannot be overemphasized for economic reasons, long-term reliability, and safety. Modeling microsystem dynamics in engineering and scientific systems requires an accurate description of microsystem and dynamics. Unfortunately, this is extremely challenging as it involves complex phenomena in microscale. On the other hand, the resultant vibrations and dynamics in microsystems often exhibit various nonlinear, nonstationary, and uncertain features due to complex dynamics of microsystems. Moreover, small changes in parameters could have significant effect on the resultant vibrations and dynamics, thus the scales of influencing factors span from macroscales, microscales, to nanoscales, molecular, or even atomic scales. The boundary condition of the problems is not fixed or given in prior, it is dependent on environmental conditions, operation conditions, system interactions, and dependent on time. Because of the complexity of the problems, tremendous efforts have been made in many engineering and scientific communities.

The purpose of this book is to present the principles of microsystem dynamics and their relevance to various applications. This book offers a combined treatment of the modeling, analysis, and testing of many microsystem dynamics problems that application engineers and scientists are trying to solve. After delineating these mathematical characterizations, it presents several applications in use today for analyzing microsystem dynamics. Emphasis is put on the contemporary knowledge and perspectives of microsystem dynamics.

1

Introduction

In this introductory chapter, the concepts of microsystems, dynamics/vibrations, and microsystem dynamics are described. Then the significance of microsystem dynamics in engineering, science, and everyday life is presented. In the last section, the organization of the book is introduced.

1.1 Definition of Microsystem, Vibrations and Dynamics

Microsystems or microelectromechanical systems (MEMS) are miniaturized devices with components measured in micrometers that perform micro- to nanometer scale electronic machine functions such as sensing and actuation. Typical microsystems have both mechanical and electrical parts, like read-write heads in computer storage devices, or bending cantilevers in atomic force microscopes [1–13].

Vibration studies the oscillatory motion of an object around an equilibrium point and the forces associated with it. The oscillations may be periodic or random. The associated forces may be linear or nonlinear. Vibration is usually detrimental, and occasionally “desirable” for engineering systems. Dynamics studies the movement of systems of interconnected bodies under the action of external forces. For rigid-body dynamics, the moving bodies are assumed to be rigid, which simplifies the analysis by reducing the parameters that describe the configuration of the system to the translation and rotation of reference frames attached to each body. The dynamics of a rigid body system are defined by its equations of motion, which are derived using either Newton–Euler equations or Lagrangian equations. The dynamics of a flexible body system or structural dynamics have general dynamical equations of motions, including stress and strain relations.

Unlike most conventional engineering systems, in microsystems, surface-related forces play significant roles and are not ignorable compared with body forces. A microsystem with moving parts functionally operates with varied movements and thus involves vibrations and dynamics. Numerous models have been developed for varied microsystems under individual conditions. Microsystems and vibration/dynamics used to be two distinct fields. However, with the recent rapid developments in dynamical microsystems – especially the extensive applications of dynamical microsystems in IT hardware, telecommunications, biomedical technology, manufacturing and robotic systems, transportation, and aerospace, engineers are turning to combining microsystem and dynamics/vibrations for integrated and efficient methods to handle and analyze the vast amounts of practical cases.

Microsystem Dynamics: Principles and Applications, First Edition. Gang S. Chen, Jianfeng Xu and Wei Hua.

This book, *Microsystem Dynamics*, offers a combined treatment of the modeling, analysis, and testing of many problems that application engineers are trying to solve. After delineating these mathematical characterizations, it presents several applications in use today for analyzing microsystem dynamics. Emphasis is put on the contemporary knowledge and perspectives of microsystem dynamics.

1.2 Engineering and Scientific Significance of Microsystem Dynamics

Several decades have passed since the discovery and development of microsystems. Microsystem technology is beginning to explode with extensive applications.

Varied microsystems are used in numerous scientific and engineering systems and our everyday life. Just to name a few: active sliders in computer hard disk drives; accelerometers and pressure sensors in vehicles; lithium-ion batteries in electric vehicles; micromirrors in TVs; radiofrequency switches and MEMS microphones in cell phones; varied microactuators, such as MEMS valves, pumps, and microfluidics; electrical and optical relays and switches; MEMS grippers, tweezers, and tongs; MEMS linear and rotary motors; inkjet printer heads; microvehicles (e.g. microaircraft, microcars). After several decades of development, the fabrication methods of bulk and surface micromachining for microsystems are now matured and almost standardized.

The examples of microsystem dynamics phenomena cover numerous mechanisms in science and engineering. Even in laptops, we rely on dynamic microsystems for data storage and retrieval. Microsystem dynamics extend beyond engineering applications, it includes numerous phenomena in science and nature. This book considers microsystem dynamics in its broader meaning yet concentrates on fundamentals and engineering applications.

To give some examples of the problems treated in the book, let's consider the immense efforts that are being put into dealing with microsystem dynamics in the information storage industry, lithium-ion battery industry, and microactuator industry.

We are living in an information age. The needs for information storage systems are tremendously high and ever-increasing. There are a variety of information storage systems with varying degrees of development and commercialization. To date, magnetic information storage technology, particularly hard disk drives, is the most widely used. We are all familiar with computers in which the hard disk is one of key components. The worldwide hard disk drive revenue had reached $50 billion. Magnetic hard disk drives are based on the same fundamental principles of magnetic recording which involves a recording head and a recording medium. The former is on a suspension-supported slider, while the latter is on a spinning disk. The slider is flying on the spinning disk with the air gap. The operation of the hard disk drive is based on a self-pressurized air-bearing between the slider and the spinning disk, which maintains a constant separation called flying height. The state-of-the-art flying height is on the order of below 10 nm, while the relative speed between the slider and disk is extremely high (20 m/s or higher). The mechanical spacing between the slider and the disk must be further reduced to less than 2 nm to achieve an areal density beyond 1 Tbit/in^2. In these regimes, microsystem dynamics have been the most challenging and critical problem for the products. On the other hand, over the last decade, the microsystem dynamics technique have been one of the most important techniques to advance slider disk interface and hard disk drive technology.

