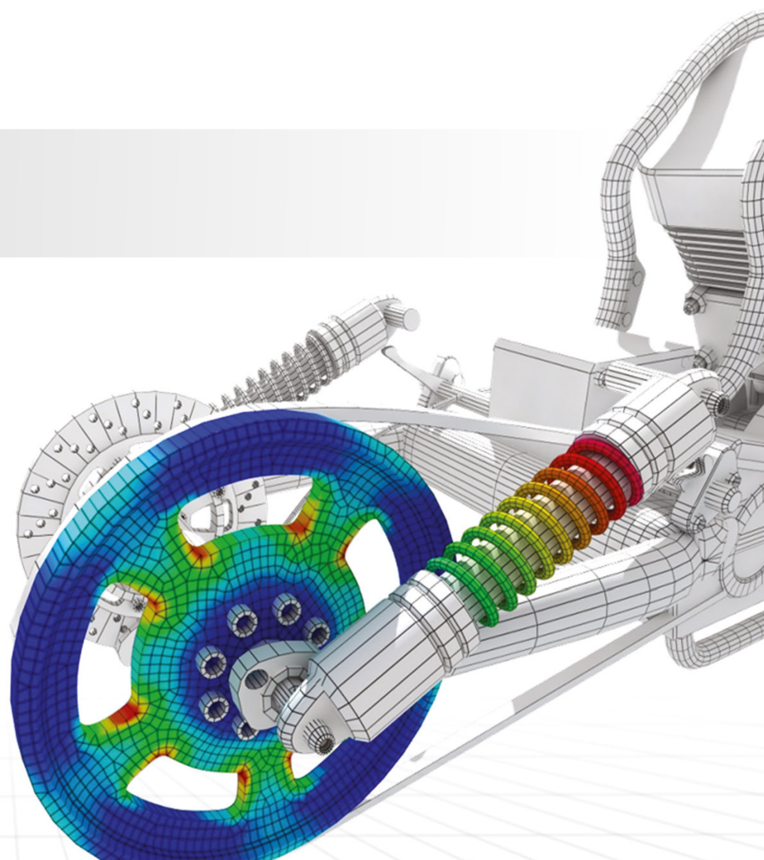
INTRODUCTION TO FINITE ELEMENT ANALYSIS AND DESIGN

THIRD EDITION



Nam-Ho Kim
Bhavani V. Sankar
Ashok V. Kumar

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Nam-Ho Kim, Bhavani V. Sankar,
and Ashok V. Kumar
University of Florida
Gainesville, FL

WILEY

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To our children, Hyesu and Jinsu, Dhyana, and Devika

Contents

Preface

The finite element method (FEM) is a numerical method for solving differential equations that describe many engineering problems. One of the reasons for FEM's popularity is that the method results in versatile computer programs that can solve many practical problems with a small amount of training. As there is a risk associated with using computer programs without a proper understanding of the theory behind them, thorough understanding of the theory behind the FEM is required.

Many universities teach the FEM to junior-/senior-level students. One of the biggest challenges to the instructor is finding a textbook appropriate for students' level. In the past, the FEM was taught only to graduate students who would carry out research in that field. Accordingly, many textbooks focus on theoretical development and numerical implementation of the method. However, the goal of an undergraduate FEM course is to introduce the basic concepts so that the students can use the method efficiently and interpret the results properly. Furthermore, the theoretical aspects of FEM must be presented without too many mathematical niceties. Practical applications through several design projects can help students understand the method clearly.

This book is suitable for junior-/senior-level undergraduate students and beginning graduate students in engineering mechanics, mechanical, civil, aerospace, biomedical, and industrial engineering as well as researchers and design engineers in the above fields. The textbook is organized into 10 chapters.

The book begins with the introduction of finite element (FE) concepts via the direct stiffness method using spring elements. The concepts of nodes, elements, internal forces, equilibrium, assembly, and applying boundary conditions are presented in detail. The spring element is then extended to the uniaxial bar element without introducing interpolation. The concept of local (elemental) and global coordinates and their transformations and element connectivity tables are introduced using 2D and 3D truss elements. In addition, the concept of thermal strain and thermal load is presented using truss elements. Four design projects are provided at the end of the chapter so that students can apply the method to real-life problems. The direct method in Chapter 1 provides a clear physical insight into the FEM and is preferred in the beginning stages of learning the principles. However, it can be used to solve 1D problems only.

The 1D formulation is further extended to beams and plane frames in Chapter 2. At this point, the direct method is not useful because the stiffness matrix generated from the direct method cannot provide a clear physical interpretation. Accordingly, we use the principle of minimum potential energy to derive the matrix equation at the element level. The 1-D beam element is extended to the 2D frame element using coordinate transformation. A 2D bicycle frame design project is included at the end of this chapter. An important failure mode, buckling of beams and plane frames, is included in this chapter. First, the concept of linear buckling of a beam is introduced, and then, the corresponding energy terms are derived in the FE context.

The FE formulation is extended to the steady-state heat transfer problem in Chapter 3. Both the direct and Galerkin methods are used for 1D heat conduction problems with temperature, heat flux, and convective boundary conditions. 2D heat transfer problems are used to demonstrate the application of the Galerkin method for higher dimensions. Both classical and isoparametric triangular elements are introduced and utilized for solving 2D heat conduction problems. Practical issues in modeling 2D heat transfer problems are also discussed.

Before proceeding to solid elements in Chapter 5, a review of solid mechanics is provided in Chapter 4. The concepts of stress and strain are presented followed by constitutive relations and equilibrium equations. We limit our interest to linear, isotropic materials in order to make the concepts simple and clear. However, advanced concepts such as the transformation of stress and strain, and the eigenvalue problem for calculating the principal values, are also included. Since, in practice, the FEM is used mostly for designing a structure or a mechanical system, failure/yield criteria are also introduced in this chapter.

