Trends in the History of Science

Danuta Ciesielska
Henning Heller
Renate Tobies
Joanna Zwierzyńska
Editors

Felix Klein's Foreign Students

Opening Up the Way for Transnational Mathematics

Birkhäuser

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Preface

Felix Klein was one of the most prolific mathematicians of the nineteenth and twentieth centuries, both for his original mathematical contributions, but also as a mathematics educator and organizer. Through his efforts, Göttingen became a world center of mathematics, attracting students and researchers from all over Germany, Europe, and overseas, thus exerting a great influence on the development of mathematics in countries outside Germany.

The 16 chapters collected in this book testify to the enormous influence Klein had on and through his international students, and give witness to the process of internationalization and collaboration of mathematics under the aegis of Klein. We particularly look at mathematicians of foreign origin who have received little attention so far (British, Czech, Hungarian, Polish, Russian, Ukrainian, Greek, Spanish, Japanese mathematicians) and uncover how Klein guided them through his lectures and seminars. Although there are already good studies on Klein's influence on mathematicians in the USA and Italy, as well as on his relationship with mathematicians in France, Austria, and Switzerland, we present new findings on these topics. We also add some new insights into Klein's support of women mathematicians. Our book also contains a chapter on how Felix Klein tried to promote mathematical methods of probability, which had not been taught at German universities for a long time, with his own seminars in Göttingen and details the extent to which foreign mathematicians were involved.

Our approach has made it possible to unravel Felix Klein's international network. The list of people who financially supported the portrait of Felix Klein created by the famous artist Max Liebermann is included in the first chapter and indicates the breadth of his contacts. As well as making an impact through his students, Klein made an impact through many other ways: his publications, his lectures (which were lithographed, made available in the Göttingen reading room, and sent internationally as far away as Japan), his travel within Europe and the USA, and his correspondence.

In addition, Klein consciously promoted the publications of foreign mathematicians (women among them), included them in the publications of the scientific societies (in Erlangen, Leipzig, and Munich) and published more detailed contributions in the journal *Mathematische Annalen*, which he edited. Klein also had an influence on numerous mathematicians abroad by involving them in the *Encyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* (*Encyclopaedia of Mathematical Sciences including its applications*), where some mathematical

areas were presented systematically and in detail for the first time and for which he also initiated a French edition and coordinated the authors.

One of the main sources are the 29 volumes in which Felix Klein had the presentations given in the seminars handwritten down for around 40 years. The last chapter provides an overview of the participation of foreign mathematicians in these seminars, including those from Bulgaria, Romania, Serbia, Scandinavia, and North America, whom we do not consider elsewhere.

The starting point for this book was the mini-workshop No. 47/2023 at the *Mathematisches Forschungsinstitut Oberwolfach* on *Felix Klein's Foreign Students: Opening Up the Way for Transnational Mathematics*, organized by Danuta Ciesielska and Renate Tobies. The majority of the participants in the mini-workshop were able to elaborate their insights for this book. In addition, we were able to recruit two further authors who contributed their expertise on Klein's close links with mathematicians at the University of Turin (Italy), and on Klein's commitment to promoting probability theory and actuarial mathematics.

The editors would like to thank all those involved in producing this book, the *Mathematisches Forschungsinstitut Oberwolfach* for the opportunity to hold the mini-workshop. We would like to thank all employees of the archives used for their support. Special thanks go to Bärbel Mund, Manuscript Department of the Niedersächsische Staats- und Universitätsbibliothek, and Philipp Kastendieck, Mathematical Institute of the University of Göttingen, for their outstanding help in using documents from Felix Klein's estate. Last but not least, thanks are due to Thomas Hempfling, Verlag Birkhäuser (Springer Nature Switzerland), for including the book in the series *Trends in the History of Science*, and to Frida Trotter for her constant support during the working process.