Rechargeable lithium-ion batteries (LIBs) have been used for a wide variety of applications from small-scale portable electronics to massive-scale energy storage systems. Particularly, electric vehicle battery building has been booming worldwide for the last several years. The market value of the LIBs industry was about $55 billion in 2023. With the enhanced demand for LIBs, experts predict this market will grow steadily, with a compound annual growth rate of around 20% from 2024 to 2030. A typical LIB cell is made up of an anode, cathode, separator, electrolyte, and two current collectors (positive and negative). The anode and cathode (both with thickness between 50 and 100 μm) store the lithium. The electrolyte carries positively charged lithium ions from the anode to the cathode and vice versa through the separator. The movement of the lithium ions creates free electrons in the anode which creates a charge at the positive current collector. The electrical current then flows from the current collector through a device being powered (cell phone, computer, vehicle motor, etc.) to the negative current collector. The separator (with a thickness between 20 and 30 μm) blocks the flow of electrons inside the battery. While LIB is discharging and providing an electric current, the anode releases lithium ions to the cathode, generating a flow of electrons from one side to the other. When plugging in the used LIB to the electrical grid for charging, the opposite happens: lithium ions are released by the cathode and received by the anode. However, the existing problems of LIBs limit their reliable applications in vehicles due to the stringent safety standards. The limitations of current battery technology include underutilization, capacity fade, thermal runaway, stress-induced fracture, and microscale material damage. To overcome these challenges, understanding the complex multiphysics and multiscale dynamics of LIBs is indispensable.

Microactuators or MEMS actuators are devices that convert electrical energy to mechanical motion, which comprise more than 50% of the rapidly growing microsystems/MEMS market which has a worldwide revenue of about $20 billion. Microactuators are widely used in science and engineering. Examples include variable capacitors, microrelays for low-power VLSI, optical phase shifters, next-generation displays, microgrippers for robotic surgery, and focusing mechanisms for cameras in mobile devices. There are various microactuators using different dynamical systems, which are characterized by microsystem dynamics with various electrostatic, thermal, piezoelectric, and magnetic features.

Understanding the nature of microsystem dynamics and solving the technological problems associated with microsystem dynamics are the essence of these fields.

Modeling of microsystem dynamics in engineering and scientific systems requires an accurate description of microsystems and dynamics. Unfortunately, this is extremely challenging as it involves complex surface phenomena in microscales. On the other hand, the resultant vibrations and dynamics in microsystems often exhibit various nonlinear, nonstationary, and uncertain features due to complex surface or interfacial forces. Moreover, small changes in interfacial parameters could have significant effect on the resultant vibrations and dynamics, thus, the scales of influencing factor span from macro-, micro-, to nanometer levels. The boundary condition of the problems is not fixed or given in prior, it is dependent on environmental conditions, operation conditions, system interactions, and dependent on time. Because of the complexity of the problems, tremendous efforts have been made in many engineering and scientific communities.

The recent extensive efforts in modeling, analytical, and experimental investigation have made a lot of substantial progress in many practical applications. Many advanced techniques of measurement, nonlinear dynamics, signal processing, computational intelligence, and system identification have been used as efficient means to address nonlinear, nonstationary, and uncertain vibrations and dynamics, which enables to efficient quantify microsystem dynamics and leads to the development of the insight of microsystem in micro, nano, molecular, or even atomic scales.

The purpose of research in microsystem dynamics lies in many ways, just to name a few: to develop a fundamental understanding of dynamical microsystems; to make use of its principle to design novel engineering microsystems; to make use of the microsystem dynamics to explore phenomena in complex process where the other means are not accessible; and understandably to reduce and eliminate the instability and vibrations in engineering microsystems thus enhance the durability, reliability and safety of products.

1.3 Organization of the Book

The book has been set out with a twofold aim in view. The first aim is to give a general introduction to the theory of vibrations, dynamics, and surface and interface interactions at the microscale, offering a physical picture of the fundamental theory. The second aim is to give a series of examples of the applications of the theoretical approaches. The author is expected to provide contemporary coverage of the primary concepts and techniques in the treatment of microsystem dynamics.

This book brings together into one accessible text the fundamentals of the many disciplines needed by today's engineers working in the field of microsystem dynamics. This book consists of six chapters. The basic principles of vibrations, dynamics, surface forces and interactions, and critical applications of microsystem dynamics are introduced in these chapters. Chapter 1 introduces the whole book. Chapter 2 provides a comprehensive introduction to the analysis of vibrations and dynamics, from vibrations of linear systems, and random excited systems to nonlinear systems and rigid-body dynamics by covering most required areas and applications. Chapter 3 describes the principles of surface forces and interface interactions in macro- and microscales. In Chapter 4, the concepts and methods are extended to the most critical engineering applications of microsystems in the IT industry. It presents the microsystem dynamics of microscale air-bearing slider systems and nanoscale MEMS slider systems, which are widely used computer hard disk drive. Chapter 5 presents the microdynamics of lithium-ion batteries in the context of multiphysics, which has been widely investigated in science and engineering communities for extensive applications recently. In Chapter 6, the microsystem dynamics of varied MEMS actuators are presented.

Complete references given in the book provide a comprehensive perspective on the developments in microsystem dynamics, as well as covering various applications. For didactical reasons, the text is not interrupted by the inclusion of references. However, at the end of each chapter, the relevant literatures published are cited.