In Chapter 5, we introduce 2D solid elements, including plane strain, plane stress, and axisymmetric problems. The governing variational equation is developed using the principle of minimum potential energy. The FE concepts are explained in detail using only triangular and rectangular elements. The numerical performance of each element is discussed through examples.

The concept of isoparametric mapping is introduced in a separate chapter (Chapter 6) as most practical problems require irregular elements such as linear or higher-order quadrilateral elements. The properties of the mapping and its Jacobian are discussed. Shear-deformable Timoshenko beam element is introduced as a 1D example of an isoparametric element. Various 2D and 3D isoparametric elements are also introduced, including higher-order elements. Numerical integration and FE modeling practices for isoparametric elements are also included.

Many mechanical systems are made of thin plates such as sheet metals. Plate and shell elements are useful for modeling such systems. Chapter 7 introduces plate and shell elements. Since the theories and FE formulations for plates and shells are complicated, we minimize deriving equations. Instead, we focus on how modeling plates and shells is different from that of solids. Both the classical plate and shear-deformable plate formulations are discussed. Some remedies to prevent shear locking are discussed, which is an important issue in shear-deformable plates.

Chapter 8 introduces the FE formulation of dynamic problems. The concept of free vibration, calculation of natural frequencies and mode shapes, various time integration methods, and mode superposition method are all explained using 1D structural elements such as uniaxial bars and beams.

In Chapter 9, we discuss traditional finite element analysis (FEA) procedures, including preliminary analysis, pre-processing, solving matrix equations, and post-processing. Emphasis is on the selection of element types, approximating the part geometry, different types of meshing, convergence, and taking advantage of symmetry. A design project involving 2D analysis is provided at the end of the chapter.

As one of the important goals of the FEM is to apply it to engineering design, the last chapter (Chapter 10) is dedicated to the topic of structural design using the FEM. The basic concept of design parameterization and the standard design problem formulation are presented. This chapter is self-contained and can be skipped depending on the schedule and content of the course.

Due to page restrictions, several contents were moved to an online companion site of the book. The first online chapter is of mathematical preliminaries that are repeatedly used in the text. The chapter is by no means a comprehensive mathematical treatment of the subject. Rather, it provides a common notation and basic mathematical knowledge that will be required in using the book effectively. The second online chapter is tutorials on several commercial FE software, which include Ansys, Abaqus, Autodesk Nastran, and Matlab toolbox. The last online chapter summarizes several theories related to FE formulations, including the weighted residual method, the Galerkin method, the principle of minimum potential energy, and the principle of virtual work. We moved this chapter online because it can be left out in elementary-level courses.

Each chapter contains a comprehensive set of homework problems, some of which require commercial FEA programs. A total of nine design projects are provided in the book.

We are thankful to several instructors across the country who used the first edition and provided feedback. We are grateful for their valuable suggestions, especially regarding the examples and exercise problems.

October 2024
Nam-Ho Kim, Bhavani V. Sankar, and Ashok V. Kumar

About the Companion Website

This book is accompanied by a companion website:

www.wiley.com/go/introFEA3e

The website includes:

1. The author's web URL https://web.mae.ufl.edu/nkim/introFEA/ contains Lecture slides, figures, and tutorials using commercial software.
2. Chapter 11 and the Appendix.

Chapter 1

Direct Method—Springs, Bars, and Truss Elements

The ability to predict the behavior of machines in engineering systems, in general, is of great importance at every stage of engineering processes, including design, manufacture, and operation. Such predictive methodologies are possible because engineers and scientists have made tremendous progress in understanding the physical behavior of materials and structures and have developed mathematical models, albeit approximate, in order to describe their physical behavior. Most often the mathematical models result in algebraic, differential, or integral equations or combinations thereof. Seldom can these equations be solved in closed form, and hence numerical methods are used to obtain solutions. The finite difference method is a classical method that provides approximate solutions to differential equations with reasonable accuracy. There are other methods of solving mathematical equations that are covered in traditional numerical methods courses.[1]

The finite element method (FEM) is one of the numerical methods for solving differential equations. The FEM, originated in the area of structural mechanics, has been extended to other areas of solid mechanics and later to other fields such as heat transfer, fluid dynamics, and electromagnetism. In fact, FEM has been recognized as a powerful tool for solving partial differential equations and integro-differential equations, and it has become the numerical method of choice in many engineering and applied science areas. One of the reasons for FEM's popularity is that the method results in computer programs versatile in nature that can solve many practical problems with the least amount of training. Obviously, there is a danger in using computer programs without a proper understanding of the theory behind them, and that is one of the reasons to have a thorough understanding of the theory behind the FEM.

The basic principle of FEM is to divide or *discretize* the system into a number of smaller elements called finite elements (FEs), to identify the degrees of freedom (DOFs) that describe its behavior, and then to write down the equations that describe the behavior of each element and its interaction with neighboring elements. The element-level equations are assembled to obtain global equations, often a linear system of equations, which are solved for the unknown DOFs. The phrase *finite element* refers to the fact that the elements are of a finite size as opposed to the infinitesimal or

[1] Atkinson, K. E. 1978. *An Introduction to Numerical Analysis*. Wiley, New York, NY.

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differential element considered in deriving the governing equations of the system. Another interpretation is that the FE equations deal with a finite number of DOFs as opposed to the infinite number of DOFs of a continuous system.

In general, solutions to practical engineering problems are quite complex, and they cannot be represented using simple mathematical expressions. An important concept of the FEM is that the solution is approximated using simple polynomials, often linear or quadratic, within each element. Since elements are connected throughout the system, the solution of the system is approximated using piecewise polynomials. Such approximation may contain errors when the size of an element is large. As the size of the element reduces, however, the approximated solution will converge to the exact solution.