Warsaw, Poland — Danuta Ciesielska
Bonn, Germany — Henning Heller
Jena, Germany — Renate Tobies
Katowice, Poland — Joanna Zwierzyńska
March 2025

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About the Editors

Danuta Ciesielska is a historian of mathematics and science at the L. & A. Birkenmajer Institute for the History of Science of Polish Academy of Sciences (IHS PAS). She had completed her doctoral degree in pure mathematics at the Jagiellonian University in Kraków, and her habilitation in history at the IHS PAS in Warsaw. Her research focuses on the history of mathematics and mathematical education in universities in the nineteenth and twentieth centuries, as well as women in science, and the development of geometry in the nineteenth century. She published more than 60 articles, chapters, and five books on the history of science, mathematical education, and pure mathematics. She is an associate editor of the *Quarterly Journal of the History of Science and Technology*, and was an associate editor of the *Antiquitates Mathematicae*. She is a member of the European Society for the History of Science and two Commissions of the Polish Academy of Art and Sciences in Kraków: History of Science and Biographical. In 2020 Ciesielska received the Samuel Dickstein Main Prize of the Polish Mathematical Society and now she chairs the Jury of the Prize.

Henning Heller is a postdoctoral researcher in the working group Mathematics Education at the University of Bonn. He studied mathematics in Jena (BSc) and philosophy of mathematics in Bristol (MA) and Vienna (PhD). He also holds a Belgian (Flanders) license for teaching mathematics at secondary schools (MSc). Heller is interested in bringing together history, philosophy, and education of mathematics. This combined interest naturally led to the figure of Felix Klein, who was the main actor of his PhD thesis, *Integrating History and Philosophy of Mathematics: The Case of Modern Algebra* (2022). In Bonn, Heller teaches at the Master program for mathematics teachers and researches on topics of history of mathematics education and university education.

Renate Tobies is a historian of mathematics and science at the Friedrich Schiller University in Jena (Germany). She had completed her academic degrees at the University of Leipzig. Throughout her career, she has held visiting professorships in Kaiserslautern, Braunschweig, Saarbrücken, Jena, and in Austria (Linz and Graz), and an interim professor of the history of science and technology at the University of Stuttgart. In parallel, she served as the managing editor of NTM: *Journal for the History of Science, Technology, and Medicine* for twenty years. Her research

focusses on the history of mathematics and its applications in the nineteenth and twentieth centuries as well as women in mathematics, science, and technology. She has published more than ten books and hundreds of articles. A second expanded edition of her book *Felix Klein: Visionen für Mathematik, Anwendungen und Unterricht* (German 2019, English 2021) was published in June 2025, Springer Spektrum). She is a full member of the Académie Internationale d'Histoire des Sciences (Paris) and a foreign member of the Agder Academy of Sciences and Letters (Kristiansand, Norway).

Joanna Zwierzyńska, MSc in Mathematics (University of Silesia in Katowice), a graduate of Postgraduate Studies in Editing (Faculty of Polish Philology, Jagiellonian University) and PhD studies in the History of Science (L. & A. Birkenmajer Institute of the History of Science, Polish Academy of Sciences), where she is preparing her doctoral thesis. Her research interests include the history of mathematics of the nineteenth, twentieth, and twenty-first centuries, the history of Polish mathematics, women in mathematics, editing, scientific correspondence, and biography. Co-author (with Danuta Ciesielska and Lech Maligranda) of the monograph *W świątyni nauki, mekce matematyków* about studies and research of Polish mathematicians, physicists, and astronomers at the University of Göttingen 1884–1933 (distinguished in the competition for the Jan Jędrzejewicz Prize for the best Polish book on the history of science 2021), co-editor (with Gabriela Besler) of the book *Gottlob Frege. Korespondencja naukowa*, author of scientific and popular science texts related to mathematics, history of mathematics, and editing. Member of the Polish Mathematical Society, European Mathematical Society, and European Society for the History of Science.