References

1 Senturia, S.D., *MEMS & Microsystems: Design, Manufacture, and Nanoscale Engineering*. Springer, 2000.

2 Menz, W., Mohr, J., and Paul, O., *Microsystem Technology*. Wiley, 2001.

3 Hsu, T.R., *MEMS and Microsystems: Design, Manufacture, and Nanoscale Engineering*. Wiley, 2008.

4 Madou, M.J., *Fundamentals of Microfabrication: The Science of Miniaturization*. Wiley, 2011.

5 Geschke, O., Klan, H., and Telleman, P., *Microsystem Engineering of Lab-on-a-chip Devices*. John Wiley & Sons, 2004.

6 Lobontiu, N., *Dynamics of Microelectromechanical Systems*. Springer, 2007.

7 Ostasevicius, V., and Dauksevicius, R., *Microsystems Dynamics*. Springer, 2011.

8 Younis, M.I., *MEMS Linear and Nonlinear Statics and Dynamics*. Springer, 2011.

9 Magrab, E.B., *Vibrations of Elastic Systems: With Applications to MEMS and NEMS*. Springer, 2012.

10 Reifenberger, R., *Fundamentals of Atomic Force Microscopy*. Word Scientific, 2015.

11 Haustad, G., *Atomic Force Microscopy Understanding Basic Modes and Advanced Applications*. Wiley, 2012.

12 Marchon, B., Pitchford, T., Hsia, Y.T., et al., The head-disk interface roadmap to an areal density of 4 Tbit/in2. *Adv. Tribol.* 521086:1–9, 2013.

13 Korthauer, R., *Lithium-Ion Batteries: Basics and Applications*. Springer, 2018.

2

Vibrations and Dynamics

2.1 Introduction

We present the vibrations and advanced dynamics in this chapter. After the introduction, Section 2.2 is devoted to the analysis of linear deterministic vibration systems, and the Section 2.3 describes the random vibrations. Section 2.4 is dedicated to the fundamentals of nonlinear vibrations, and Section 2.5 presents advanced dynamics [1–9].

2.2 Vibration of Linear System Under Deterministic Excitations

2.2.1 Vibration of Linear Discrete and Continuous Systems

Vibration is the oscillatory motion of a body or structure. Vibration takes places when a body is displaced from its stable equilibrium position by a restoring force.

A vibration system having a finite number of unknown variables is said to be discrete, while a system whose variables are functions of location and time is called continuous. Real systems are continuous, and their parameters are distributed. In many situations, it is possible to approximate the continuous system by discrete ones.

The analytical description of the vibrations of the discrete case is a set of ordinary differential equations, while for the continuous case, it is a set of partial differential equations. If the dependent variables in the differential equation are to the first power, then the system is linear. If there are fractional or higher powers, then the system is nonlinear. The superposition principle holds only for linear systems.

The independent coordinates required to quantify the configuration of a vibration system are called generalized coordinates. The number of generalized coordinates is defined as the number of degrees of freedom of the system. A discrete model of a dynamic system possesses a finite number of degrees of freedom, whereas a continuous model has an infinite number of degrees of freedom.

The excitation of a vibration system is usually a function of time. The vibratory motion of the system caused by excitations is referred to as the response. If the vibratory motion is periodic, the system repeats its motion at equal time intervals. The minimum time required for the system to repeat its motion is called a period; this is the time to complete one cycle of motion. Frequency is defined as the number of times that the motion repeats itself per unit time. Free vibrations describe the natural behavior of vibration of a system. Many systems need to be treated as damped systems due to the dissipation of motion energy.

Microsystem Dynamics: Principles and Applications, First Edition. Gang S. Chen, Jianfeng Xu and Wei Hua.

The excitation may be either deterministic or a random function of time. In deterministic vibrations, the response at any designated future time can be completely predicted from past history; random forced vibrations are defined statistically, and only the probability of occurrence of designated magnitudes and frequencies can be predicted.

2.2.2 Vibration of Linear Discrete Systems: Single-degree-of-freedom System

Consider a sinusoidal periodic motion

$$x(t) = X\sin(2\pi f_0 t + \theta) \tag{2.1}$$

where X is amplitude, f_0 is cyclical frequency in cycles per unit time, θ is initial phase angle with respect to the time origin in radians, and $x(t)$ is the instantaneous value at time t.

The time interval required for one full cycle of sinusoidal motion is called period T_P. The number of cycles per unit time is called the frequency f_0. The frequency and period are related by $T_p = 1/f_0$.

Complex periodic motion can be defined mathematically by a time-varying function whose waveform exactly repeats itself at regular intervals such that

$$x(t) = x(t \pm nT_P) \quad n = 1, 2, 3, \ldots$$

The complex periodic motion can be expanded into a Fourier series as

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi n f_1 t + \sin 2\pi n f_1 t) \tag{2.2}$$

where $f_1 = 1/T_P$

$$a_n = \frac{2}{T_p}\int_0^{T_p} x(t)\cos 2\pi n f_1 t dt \qquad n = 0, 1, 2, \ldots$$

$$b_n = \frac{2}{T_p}\int_0^{T_p} x(t)\sin 2\pi n f_1 t dt \qquad n = 0, 1, 2, \ldots$$

Another way of expressing the Fourier series for complex periodic data is

$$x(t) = X_0 + \sum_{n=1}^{\infty} X_n \sin(2\pi n f_1 t - \theta_n) \tag{2.3}$$

where $X = a_0/2$, $X_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \tan^{-1}(a_n/b_n)$, $n = 0,\ 1,\ 2,\ \ldots$. Equation (2.3) implies that complex periodic data consist of a static component, X_0, and an infinite number of sinusoidal components called harmonics, which have amplitudes X_n and phase θ_n. The frequencies of the harmonic components are all integral multiples of f_1. The phase angles are often ignored when periodic data is analyzed in practice. For this case, Eq. (2.3) can be characterized by a discrete spectrum.