There are three methods that can be used to derive the FE equations of a problem: (i) direct method, (ii) variational method, and (iii) weighted residual method. The direct method provides a clear physical insight into the FEM and is preferred in the beginning stages of learning the principles. However, it is limited in its application in that it can be used to solve 1D problems only. The variational method is akin to the methods of calculus of variations and is a powerful tool for deriving the FE equations. However, it requires the existence of a functional, whose minimization results in the solution of the differential equations. The Galerkin method is one of the popular weighted residual methods and is applicable to most problems. If a variational function exists for the problem, then the variational and Galerkin methods yield identical solutions.

In this chapter, we will illustrate the direct method of FE analysis using 1D elements such as linear spring, uniaxial bar, and truss elements. The emphasis is on the construction and solution of the FE equations and interpretation of the results, rather than the rigorous development of the general principles of the FEM.

1.1 ILLUSTRATION OF THE DIRECT METHOD

Consider a system of rigid bodies connected by springs as shown in figure 1.1. The bodies move only in the horizontal direction. Furthermore, we consider only the static problem and, hence, the mass effects (inertia) will be ignored. External forces, F_2, F_3, and F_4, are applied on the rigid bodies as shown. The objectives are to determine the displacement of each body, forces in the springs, and support reactions.

We will introduce the principles involved in the FEM through this example. Notice that there is no need to discretize the system as it already consists of discrete elements, namely, the springs. The elements are connected at the nodes. In this case, the rigid bodies are the nodes. Of course, the two walls are also the nodes as they connect to the elements. Numbers inside the little circles mark the nodes. The system of connected elements is called the mesh and is best described using a connectivity table that defines which nodes an element is connected to as shown in table 1.1. It is noted that in this 1D problem, LN1 is the node on the left, and LN2 is the node on the right. Such a connectivity table is included in input files for FE analysis software to describe the mesh.

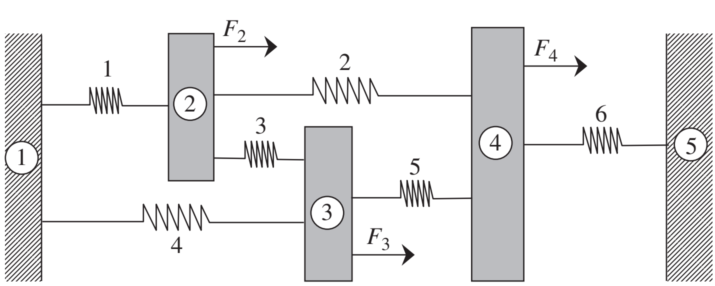

Figure 1.1 Rigid bodies connected by springs

Table 1.1 Connectivity table for figure 1.1

Element	LN1 (i)	LN2 (j)
1	1	2
2	2	4
3	2	3
4	1	3
5	3	4
6	4	5

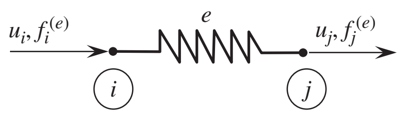

Figure 1.2 Spring element (e) connected by node i and node j

In this 1D system, each node is allowed to move in the horizontal direction. Such a movement is referred to as *DOF*. Since nodes 2, 3, and 4 are free to move, they are referred to as free DOFs, while nodes 1 and 5 are fixed DOFs. The displacements of the fixed DOFs are given (zero in this case), and they are referred to as boundary conditions. Those nodes on the boundary conditions have unknown reaction forces, which need to be calculated by solving the system of equations. The displacements of the free DOFs are unknown, which also need to be calculated, but the applied forces at the free DOFs are all known. This includes those nodes that do not have applied forces (or it can be considered as applying a zero force). Applying forces on nodes is referred to as loading conditions.

Consider the free-body diagram of a typical element (e) as shown in figure 1.2. It has two nodes, i and j. They will also be referred to as the first and the second node or local node 1 (LN1) and local node 2 (LN2), respectively, as shown in the connectivity table. Assume a coordinate system going from left to right. The convention for the first and second nodes is that $x_i < x_j$. The forces acting at the nodes are denoted by $f_i^{(e)}$ and $f_j^{(e)}$. In this notation, the subscripts denote the node numbers and the superscript the element number. This notation is adopted because multiple elements can be connected at a node, and each element may have different forces at the node. We will refer to them as *internal forces*. In figure 1.2, the forces are shown in the positive direction. The unknown displacements (i.e., DOFs) of nodes i and j are u_i and u_j, respectively. Note that there is no superscript for u, as the displacement is unique to the node denoted by the subscript. We would like to develop a relationship between the nodal displacements u_i and u_j and the internal forces $f_i^{(e)}$ and $f_j^{(e)}$.

The elongation of the spring is denoted by $\Delta^{(e)} = u_j - u_i$. Then the force of the spring is given by

$$P^{(e)} = k^{(e)}\Delta^{(e)} = k^{(e)}\left(u_j - u_i\right) \tag{1.1}$$

where $k^{(e)}$ is the spring rate or *stiffness* of element (e). In this text, the force in the spring, $P^{(e)}$, is referred to as *element force*. If $u_j > u_i$, then the spring is elongated, and the force in the spring is positive (tension). Otherwise, it is in compression. The spring element force is related to the internal force by

$$f_j^{(e)} = P^{(e)} \tag{1.2}$$

Note that the sign of $f_i^{(e)}$ and $f_j^{(e)}$ is determined based on the direction that the force is applied, while the sign of $P^{(e)}$ is determined based on whether the element is in tension or compression. For equilibrium, the sum of the forces acting on element (e) must be equal to zero, that is,

$$f_i^{(e)} + f_j^{(e)} = 0 \quad \text{or} \quad f_i^{(e)} = -f_j^{(e)} \tag{1.3}$$

Therefore, the two forces are equal, and they are applied in opposite directions. When $f_j^{(e)}$ is positive, the element is in tension, and, thus, $P^{(e)}$ is positive.