1 Felix Klein's Vision: A School for Mathematical Productivity Regardless of Nationality, Gender, and Area of Research

Renate Tobies

Abstract

In 1912, some 330 people donated for a portrait of Felix Klein by the important (Jewish) Impressionist painter Max Liebermann. The list of donors (Fig. 1.1, 1.2, 1.3, and 1.4) includes many foreign mathematicians (many of them were Jewish). Klein supported them regardless of their religion and nationality. Some of their personal connections with Klein go back to his time as a *Privatdozent* in Göttingen. This article considers Klein's impact on foreign mathematicians under the following aspects. Firstly, we look at Klein's aim to create a school of scientific productivity. We briefly compare Klein's way of leading young mathematicians to their own new results with the behaviour of other important German mathematicians in this respect. Secondly, we give an overview of the foreign mathematicians he inspired to write dissertations or make mathematical findings. Some relationships with mathematicians from Italy, Hungary, and Russia are examined in more detail. Thirdly, we investigate what influence Klein's interdisciplinary/joint research seminars had on the results of mathematicians from abroad who participated in these seminars in Göttingen.

1.1 A School for Mathematical Productivity

Felix Klein expressed his vision of founding a school of mathematical productivity early on. Shortly after taking up his first full professorship at the University of Erlangen in 1872 at the age of 23, he wrote to the French mathematician Gaston