Transient motion is defined as all nonperiodic motion other than the almost-periodic data discussed above. Transient motions include all motion which can be described by some suitable time-varying function. Physical phenomena which produce transient data are numerous and diverse. The important characteristic of transient motion is its continuous spectral representation, which can be obtained in most cases from a Fourier integral given by

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \tag{2.4}$$

The Fourier spectrum $X(f)$ is generally a complex number, which can be expressed in complex polar notation as $X(f) = |X(f)|\, e^{-j\theta(f)}$, where $|X(f)|$ is the magnitude of $X(f)$ and $\theta(f)$ is the argument.

Next, we discuss the vibration of a single-degree-of-freedom (SDOF) model, as shown in Figure 2.1. From Newton's law, we obtain

$$F(t) - F_s(t) - F_d(t) = m\ddot{x}(t) \tag{2.5}$$

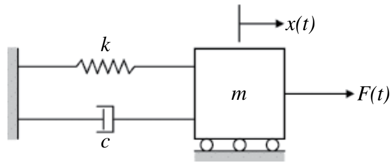


Figure 2.1 Single-degree-of-freedom system.

where $F(t)$, $F_s(t)$, and $F_d(t)$ are the exciting, spring, and damping forces, respectively; m denotes the mass of the body and $\ddot{x}(t)$ its acceleration. Because $F_s(t) = kx(t)$ and $F_d(t) = c\dot{x}(t)$, Eq. (2.5) becomes

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \tag{2.6}$$

where c and k are the viscous damping and stiffness coefficients, respectively.

Equation (2.6) is the equation of motion of the linear SDOF system and is a second-order linear differential equation with constant coefficients.

In the case of the free vibration of an SDOF system, the exciting force $F(t) = 0$ and the equation of motion is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = 0 \tag{2.7}$$

If we define $\omega_n^2 = k/m$ and $\xi = c/2m\omega_n$, Eq. (2.7) can be written as

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = 0 \tag{2.8}$$

To solve Eq. (2.8), we assume

$$x(t) = Ae^{st} \tag{2.9}$$

where A is a constant and s is a parameter to be determined. By substituting (2.9) into (2.8), one obtains

$$\left(s^2 + 2\xi\omega_n s + \omega_n^2\right)A\,e^{st} = 0 \tag{2.10}$$

Since $A\,e^{st} \neq 0$, then

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0 \tag{2.11}$$

Equation (2.11) is known as the characteristic equation of the system. This equation has the following two roots

$$s_1, s_2 = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n \tag{2.12}$$

For case a, $\xi < 1$ (underdamped condition)

$$s_1, s_2 = (-\xi \pm i\sqrt{1 - \xi^2})\omega_n$$

$$x(t) = A\,\exp(-i\,\omega_n t)\cos\left(\omega_n\sqrt{1 - \xi^2}\,t - \phi\right) \tag{2.13}$$

$$x(t) = A\,\exp(-i\,\omega_n t)\cos(\omega_d t - \phi) \tag{2.14}$$

where ω_n is the natural circular frequency, ξ the damping factor, and $\omega_d = \omega_n\sqrt{1 - \xi^2}$, the damped frequency of the system. Constants A and ϕ are determined from the initial conditions.

For case b, $\xi > 1$ (overdamped condition)

$$s_1, s_2 = \left(-\xi \pm \sqrt{\xi^2 - 1}\right)\omega_n$$

$$x(t) = A_1\,\exp\left(-\xi + \sqrt{\xi^2 - 1}\right)\omega_n t + A_2\,\exp\left(-\xi - \sqrt{\xi^2 - 1}\right)\omega_n t \tag{2.15}$$

The motion is aperiodic and decays exponentially with time. Constants A_1 and A_2 are determined from the initial conditions.

For case c, $\xi = 1$ (critically damped condition)

$$s_1 = s_2 = -\omega_n$$

$$x(t) = (A_1 + A_2)\exp(-\omega_n t) \tag{2.16}$$

Equation (2.16) represents an exponentially decaying response. The constants A_1 and A_2 depend on the initial conditions. For this case, the coefficient of viscous damping has the value $c_c = 2m\omega_n = 2\sqrt{k/m}$.

$$\xi = c/c_c \tag{2.17}$$

We consider the undamped condition in which t_1 and t_2 denote the times corresponding to the consecutive displacements x_1 and x_2, measured one cycle apart. By using Eq. (2.16), we can write

$$\frac{x_1}{x_2} = \frac{A\ \exp(-i\,\omega_n t_1)\cos(\omega_d t_1 - \phi)}{A\ \exp(-i\,\omega_n t_2)\cos(\omega_d t_2 - \phi)} \tag{2.18}$$

Since $t_2 = t_1 + T = t_1 + 2\pi/\omega_d$, then $\cos(\omega_d t_1 - \phi) = \cos(\omega_d t_2 - \phi)$. Equation (2.18) then reduces to

$$x_1/x_2 = \exp(\xi\omega_n T)$$

We define logarithmic decrement as

$$\delta = ln(x_1/x_2) = \xi\omega_n T = 2\pi\xi/\sqrt{1 - \xi^2} \tag{2.19}$$

To determine the amount of damping in the system, it is sufficient to measure any two consecutive displacements x_1 and x_2 and obtain ξ from the equation

$$\xi = \delta/\sqrt{(2\pi)^2 + \delta^2} \tag{2.20}$$

We now consider the response of an SDOF system to a harmonic excitation, for which the equation of motion is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F_0 \cos\omega t \tag{2.21}$$

where F_0 is the amplitude and ω the frequency of the excitation. Equation (2.21) can be simplified as

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = (F_0/k)\omega_n^2 \cos\omega t \tag{2.22}$$

The solution of Eq. (2.22) consists of two parts, the complementary function, which is the solution of the homogeneous equation, and the particular integral. The complementary function dies out with time for $\xi > 0$ and is often called the transient solution, whereas the particular solution does not vanish for a large t and is referred to as the steady-state solution to the harmonic excitation. We assume a solution of the form

$$x(t) = X\cos(\omega t - \phi) \tag{2.23}$$

where X and ϕ are the amplitude and phase angle of response, respectively.