From eqs. (1.1) to (1.3), we can obtain a relation between the internal forces and the displacements as

$$\begin{aligned} f_i^{(e)} &= k^{(e)}\left(u_i - u_j\right) \\ f_j^{(e)} &= k^{(e)}\left(-u_i + u_j\right) \end{aligned} \tag{1.4}$$

Equation (1.4) can be written in the matrix forms as

$$k^{(e)}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix} \tag{1.5}$$

We also write eq. (1.5) in a shorthand notation as

$$\left[\mathbf{k}^{(e)}\right]\begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix}$$

or

$$\boxed{\left[\mathbf{k}^{(e)}\right]\left\{\mathbf{q}^{(e)}\right\} = \left\{\mathbf{f}^{(e)}\right\}} \tag{1.6}$$

where $[\mathbf{k}^{(e)}]$ is the element stiffness matrix, $\{\mathbf{q}^{(e)}\}$ is the vector of DOFs associated with element (e), and $\{\mathbf{f}^{(e)}\}$ is the vector of internal forces. Sometimes, we will omit the superscript (e) with the understanding that we are dealing with a generic element. Equation (1.6) is called the *element equilibrium equation*.

The element stiffness matrix $[\mathbf{k}^{(e)}]$ has the following properties:

1. It is square as it relates to the same number of forces as the displacements.
2. It is symmetric (a consequence of the Betti–Rayleigh reciprocal theorem in solid and structural mechanics[2]).
3. It is singular, *that is*, its determinant is equal to zero, so it cannot be inverted.
4. It is positive semidefinite.

Properties 3 and 4 are related to each other, and they have physical significance. In eq. (1.6), if the nodal displacements u_i and u_j of a spring element are given, then it should be possible to predict the force $P^{(e)}$ in the spring from its change in length ($u_j - u_i$), and, hence, the forces $\{\mathbf{f}^{(e)}\}$ acting at its nodes can be predicted. In fact, the internal forces can be computed by performing the matrix multiplication $[\mathbf{k}^{(e)}]\{\mathbf{q}^{(e)}\}$. On the other hand, if the two internal forces are given (they must have equal magnitude but opposite directions), the nodal displacements cannot be determined uniquely, as a rigid-body displacement (equal u_i and u_j) can be added without affecting the spring force. If $[\mathbf{k}^{(e)}]$ were to have an inverse as opposed to Property 3, then it would have been possible to solve for $\{\mathbf{q}^{(e)}\} = [\mathbf{k}^{(e)}]^{-1}\{\mathbf{f}^{(e)}\}$ uniquely in violation of the physics. Property 4 has also a physical interpretation, which will be discussed in conjunction with energy methods.

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[2] Fung, Y. C. 1965. *Foundations of Solid Mechanics*. Prentice-Hall, Englewood Cliffs, NJ.

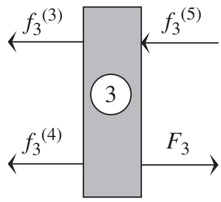

Figure 1.3 Free-body diagram of node 3 in the example is shown in figure 1.1. The external force, F_3, and the internal forces, $f_3^{(e)}$, exerted by the springs attached to the node are shown. Note the internal forces $f_3^{(e)}$ act in the negative direction.

In the next step, we develop a relationship between the internal forces $f_i^{(e)}$ and the known external forces F_i. For example, consider the free-body diagram of node 3 (or the rigid body in this case) in figure 1.1. The forces acting on the node are the external force F_3 and the internal forces from the springs connected to node 3 as shown in figure 1.3.

For equilibrium of the node, the sum of the forces acting on the node should be equal to zero:

$$F_i - \sum_{e=1}^{i_e} f_i^{(e)} = 0$$

or

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)}, \quad i = 1, \ldots, ND \tag{1.7}$$

where i_e is the number of elements connected to node i, and ND is the total number of nodes in the model. Equation (1.7) is the equilibrium between externally applied forces at a node and internal forces from connected elements. If there is no externally applied force at a node, then the sum of internal forces at the node must be zero. Such equations can be written for each node including the boundary nodes, such as nodes 1 and 5 in figure 1.1. The internal forces $f_i^{(e)}$ in eq. (1.7) can be replaced by the unknown DOFs $\{\mathbf{q}\}$ by using eq. (1.6). For example, the force equilibrium for the springs in figure 1.1 can be written as

$$\begin{cases} F_1 = f_1^{(1)} + f_1^{(4)} = k^{(1)}(u_1 - u_2) + k^{(4)}(u_1 - u_3) \\ F_2 = f_2^{(1)} + f_2^{(3)} + f_2^{(2)} = k^{(1)}(u_2 - u_1) + k^{(3)}(u_2 - u_3) + k^{(2)}(u_2 - u_4) \\ F_3 = f_3^{(3)} + f_3^{(4)} + f_3^{(5)} = k^{(3)}(u_3 - u_2) + k^{(4)}(u_3 - u_1) + k^{(5)}(u_3 - u_4) \\ F_4 = f_4^{(2)} + f_4^{(5)} + f_4^{(6)} = k^{(2)}(u_4 - u_2) + k^{(5)}(u_4 - u_3) + k^{(6)}(u_4 - u_5) \\ F_5 = f_5^{(6)} = k^{(6)}(u_5 - u_4) \end{cases} \tag{1.8}$$

This will result in ND number of linear equations for the ND number of DOFs:

$$[\mathbf{K}_s]\begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{ND} \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{ND} \end{Bmatrix} \tag{1.9}$$