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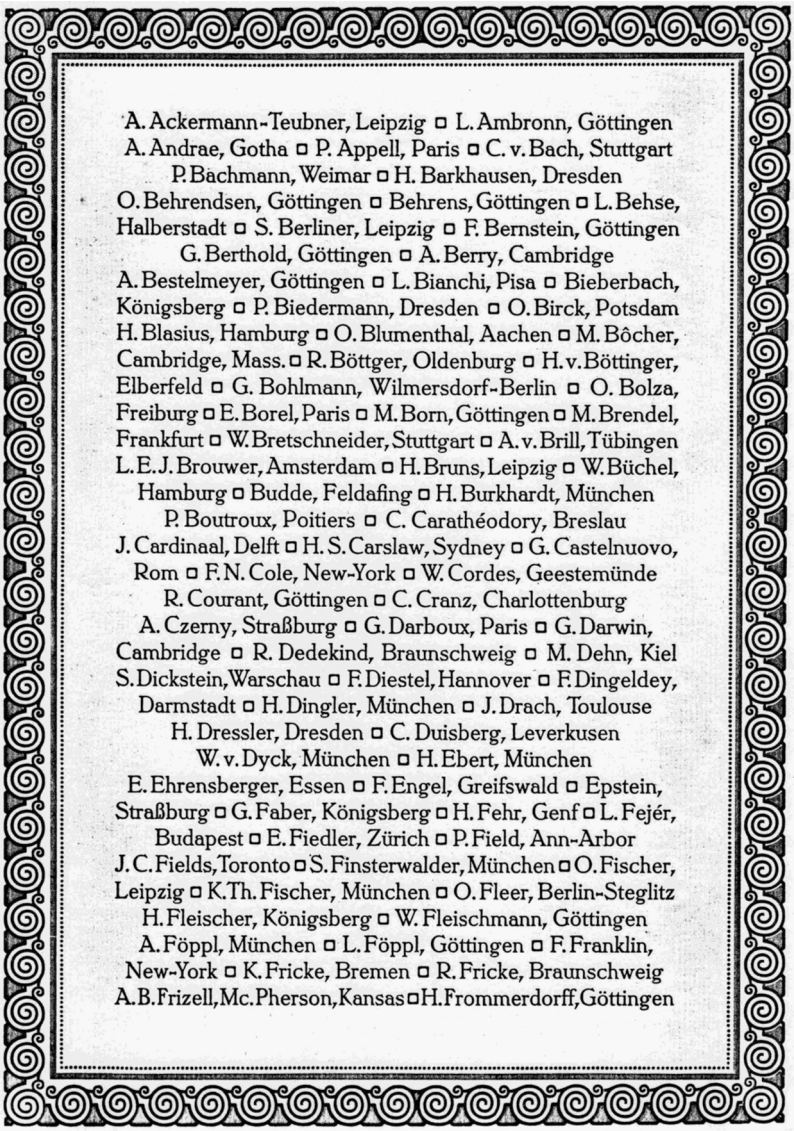
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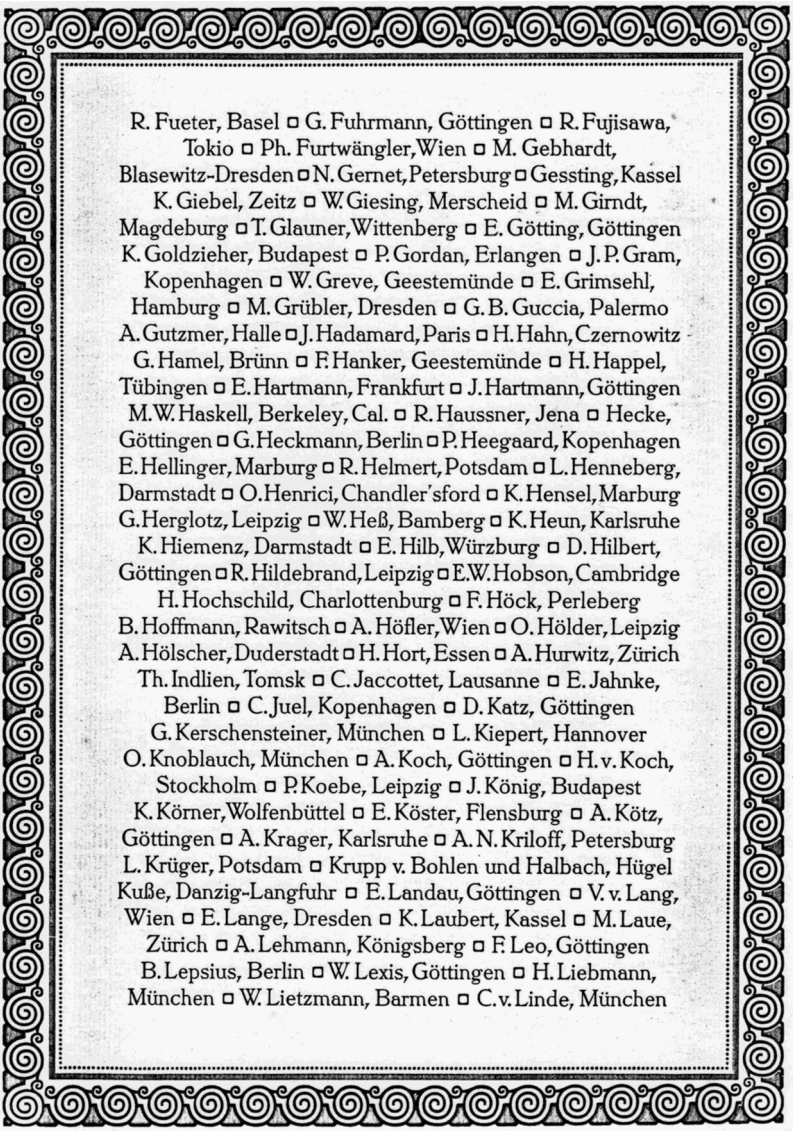
O. Behrendsen, Göttingen ◻ Behrens, Göttingen ◻ L. Behse,
Halberstadt ◻ S. Berliner, Leipzig ◻ F. Bernstein, Göttingen
G. Berthold, Göttingen ◻ A. Berry, Cambridge
A. Bestelmeyer, Göttingen ◻ L. Bianchi, Pisa ◻ Bieberbach,
Königsberg ◻ P. Biedermann, Dresden ◻ O. Birck, Potsdam
H. Blasius, Hamburg ◻ O. Blumenthal, Aachen ◻ M. Bôcher,
Cambridge, Mass. ◻ R. Böttger, Oldenburg ◻ H. v. Böttinger,
Elberfeld ◻ G. Bohlmann, Wilmersdorf-Berlin ◻ O. Bolza,
Freiburg ◻ E. Borel, Paris ◻ M. Born, Göttingen ◻ M. Brendel,
Frankfurt ◻ W. Bretschneider, Stuttgart ◻ A. v. Brill, Tübingen
L. E. J. Brouwer, Amsterdam ◻ H. Bruns, Leipzig ◻ W. Büchel,
Hamburg ◻ Budde, Feldafing ◻ H. Burkhardt, München
P. Boutroux, Poitiers ◻ C. Carathéodory, Breslau
J. Cardinaal, Delft ◻ H. S. Carslaw, Sydney ◻ G. Castelnuovo,