By substituting Eq. (2.23) into Eq. (2.22), we obtain

$$X\left[(\omega_n^2 - \omega^2)\cos(\omega t - \phi) - 2\xi\omega_n\omega\ \sin(\omega t - \phi)\right] = (F_0/k)\omega_n^2\cos\omega t \tag{2.24}$$

By developing the terms in Eq. (2.24), and equating the coefficients of cos ωt and sin ωt on both sides of the equation, we obtain

$$X\left[(\omega_n^2 - \omega^2)\cos\phi + 2\xi\omega_n\omega\sin\phi\right] = (F_0/k)\omega_n^2 \tag{2.25a}$$

$$X\left[(\omega_n^2 - \omega^2)\sin\phi + 2\xi\omega_n\omega\cos\phi\right] = 0 \tag{2.25b}$$

By solving Eqs. (2.25), we get

$$X/(F_0/k) = \left\{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\xi(\omega/\omega_n)\right]^2\right\}^{-1/2} \tag{2.26}$$

and

$$\phi = \tan^{-1}\left\{\left[2\xi(\omega/\omega_n)\right]/\left[1 - (\omega/\omega_n)^2\right]^2\right\} \tag{2.27}$$

Equations (2.26) and (2.27) indicate that the nondimensional amplitude $X/(F_0/k)$ and the phase angle ϕ are functions of the frequency ratio ω/ω_n and the damping ratio ξ. For ω/ω_n much less than one, both the inertia and damping forces are small, and this results in a small phase angle ϕ, with $X/(F_0/k) \cong 1$. However, for ω/ω_n much greater than one, the phase angle $\phi \to 180°$ and $X/(F_0/k) \to 0$. For $\omega/\omega_n = 1$, the phase angle $\phi = 90°$ and $X/(F_0/k) = 1/2\xi$. In summary, the complete solution of Eq. (2.22) is given as

$$x(t) = A_1\, exp(-i\,\omega_n t)\cos(\omega_d t + \phi_1) + \frac{F_0}{k}\frac{\cos(\omega t - \phi)}{\sqrt{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\xi(\omega/\omega_n)\right]^2}} \tag{2.28}$$

where the constants A_1 and ϕ_1 are determined by the initial conditions.

Let us reconsider Eq. (2.22) and represent the excitation by the complex form

$$(F_0/k)\omega_n^2 e^{-i\omega t} = X_s\omega_n^2 e^{-i\omega t} \tag{2.29}$$

where $X_s = F_0/k$ and is referred to as a static response. We assume a solution in the form

$$x(t) = Xe^{-i\omega t} \tag{2.30}$$

By substituting Eq. (2.30) into Eq. (2.22), we get

$$\left[\omega_n^2 - \omega^2 - 2i\xi\omega_n\omega\right]Xe^{-i\omega t} = X_s\omega_n^2 e^{-i\omega t} \tag{2.31}$$

$$X/X_s = \left[1 - (\omega/\omega_n)^2 - 2i\xi(\omega/\omega_n)\right]^{-1} = H(w) \tag{2.32}$$

where $H(\omega)$ is known as the complex frequency response function. Its magnitude $|H(\omega)|$ refers to a magnification factor and is given as

$$|H(\omega)| = \left\{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\xi(\omega/\omega_n)\right]^2\right\}^{-1/2} \tag{2.33}$$

The phase angle ϕ will be

$$\phi = \tan^{-1}\left[\frac{2\xi(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}\right] \tag{2.34}$$

The excitation considered thus far has been a simple harmonic force. We can generalize the results when the exciting force is periodic, because periodic force can be expanded in the terms of the Fourier series, as follows

$$F(t) = a_1\sin\omega t + b_1\cos\omega t + a_2\sin 2\omega t + b_2\cos 2\omega t + \cdots + a_n\sin n\omega t + b_n\cos n\omega t \tag{2.35}$$

where a_n and b_n are the coefficients of the Fourier series expansion and it has been assumed that the constant $b_0 = 0$. Because

$$a_n \sin n\omega t + b_n \cos n\omega t = f_n \sin(n\omega t + \alpha_n) \tag{2.36}$$

where $f_n = \sqrt{a_n^2 + b_n^2}$ and $\alpha_n = \tan^{-1}(b_n/a_n)$, it follows that

$$F(t) = f_1 \sin(\omega t + a_1) + f_2 \sin(2\omega t + a_2) + \cdots + f_n \sin(n\omega t + a_n) \tag{2.37}$$

Because superposition is valid, we can consider each term on the right-hand side of Eq. (2.37) as a separate forcing function and obtain the steady-state response by adding individual responses due to each forcing function acting separately. Hence, it follows that

$$x(t) = X_1 \cos(\omega t + \alpha_1 - \phi_1) + X_2 \cos(2\omega t + \alpha_2 - \phi_2) + \cdots + X_n \cos(n\omega t + \alpha_n - \phi_n) \tag{2.38}$$

in which

$$X_n = \frac{f_n/k}{\left\{\left[1 - (n\omega/\omega_n)^2\right]^2 + \left[2\xi(n\omega/\omega_n)\right]^2\right\}^{1/2}}$$

and

$$\phi_n = \tan^{-1}\frac{2\xi(n\omega/\omega_n)}{1 - (n\omega/\omega_n)^2} \qquad n = 1, 2, \ldots$$

Hence, the steady-state response is also periodic, with the same period as the forcing function but with a different amplitude and an associated phase lag.