Or, in shorthand notation $[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}$, where $[\mathbf{K}_s]$ is the structural stiffness matrix, $\{\mathbf{Q}_s\}$ is the vector of displacements of all nodes in the model, and $\{\mathbf{F}_s\}$ is the vector of external forces, including the unknown reactions. The expanded form of eq. (1.9) is given in the following equation:

$$\begin{bmatrix} k^{(1)}+k^{(4)} & -k^{(1)} & -k^{(4)} & 0 & 0 \\ -k^{(1)} & k^{(1)}+k^{(2)}+k^{(3)} & -k^{(3)} & -k^{(2)} & 0 \\ -k^{(4)} & -k^{(3)} & k^{(3)}+k^{(4)}+k^{(5)} & -k^{(5)} & 0 \\ 0 & -k^{(2)} & -k^{(5)} & k^{(2}+k^{(5)}+k^{(6)} & -k^{(6)} \\ 0 & 0 & 0 & -k^{(6)} & k^{(6} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{Bmatrix}$$

or,

$$[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\} \tag{1.10}$$

The properties of the structural stiffness matrix $[\mathbf{K}_s]$ are similar to that of the element stiffness matrix: square, symmetric, singular, and positive semidefinite. In addition, when nodes are numbered properly, $[\mathbf{K}_s]$ will be a banded matrix. It should be noted that when the boundary displacements in $\{\mathbf{Q}_s\}$ are known (usually equal to zero[3]), the corresponding forces in $\{\mathbf{F}_s\}$ are unknown reactions. In the present illustration, $u_1 = u_5 = 0$ and corresponding forces (reactions) F_1 and F_5 are unknown. It should also be noted that when displacements in $\{\mathbf{Q}_s\}$ are unknown, the corresponding forces in $\{\mathbf{F}_s\}$ should be known (either a given value or zero when no force is applied). In a well-posed problem, either the displacement or the external force should be known as a node, but not both. When the displacement at a node is given (boundary condition), the external force at the node becomes an unknown reaction. When a node has a known external force (and often it is zero), its displacement should be unknown. When a system of springs has *ND* numbers of nodes, there should be the same number of unknowns, either unknown displacements or unknown reactions.

We will impose the boundary conditions as follows. First, we ignore the equations for which the right-hand side (RHS) forces are unknown (i.e., reaction forces) and strike out the corresponding rows in $[\mathbf{K}_s]$. This is called *striking-the-rows*. Then we eliminate the columns in $[\mathbf{K}_s]$ that are multiplied by the zero values of displacements of the boundary nodes. This is called *striking-the-columns*. It may be noted that if the *n*th row is eliminated (struck), then the *n*th column will also be eliminated (struck). This process is equivalent to removing all unknown reactions and known displacement, such that only the unknown displacements and known forces remain. This process results in a system of equations given by $[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$, where $[\mathbf{K}]$ is the global stiffness matrix, $\{\mathbf{Q}\}$ is the vector of unknown DOFs, and $\{\mathbf{F}\}$ is the vector of known forces. The global stiffness matrix will be square, symmetric, and **positive definite** and, hence, nonsingular. Usually, $[\mathbf{K}]$ will also be banded. In large systems, that is, in models with large numbers of DOFs, $[\mathbf{K}]$ will be a sparse matrix with a small proportion of nonzero numbers in a diagonal band.

After striking-the-rows and -columns corresponding to zero DOFs (u_1 and u_5) in eq. (1.10), we obtain the global equations as follows:

$$\begin{bmatrix} k^{(1)}+k^{(2)}+k^{(3)} & -k^{(3)} & -k^{(2)} \\ -k^{(3)} & k^{(3)}+k^{(4)}+k^{(5)} & -k^{(5)} \\ -k^{(2)} & -k^{(5)} & k^{(2)}+k^{(5)}+k^{(6)} \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

[3] Nonzero or prescribed DOFs will be dealt with in chapter 3.

or,

$$[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\} \tag{1.11}$$

It is noted that the global system of equations in eq. (1.11) only includes the unknown displacements and known forces.

In principle, the solution can be obtained as $\{\mathbf{Q}\} = [\mathbf{K}]^{-1}\{\mathbf{F}\}$. Once the unknown DOFs are determined, the spring forces of individual elements can be obtained using eq. (1.1). The support reactions can be obtained from either the nodal equilibrium equations (1.7) or the structural equations (1.10). Therefore, the FE solution procedure is to solve for unknown DOFs first, followed by unknown reactions.

EXAMPLE 1.1 *Rigid body-spring system*

Find the displacements of the rigid bodies shown in figure 1.1. Assume that the only nonzero force is $F_3 = 1{,}000$ N. Determine the element forces (tensile/compressive) in the springs. What are the reactions on the walls? Assume the bodies can undergo only translation in the horizontal direction. The spring constants (N/mm) are $k^{(1)} = 500$, $k^{(2)} = 400$, $k^{(3)} = 600$, $k^{(4)} = 200$, $k^{(5)} = 400$, and $k^{(6)} = 300$.