Rom ◻ F. N. Cole, New-York ◻ W. Cordes, Geestemünde
R. Courant, Göttingen ◻ C. Cranz, Charlottenburg
A. Czerny, Straßburg ◻ G. Darboux, Paris ◻ G. Darwin,
Cambridge ◻ R. Dedekind, Braunschweig ◻ M. Dehn, Kiel
S. Dickstein, Warschau ◻ F. Diestel, Hannover ◻ F. Dingeldey,
Darmstadt ◻ H. Dingler, München ◻ J. Drach, Toulouse
H. Dressler, Dresden ◻ C. Duisberg, Leverkusen
W. v. Dyck, München ◻ H. Ebert, München
E. Ehrensberger, Essen ◻ F. Engel, Greifswald ◻ Epstein,
Straßburg ◻ G. Faber, Königsberg ◻ H. Fehr, Genf ◻ L. Fejér,
Budapest ◻ E. Fiedler, Zürich ◻ P. Field, Ann-Arbor
J. C. Fields, Toronto ◻ S. Finsterwalder, München ◻ O. Fischer,
Leipzig ◻ K. Th. Fischer, München ◻ O. Fleer, Berlin-Steglitz
H. Fleischer, Königsberg ◻ W. Fleischmann, Göttingen
A. Föppl, München ◻ L. Föppl, Göttingen ◻ F. Franklin,
New-York ◻ K. Fricke, Bremen ◻ R. Fricke, Braunschweig
A. B. Frizell, Mc. Pherson, Kansas ◻ H. Frommerdorff, Göttingen