For transient excitation $F(t)$, the response is

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)\exp[-\xi\omega_n(t-\tau)]\sin\omega_d(t-\tau)d\tau \tag{2.39}$$

2.2.3 Vibrations of Linear Discrete Systems: Multiple-degree-of-freedom System

Next, we consider the multiple-degree-of-freedom (MDOF) discrete system shown in Figure 2.2. Its general equations of motion are written as

$$[m]\{\ddot{x}(t)\} + [c]\{\dot{x}(t)\} + [k]\{x(t)\} = \{F(t)\} \tag{2.40}$$

where $\{F(t)\}$ denotes the externally applied force. In Eq. (2.40), $[m]$, $[c]$, and $[k]$ are $n \times n$ mass, damping, and stiffness matrices, respectively. For linear systems, these matrices are constant, whereas, for nonlinear systems, the elements of these matrices are functions of generalized displacements and velocities that are time-dependent.

The response $\{x(t)\}$ of Eq. (2.40) consists of two parts: first, $\{x_h(t)\}$, the homogeneous solution which is the transient response; and second, $\{x_p(t)\}$, the particular solution which is the steady state or forced response.

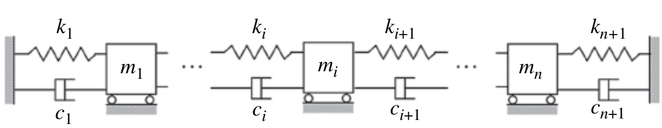


Figure 2.2 Linear multiple-degree-of-freedom system.

2.2.3.1 Eigenvalues and Eigenvectors

Next, we discuss the simplest case, the equations of motion for the free vibration of an undamped MDOF system, focusing on its eigenvalues and eigenvectors. By setting $[c]$ and $\{F(t)\}$ in Eq. (2.40) to zero, it follows that

$$[m]\{\ddot{x}(t)\} + [k]\{x(t)\} = \{0\} \tag{2.41}$$

We use a linear transformation to replace $\{x\}$

$$\{x\} = [\phi]\{y\} \tag{2.42}$$

where $[\phi]$ is a constant nonsingular square matrix to be specified in the following analysis. It is referred to as a transformation matrix

$$\{\ddot{x}(t)\} = [\phi]\{\ddot{y}(t)\} \tag{2.43}$$

Substituting Eqs. (2.42) and (2.43) into Eq. (2.41), we obtain

$$[m][\phi]\{\ddot{y}(t)\} + [k][\phi]\{y(t)\} = \{0\} \tag{2.44}$$

Premultiply both sides of Eq. (2.44) by $[\phi]^T$ to yield

$$[\phi]^T[m][\phi]\{\ddot{y}(t)\} + [\phi]^T[k][\phi]\{y(t)\} = \{0\} \tag{2.45}$$

From Eq. (2.45), it follows that

$$\left[m^*\right]\{\ddot{y}(t)\} + \left[k^*\right]\{y(t)\} = \{0\} \tag{2.46}$$

where $[m^*]$, $[k^*]$ are diagonal matrices known as the generalized mass and the stiffness matrix, respectively. Equation (2.46) refers to the uncoupled homogeneous equations of motion of the system. It follows that the uncoupled equation of motion for the ith degree of freedom is

$$\ddot{y}_i + \omega_i^2 y_i = 0 \tag{2.47}$$

where ω_i is the frequency corresponding to the ith mode of vibration. The solution of Eq. (2.47) is given as

$$y_i(t) = A_i \sin \omega_i t + A_i^* \cos \omega_i t \tag{2.48}$$

where arbitrary constants A_i and A_i^* are determined by the initial conditions $x_i(0)$ and $\dot{x}_i(0)$.

We now consider Eq. (2.41) and pre-multiply both sides by $[m]^{-1}$ to yield

$$[m]^{-1}[m]\{\ddot{x}(t)\} + [m]^{-1}[k]\{x(t)\} = \{0\} \tag{2.49}$$

Equation (2.49) can be written as

$$[I]\{\ddot{x}(t)\} + [D]x(t)\} = \{0\} \tag{2.50}$$

where $[I]$ is the unit matrix and $[D] = [m]^{-1}[k]$, which is known as the dynamic matrix.

Let us assume a harmonic motion so that

$$\{x\} = \{A\} e^{i\omega t} \tag{2.51}$$

Equation (2.51) yields

$$\{\ddot{x}(t)\} = -\omega^2\{A\} e^{i\omega t} = -\lambda\{x(t)\} \tag{2.52}$$

where $\lambda = \omega^2$. Substituting Eqs. (2.51) and (2.52) into Eq. (2.49) results in

$$[[D] - \lambda[I]]\{x\} = \{0\} \tag{2.53}$$

The characteristic equation of the system is then, with the determinant being zero

$$|[D] - \lambda[I]| = 0 \tag{2.54}$$

The roots λ_i of the characteristic equation are called eigenvalues. The natural frequencies of the system are determined from

$$\lambda_i = \omega_i^2 \tag{2.55}$$

By substituting λ_i into the matrix, Eq. (2.53), we obtain the corresponding mode shapes, which are called the eigenvectors. Thus, for an n-degree-of-freedom system, there are n eigenvalues and n eigenvectors.