SOLUTION The element equilibrium equations are as follows:

$$\begin{gathered}
\begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix} = 500\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}; \quad \begin{Bmatrix} f_2^{(2)} \\ f_4^{(2)} \end{Bmatrix} = 400\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_4 \end{Bmatrix} \\
\begin{Bmatrix} f_2^{(3)} \\ f_3^{(3)} \end{Bmatrix} = 600\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}; \quad \begin{Bmatrix} f_1^{(4)} \\ f_3^{(4)} \end{Bmatrix} = 200\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} \\
\begin{Bmatrix} f_3^{(5)} \\ f_4^{(5)} \end{Bmatrix} = 400\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_3 \\ u_4 \end{Bmatrix}; \quad \begin{Bmatrix} f_4^{(6)} \\ f_5^{(6)} \end{Bmatrix} = 300\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_4 \\ u_5 \end{Bmatrix}
\end{gathered} \tag{1.12}$$

The nodal equilibrium equations are

$$\begin{aligned}
f_1^{(1)} + f_1^{(4)} &= F_1 = R_1 \\
f_2^{(1)} + f_2^{(2)} + f_2^{(3)} &= F_2 = 0 \\
f_3^{(3)} + f_3^{(4)} + f_3^{(5)} &= F_3 = 1{,}000 \\
f_4^{(2)} + f_4^{(5)} + f_4^{(6)} &= F_4 = 0 \\
f_5^{(6)} &= F_5 = R_5
\end{aligned} \tag{1.13}$$

where R_1 and R_5 are unknown reaction forces at nodes 1 and 5, respectively. In the above equation, F_2 and F_4 are equal to zero because no external forces act on those nodes. Combining eqs. (1.12) and (1.13), we obtain the equation $[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\}$,

$$100\begin{bmatrix} 7 & -5 & -2 & 0 & 0 \\ -5 & 15 & -6 & -4 & 0 \\ -2 & -6 & 12 & -4 & 0 \\ 0 & -4 & -4 & 11 & -3 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ 0 \\ 1{,}000 \\ 0 \\ R_5 \end{Bmatrix} \tag{1.14}$$

After implementing the boundary conditions at nodes 1 and 5 (striking-the rows and columns corresponding to zero displacements), we obtain the following global equations $[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\}$:

$$100\begin{bmatrix} 15 & -6 & -4 \\ -6 & 12 & -4 \\ -4 & -4 & 11 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1{,}000 \\ 0 \end{Bmatrix}$$

By inverting the global stiffness matrix, the unknown displacements can be obtained as $u_2 = 0.854$ mm, $u_3 = 1.55$ mm, and $u_4 = 0.875$ mm.

The element forces in the springs are computed using $P^{(e)} = k^{(e)}(u_j - u_i)$:

$$P^{(1)} = 427 \text{ N}; \quad P^{(2)} = 8.3 \text{ N}; \quad P^{(3)} = 419 \text{ N}$$
$$P^{(4)} = 310 \text{ N}; \quad P^{(5)} = -271 \text{ N}; \quad P^{(6)} = -263 \text{ N}$$

Wall reactions, R_1 and R_5, can be computed either from eq. (1.14) after substituting for the displacements or from eqs. (1.12) and (1.13) as $R_1 = -737$ N; $R_5 = -263$ N. Both reactions are negative, meaning that they act on the structure (the system) from right to left.

1.2 UNIAXIAL BAR ELEMENT

The FE analysis procedure for the spring–force system in the previous section can easily be extended to uniaxial bars. Plane and space trusses consist of uniaxial bars, and, hence, a detailed study of uniaxial bar finite element will provide the basis for analysis of trusses. Typical problems that can be solved using uniaxial bar elements are shown in figure 1.4. A uniaxial bar is a slender two-force member where the length is much larger than the cross-sectional dimensions. The bar can have varying cross-sectional area, $A(x)$, and consists of different materials, that is, varying Young's modulus, $E(x)$. Both concentrated forces F and distributed force $p(x)$ can be applied. The distributed forces can be applied over a portion of the bar. The forces F and $p(x)$ are considered positive if they act in the positive direction of the x-axis. Both ends of the bar can be fixed making it a statically indeterminate problem, which is difficult to solve using the methods in the mechanics of materials course. However, statically indeterminate problems can be readily solved using FE analysis. In fact, the dimension of the matrix equation in FE analysis is reduced for statically indeterminate systems.

1.2.1 FE Formulation for Uniaxial Bar

The FE analysis procedures for the uniaxial bar are as follows:

1. **Discretize** the bar into a number of elements. The criteria for determining the size of the elements will become obvious after learning the properties of the element. It is assumed that each element has a constant axial rigidity, EA, throughout its length, although it may vary from element to element.

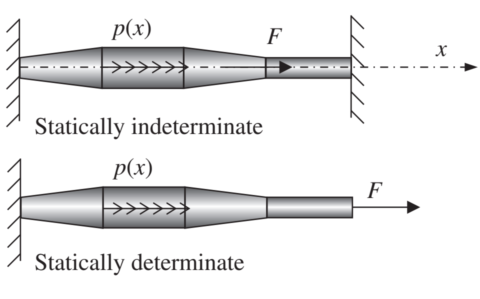

Figure 1.4 Typical 1D bar problems

2. The elements are connected at nodes. Thus, more than one element can share a node. There will be nodes at points where the bar is supported.
3. External forces are applied only at the nodes, and they must be point forces (concentrated forces). If distributed forces are applied to the bar, they have to be approximated as point forces acting at nodes. At the bar boundary, if the displacement is specified, then the reaction is unknown. The reaction will be the external force acting on the boundary node. If a specified external force acts on the boundary, then the corresponding displacement is unknown. There will be no case when both displacement and force are unknown at a node.
4. The deformation of the bar is determined by the axial displacements of the nodes. That is, the nodal displacements are the DOFs in the FEM. Thus, the DOFs are u_1, u_2, u_3, …, u_N, where N is the total number of nodes.

The objective of the FE analysis is to determine: (i) unknown DOF (u_i), (ii) axial force resultant ($P^{(e)}$) in each element (i.e., element force), and (iii) support reactions. Once the axial force resultant, $P^{(e)}$, is available, the element stress can easily be calculated by $\sigma^{(e)} = P^{(e)}/A^{(e)}$, where $A^{(e)}$ is the cross section of the element. The term 'force resultant' is often used in mechanics for the total force obtained by integrating stress over the cross section.