Let us consider two distinct solutions corresponding to the rth and the sth modes, respectively, ω_r^2, $\{\phi^{(r)}\}$ and ω_s^2, $\{\phi^{(s)}\}$ of the eigenvalue problem. Because these solutions satisfy Eq. (2.41), it follows that

$$[k]\{\phi^{(r)}\} = \omega_r^2[m]\{\phi^{(r)}\} \tag{2.56}$$

And

$$[k]\{\phi^{(s)}\} = \omega_s^2[m]\{\phi^{(s)}\} \tag{2.57}$$

We premultiply both sides of Eq. (2.56) by $\{\phi^{(s)}\}^T$ and both sides of Eq. (2.57) by $\{\phi^{(r)}\}^T$ to obtain

$$\{\phi^{(s)}\}^T[k]\{\phi^{(r)}\} = \omega_r^2\{\phi^{(s)}\}^T[m]\{\phi^{(r)}\} \tag{2.58a}$$

$$\{\phi^{(r)}\}^T[k]\{\phi^{(s)}\} = \omega_s^2\{\phi^{(r)}\}^T[m]\{\phi^{(s)}\} \tag{2.58b}$$

Now we take the transposition of Eq. (2.58b) to get

$$\{\phi^{(s)}\}^T[k]\{\phi^{(r)}\} = \omega_s^2\{\phi^{(s)}\}^T[m]\{\phi^{(r)}\} \tag{2.59}$$

By subtracting Eq. (2.59) from Eq. (2.58a), we obtain

$$(\omega_r^2 - \omega_s^2)\{\phi^{(s)}\}^T[m]\{\phi^{(r)}\} = 0 \tag{2.60}$$

Because $\omega_r \neq \omega_s$, we conclude that

$$\{\phi^{(s)}\}^T[m]\{\phi^{(r)}\} = 0 \;\; r \neq s \tag{2.61}$$

Equation (2.61) represents the orthogonality condition of modal vectors. It can also be shown that

$$\{\phi^{(s)}\}^T[k]\{\phi^{(r)}\} = 0 \;\; r \neq s \tag{2.62}$$

Thus, $[\phi]$ is composed of $\{\phi^{(i)}\}$, $i = 1, 2, \ldots, n$. If each column of the modal matrix $[\phi]$ is divided by the square root of the generalized mass M_i^*, the new matrix $[\overline{\phi}]$ is called the weighted modal matrix. It can be seen that

$$[\overline{\phi}]^T[m][\overline{\phi}] = [I] \tag{2.63}$$

and

$$[k][\overline{\phi}] = [m][\overline{\phi}]][\omega^2_{\cdot}] \tag{2.64}$$

Premultiplying (2.64) by $[\overline{\phi}]^T$ results in

$$[\overline{\phi}]^T[k][\overline{\phi}] = [\overline{\phi}]^T[m][\overline{\phi}]][\omega^2] = [\omega^2] \tag{2.65}$$

2.2.3.2 Forced Vibration Solution of an MDOF System

We consider Eq. (2.40) and first solve the undamped free-vibration problem to obtain the eigenvalues and eigenvectors, which describe the normal modes of the system and the weighted modal matrix $[\overline{\phi}]$. Let

$$\{x\} = [\overline{\phi}]\{y\} \tag{2.66}$$

Substituting Eq. (2.66) into Eq. (2.40) yields

$$[m][\overline{\phi}]\{\ddot{y}\} + [c][\overline{\phi}]\{\dot{y}\} + [k][\overline{\phi}]\{y\} = \{F(t)\} \tag{2.67}$$

Pre-multiply both sides of Eq. (2.67) by $[\overline{\phi}]^T$ to obtain

$$[\overline{\phi}]^T[m][\overline{\phi}]\{\ddot{y}\} + [\overline{\phi}]^T[c][\overline{\phi}]\{\dot{y}\} + [\overline{\phi}][k][\overline{\phi}]\{y\} = [\overline{\phi}]^T\{F(t)\} \tag{2.68}$$

Notice that the matrices $[\overline{\phi}]^T[m][\overline{\phi}]$ and $[\overline{\phi}]^T[k][\overline{\phi}]$ on the left side of Eq. (2.68) are diagonal matrices that correspond to the matrices $[I]$ and $[\omega^2]$, respectively. However, the matrix $[\overline{\phi}]^T[c][\overline{\phi}]$ is not a diagonal matrix. If $[c]$ is proportional to $[m]$ or $[k]$ or both, the $[\overline{\phi}]^T[c][\overline{\phi}]$ becomes diagonal, in which case we can say that the system has proportional damping. The equations of motion are then completely uncoupled, and the ith equation will be

$$\ddot{y}_i + 2\xi_i\omega_i\dot{y} + \omega_i^2 y_i = \bar{f}_i(t) \qquad i = 1, 2, \ldots, n \tag{2.69}$$

where $\bar{f}_i(t) = \left\{\overline{\phi}^{(i)}\right\}\{F(t)\}$. Thus, instead of n-coupled equations, we will have n uncoupled equations.

Let $[c] = \alpha\,[m] + \beta\,[k]$, in which α and β are proportionality constants. Then we have

$$[\overline{\phi}]^T[c][\overline{\phi}] = [\overline{\phi}]^T(\alpha[m] + \beta[k])[\overline{\phi}] = \alpha[I] + \beta[\omega^2] \tag{2.70}$$

This will yield the uncoupled ith equation of motion as

$$\ddot{y}_i + \left(\alpha + \beta\omega_i^2\right)\dot{y}_i + \omega_i^2 y_i = \bar{f}_i(t) \tag{2.71}$$

and the modal damping can be defined as

$$2\xi_i\omega_i = \alpha + \beta\omega_i^2 \tag{2.72}$$

The solution of Eq. (2.69) is obtained by using Eq. (2.39) with initial conditions $y_i(0)$ and $\dot{y}_i(0)$

$$\begin{aligned} y_i(t) &= \frac{1}{\omega_d}\int_0^t \bar{f}_i(\tau)\exp[-\xi_i\omega_i(t-\tau)]\sin\omega_{di}(t-\tau)d\tau \\ &\quad + \frac{y_i(0)\exp(-\xi_i\omega_i t)}{\left(1-\xi_i^2\right)^{1/2}}\cos(\omega_{di}t - \psi_i) + \frac{\dot{y}_i(0)\exp(-\xi_i\omega_i t)}{\omega_{di}}\sin\omega_{di}t \end{aligned} \tag{2.73}$$

where $\omega_{di} = \left(1-\xi_i^2\right)^{1/2}\omega_i$ and $\psi_i = \tan^{-1}\left[\xi_i/\left(1-\xi_i^2\right)^{1/2}\right]$.

Similarly, the contribution from each normal mode is calculated and substituted in Eq. (2.66) to obtain the complete response of the system. This is known as the normal mode summation method. The contributions of the higher vibration modes to the system response are often quite small, and for all practical purposes, they may be ignored in the summation procedure by considering fewer modes of vibration.

2.2.4 Vibrations of Continuous Systems

So far we have discussed discrete systems where elasticity and mass are modeled as discrete properties. Discrete systems have a finite number of degrees of freedom specifying the system finite configuration. Continuous systems are distributed systems, such as strings, cables, rods (bars), and beams, as well as plates where elasticity and mass are distributed parameters. We consider the continuous distribution of elasticity, mass, and damping and assume each of the infinite number of elements of the system can vibrate.

The displacement of these elements is described by a continuous function of position and time. The governing equations of motion for discrete systems are ordinary differential equations, whereas the governing equations are partial differential equations for the continuous systems, and exact solutions can be obtained for only a few special configurations. For the vibration analysis of systems with distributed elasticity and mass, it is necessary to assume that the material is homogeneous, isotropic, and follows Hooke's law.

2.2.4.1 Transverse Vibrations of String and Wave Equation

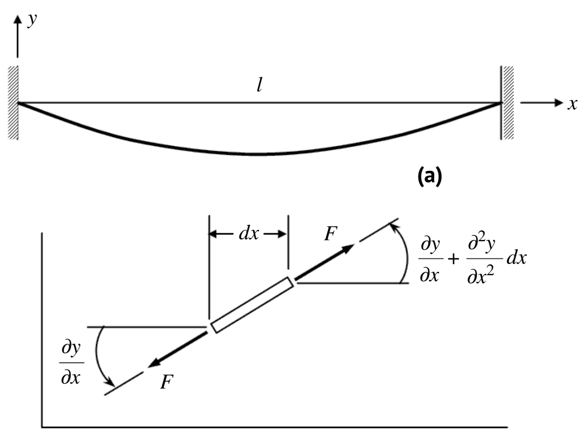
Consider a stretched flexible string of mass ρ per unit length having its end points attached to fixed surfaces. The string is free to vibrate in a vertical $(x - y)$ plane, as shown in Figure 2.3a. The coordinate of y is a function of both positions along the string x and time t, or

$$y = y(x, t) \tag{2.74}$$

The equilibrium position of the string is shown by the thick black line in Figure 2.3a, and its differential element in any possible position of motion is shown in Figure 2.3b. To develop the governing equation of motion for the string, the following assumptions are made: resistances of air and internal friction and gravitational forces are neglected in comparison with the tension in the string, which is quite large. The displacement of any point in the string is very small and occurs only in the $x - y$ plane.

Denote the tension in the string as T and the change in slope as $\frac{\partial^2 y}{\partial x^2}dx$. Considering the vertical motion of the differential element in Figure 2.3b, we can write the Newton's second law as

$$-T\sin\frac{\partial y}{\partial x} + T\sin\left(\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}dx\right) = \rho dx\frac{\partial^2 y}{\partial t^2} \tag{2.75}$$



(a)

(b)

Figure 2.3 Schematic of a string and the differential element.

Since the slopes are very small, we use the approximation $\sin\theta \approx \theta$ and Eq. (2.75) becomes

$$-T\frac{\partial y}{\partial x} + T\left(\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}dx\right) = \rho dx\frac{\partial^2 y}{\partial t^2} \tag{2.76}$$

or $$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2} \tag{2.77}$$

Equation (2.77) is a linear, second-order, partial differential equation with constant coefficients and represents the governing equation for transverse motion of the string. Rewrite Eq. (2.77) as

$$c^2\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{2.78}$$

where $c = \sqrt{T/\rho}$. Equation (2.78) is called the one-dimensional wave equation, and the constant c is called the wave speed. We assume the solution as

$$y(x, t) = F(x)G(t) \tag{2.79}$$

Then substituting the above equation into Eq. (2.78), we have

$$c^2\frac{\partial^2 F}{\partial x^2}/F = \frac{\partial^2 G}{\partial t^2}/G = \mu \tag{2.80}$$

in which μ is a constant. The initial conditions are

$$y(x, 0) = F(x)G(0) = F(x)G_0 \tag{2.81a}$$

$$\frac{\partial y(x, 0)}{\partial t} = F(x)\dot{G}(0) = F(x)\dot{G}_0 \tag{2.81b}$$

The boundary conditions are

$$y(0, t) = F(0)G(t) = 0 \tag{2.82}$$

$$y(l, t) = F(l)G(t) = 0 \tag{2.83}$$

The solution of Eq. (2.80) depends on the value of μ. It can easily be verified that $\mu < 0$ is the only possibility which satisfies both the differential equations and the boundary conditions. Hence for $\mu < 0$, we assume $\mu = -\omega^2$, Eq. (2.80) can be rewritten as

$$\ddot{G} + \omega^2 G = 0 \tag{2.84}$$

$$F'' + (\omega/c)^2 F = 0 \tag{2.85}$$

which have the solutions

$$G(t) = A\sin\omega t + B\cos\omega t \tag{2.86}$$

$$F(x) = C\sin(\omega/c)x + D\cos(\omega/c)x \tag{2.87}$$

The displacements are then given by

$$y(x, t) = (\mathrm{A}\sin\omega t + B\cos\omega t)C\sin(\omega/c)x + D\cos(\omega/c)x \tag{2.88}$$

Now applying the boundary conditions given by Eqs. (2.82) and (2.83) will give us

$$\sin(\omega/c)l = 0 \tag{2.89}$$

or $$(\omega/c)l = \pi, 2\pi, \ldots n\pi, \ldots \tag{2.90}$$