We will use the *direct stiffness method* to derive the element stiffness matrix. Consider the free-body diagram of a typical element (e), as illustrated in figure 1.5. Internal forces and displacements are defined as positive when they are in the positive x-direction. The element has two nodes, namely, i and j. Node i will be the first node and node j be the second node. The convention is that the line i–j will be in the positive direction of the x-axis. The displacements of the nodes are u_i and u_j. The element has a stiffness of $k^{(e)} = (EA/L)^{(e)}$, where EA is the axial rigidity, and L is the length of the element. This can be easily understood from the relationship between elongation and axial force of a bar, $\Delta = FL/EA$, in the mechanics of materials. It will be shown later that the stiffness $k^{(e)}$ plays exactly the same role as in the stiffness of a spring element in the previous section.

The internal forces acting at the two ends of the free body are $f_i^{(e)}$ and $f_j^{(e)}$. The superscript denotes the element number, and the subscripts denote the node numbers. The (lowercase) force f denotes the internal force as opposed to the (uppercase) external force F_i acting on the nodes. Since we do not know the direction of f, we will assume that all forces act in the positive coordinate direction. It should be noted that the nodal displacements do not need a superscript, as they are unique to the nodes. However, the internal force acting at a node may be different for different elements connected to the same node.

First, we will determine a relation between the f's and u's of the element (e). For equilibrium of the free-body diagram, we have

$$f_i^{(e)} + f_j^{(e)} = 0 \tag{1.15}$$

which means that the two forces acting on the two nodes of the element are equal and in opposite directions. Referring to figure 1.5, it is clear that when $f_j^{(e)} > 0$, the element is in tension, and when $f_j^{(e)} < 0$, the element is in compression.

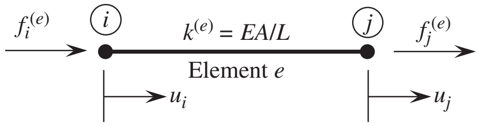

Figure 1.5 Uniaxial bar finite element

From elementary mechanics of materials, the force is proportional to the elongation of the element. The elongation of the bar element is denoted by $\Delta^{(e)} = u_j - u_i$. Then, similar to the spring element, where $f = k\Delta$, the force equilibrium of the 1D bar element can be written as

$$f_j^{(e)} = \left(\frac{AE}{L}\right)^{(e)} (u_j - u_i)$$

$$f_i^{(e)} = -f_j^{(e)} = \left(\frac{AE}{L}\right)^{(e)} (u_i - u_j)$$

where A, E, and L, respectively, are the area of the cross section, Young's modulus, and the length of the element. Using matrix notation, the above equations can be written as

$$\begin{Bmatrix} f_i^{(e)} \\ f_j^{(e)} \end{Bmatrix} = \left(\frac{AE}{L}\right)^{(e)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{1.16}$$

Equation (1.16) is called the *element equilibrium equation*, which relates the nodal forces of element (e) to the corresponding nodal displacements. Note that eq. (1.16) is similar to eq. (1.5) of the spring element if $k^{(e)} = (EA/L)^{(e)}$. Equation (1.16) for each element can be written in a compact form as

$$\left\{\mathbf{f}^{(e)}\right\} = \left[\mathbf{k}^{(e)}\right]\left\{\mathbf{q}^{(e)}\right\}, \quad e = 1,2,\ldots,N_e \tag{1.17}$$

where $[\mathbf{k}^{(e)}]$ is the element stiffness matrix of element (e), $\{\mathbf{q}^{(e)}\}$ is the vector of nodal displacements of the element, and N_e is the total number of elements in the model.

Note that the element stiffness matrix in eq. (1.16) is singular. The fact that the element stiffness matrix does not have an inverse has a physical significance. If the nodal displacements of an element are specified, then the element forces can be uniquely determined by performing the matrix multiplication in eq. (1.16). On the other hand, if the forces acting on the element are given, the nodal displacements cannot be uniquely determined because one can always translate the element by adding a rigid-body displacement without affecting the forces acting on it. Thus, it is always necessary to remove the rigid-body motion by fixing some displacements at nodes.

1.2.2 Nodal Equilibrium

Consider the free-body diagram of a typical node i. It is connected to, say, elements (e) and $(e + 1)$. Then, the forces acting on the nodes are the external force F_i and reactions to the element's internal forces as shown in figure 1.6. The internal forces are applied in the negative x-direction because they are the reaction to the forces acting on the element. The sum of the forces acting on node i must be equal to zero:

$$F_i - f_i^{(e)} - f_i^{(e+1)} = 0$$

or

$$f_i^{(e)} + f_i^{(e+1)} = F_i \tag{1.18}$$

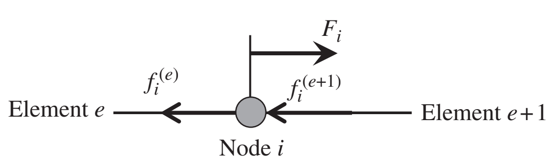

Figure 1.6 Force equilibrium at node i

In general, the external force acting on a node is equal to sum of all the internal forces acting on different elements connected to the node, and eq. (1.18) can be generalized as

$$F_i = \sum_{e=1}^{i_e} f_i^{(e)} \tag{1.19}$$

where i_e is the number of elements connected to node i, and the sum is carried out over all the elements connected to node i.

1.2.3 Assembly

The next step is to eliminate the internal forces from eq. (1.18) using eq. (1.17) in order to obtain a relation between the unknown displacements $\{\mathbf{Q}_s\}$ and known forces $\{\mathbf{F}_s\}$. This step results in a process called an *assembly* of the element stiffness matrices. We substitute for f's from eq. (1.17) into eq. (1.19) in order to find a relation between the nodal displacements and external forces. The force equilibrium in eq. (1.19) can be written for each DOF at each node yielding a relation between the external forces and displacements as

$$[\mathbf{K}_s]\{\mathbf{Q}_s\} = \{\mathbf{F}_s\} \tag{1.20}$$

Equation (1.20) is called the *structural matrix equation*. In the above equation, $[\mathbf{K}_s]$ is the structural stiffness matrix, which characterizes the load–deflection behavior of the entire structure; $\{\mathbf{Q}_s\}$ is the vector of all nodal displacements, known and unknown; and $\{\mathbf{F}_s\}$ is the vector of external forces acting at the nodes, including the unknown reactions.

There is a systematic procedure by which the element stiffness matrices $[\mathbf{k}^{(e)}]$ can be assembled to obtain $[\mathbf{K}_s]$. We will assign a row address and column address for each entry in $[\mathbf{k}^{(e)}]$ and $[\mathbf{K}_s]$. The column address of a column is the DOF that the column multiplies within the equilibrium equation. For example, the column addresses of the first and second column in $[\mathbf{k}^{(e)}]$ are u_i and u_j, respectively. The column addresses of columns 1, 2, 3 in $[\mathbf{K}_s]$ are u_1, u_2, u_3, respectively. The row addresses and column addresses are always symmetric. That is, the row address of the ith row is the same as the column address of the ith column. Having determined the row and column addresses of $[\mathbf{k}^{(e)}]$ and $[\mathbf{K}_s]$, assembly of the element stiffness matrices can be done in a mechanical way. Each of the four entries (boxes) of an element stiffness matrix is transferred to the box in $[\mathbf{K}_s]$ with corresponding row and column addresses.

It is important to discuss the properties of the structural stiffness matrix $[\mathbf{K}_s]$. After assembly, the matrix $[\mathbf{K}_s]$ has the following properties:

1. It is square.
2. It is symmetric.
3. It is positive semidefinite.
4. Its determinant is equal to zero, and, thus, it does not have an inverse (it is singular).
5. The diagonal entries of the matrix are greater than zero.

For a given $\{\mathbf{Q}_s\}$, $\{\mathbf{F}_s\}$ can be determined uniquely; however, for a given $\{\mathbf{F}_s\}$, $\{\mathbf{Q}_s\}$ cannot be determined uniquely because an arbitrary rigid-body displacement can be added to $\{\mathbf{Q}_s\}$ without affecting $\{\mathbf{F}_s\}$.

1.2.4 Boundary Conditions

Before we solve eq. (1.20), we need to impose the displacement boundary conditions using the known nodal displacements in eq. (1.20). Mathematically, it means to make the global stiffness matrix positive definite so that the unknown displacements can be uniquely determined. Let us assume that the total size of $[\mathbf{K}_s]$ is $m \times m$. From the m equations, we will discard the equations for which we do not know the right-hand side (unknown reaction forces). This is called “striking-the-rows.” The structural stiffness matrix becomes rectangular, as the number of equations is less than m. Now we delete the columns that will multiply into prescribed zero displacements in $\{\mathbf{Q}_s\}$. Usually, if the ith row is deleted, then the ith column will also be deleted. Thus, we will be deleting as many columns as we did for rows. This procedure is called “striking-the-columns.” Now the stiffness matrix becomes square with size $n \times n$, where n is the number of unknown displacements. The resulting equations can be written as

$$[\mathbf{K}]\{\mathbf{Q}\} = \{\mathbf{F}\} \tag{1.21}$$

where $[\mathbf{K}]$ is the global stiffness matrix, $\{\mathbf{Q}\}$ are the unknown displacements, and $\{\mathbf{F}\}$ are the known external forces applied to nodes. Equation (1.21) is called the *global matrix equations*. In the structural matrix equations in eq. (1.20), the vector $\{\mathbf{Q}_s\}$ includes both known and unknown displacements. However, after applying boundary conditions, that is, striking-the-rows and striking-the-columns, the vector $\{\mathbf{Q}\}$ only includes unknown nodal displacements. For the same reason, the vector $\{\mathbf{F}\}$ only includes known external forces, not support reactions. When the displacement boundary conditions are given such a way that the system does not have a rigid-body motion, the global stiffness matrix is positive definite, which has an inverse. It is symmetric and its diagonal elements are positive, that is, $K_{ii} > 0,\ \ i = 1, \dots, n$. Thus, the displacements $\{\mathbf{Q}\}$ can be solved uniquely for a given set of nodal forces $\{\mathbf{F}\}$.

1.2.5 Calculation of Element Forces and Reaction Forces

Now that all the DOFs are known, the element force (i.e., force resultant) in element (e) can be determined using eq. (1.16). The axial force resultant $P^{(e)}$ in element (e) is given by

$$P^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \Delta^{(e)} = \left(\frac{AE}{L}\right)^{(e)} \left(u_j - u_i\right) \tag{1.22}$$

The sign convention of axial force resultant is similar to that of stress. It is positive when the bar is in tension and negative when it is in compression. In fact, stress can be calculate by dividing the element force by cross-sectional area. Another method of determining the axial-force resultant distribution along an element length is as follows. Consider the element equation (1.16). At the first node or node i, the internal force is given by $P_i = -f_i$. That is, if f_i acts in the positive direction, that end is under compression. If f_i is in the negative direction, the element is under tension. On the other hand, the opposite is true at the second node, node j. In that case, $P_j = +f_j$. Then, we can modify eq. (1.16) as

$$\begin{Bmatrix} -P_i^{(e)} \\ +P_j^{(e)} \end{Bmatrix} = \left(\frac{AE}{L}\right)^{(e)} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \tag{1.23}$$

It happens that $P_i^{(e)} = P_j^{(e)}$, and, hence, we use a single variable $P^{(e)}$ to denote the axial force in an element as shown in eq. (1.22).