Fig. 1.1 A list of donors who sponsored Max Liebermann's painting of Klein's portrait in 1912 [Hillebrand]. H.Frommerdorff = Hermann Trommsdorff, teacher at the *Oberrealschule* in Göttingen (math/physics/chemistry), PhD University of Jena in 1901 [BBF]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Blasewitz-Dresden □ N. Gernet, Petersburg □ Gessting, Kassel
K. Giebel, Zeitz □ W. Giesing, Merscheid □ M. Girndt,
Magdeburg □ T. Glauner, Wittenberg □ E. Götting, Göttingen
K. Goldzieher, Budapest □ P. Gordan, Erlangen □ J. P. Gram,
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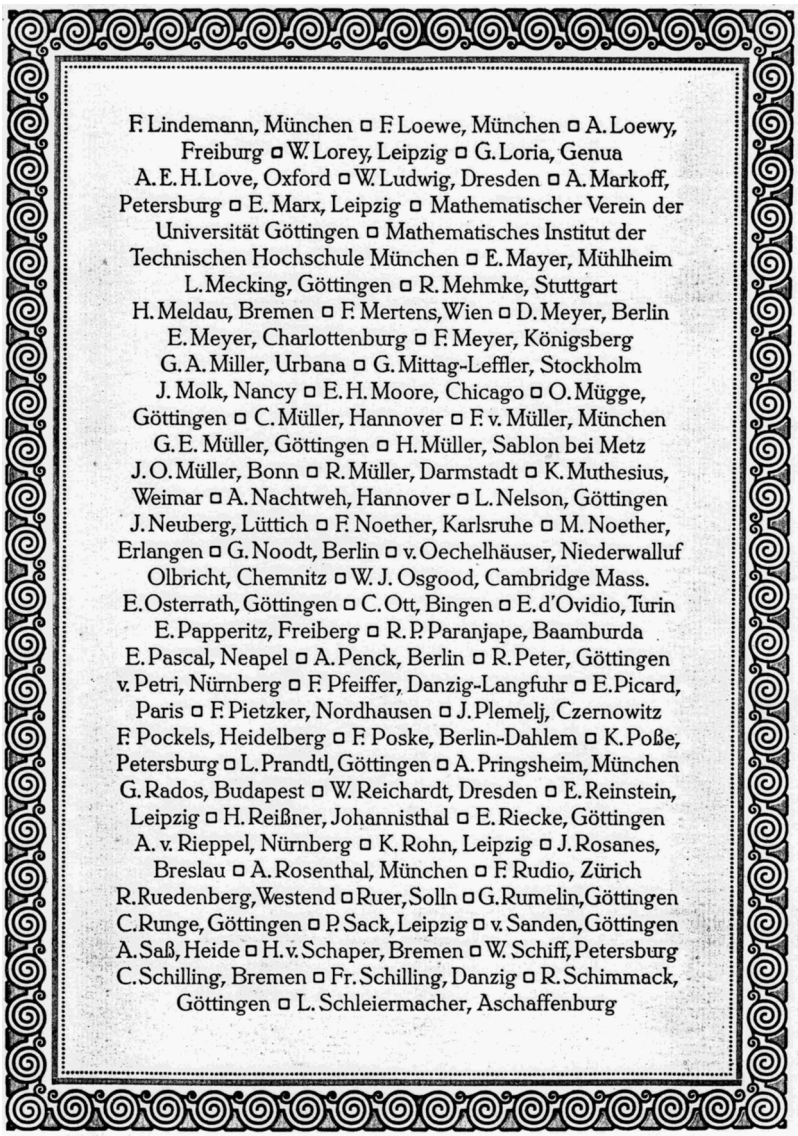
Hamburg □ M. Grübler, Dresden □ G. B. Guccia, Palermo
A. Gutzmer, Halle □ J. Hadamard, Paris □ H. Hahn, Czernowitz
G. Hamel, Brünn □ F. Hanker, Geestemünde □ H. Happel,
Tübingen □ E. Hartmann, Frankfurt □ J. Hartmann, Göttingen
M. W. Haskell, Berkeley, Cal. □ R. Haussner, Jena □ Hecke,
Göttingen □ G. Heckmann, Berlin □ P. Heegaard, Kopenhagen
E. Hellinger, Marburg □ R. Helmert, Potsdam □ L. Henneberg,
Darmstadt □ O. Henrici, Chandler'sford □ K. Hensel, Marburg
G. Herglotz, Leipzig □ W. Heß, Bamberg □ K. Heun, Karlsruhe
K. Hiemenz, Darmstadt □ E. Hilb, Würzburg □ D. Hilbert,
Göttingen □ R. Hildebrand, Leipzig □ E. W. Hobson, Cambridge
H. Hochschild, Charlottenburg □ F. Höck, Perleberg
B. Hoffmann, Rawitsch □ A. Höfler, Wien □ O. Hölder, Leipzig
A. Hölscher, Duderstadt □ H. Hort, Essen □ A. Hurwitz, Zürich
Th. Indlien, Tomsk □ C. Jaccottet, Lausanne □ E. Jahnke,
Berlin □ C. Juel, Kopenhagen □ D. Katz, Göttingen
G. Kerschensteiner, München □ L. Kiepert, Hannover
O. Knoblauch, München □ A. Koch, Göttingen □ H. v. Koch,
Stockholm □ P. Koebe, Leipzig □ J. König, Budapest
K. Körner, Wolfenbüttel □ E. Köster, Flensburg □ A. Kötz,
Göttingen □ A. Krager, Karlsruhe □ A. N. Kriloff, Petersburg
L. Krüger, Potsdam □ Krupp v. Bohlen und Halbach, Hügel
Kuße, Danzig-Langfuhr □ E. Landau, Göttingen □ V. v. Lang,
Wien □ E. Lange, Dresden □ K. Laubert, Kassel □ M. Laue,
Zürich □ A. Lehmann, Königsberg □ F. Leo, Göttingen
B. Lepsius, Berlin □ W. Lexis, Göttingen □ H. Liebmann,
München □ W. Lietzmann, Barmen □ C. v. Linde, München

Fig. 1.2 N.[adjeschda] v. Gernet had completed the Women's Courses in St Petersburg, studied under Felix Klein and David Hilbert, obtained her doctorate under the latter and was finally able to obtain a university position in St Petersburg. – Gessting = Gustav Gersting teacher examination (Göttingen 1906), teacher at the girl's secondary school in Kassel. – Goldzieher = Karl Goldziher. – "Th. Indlien" (Tomsk) = Klein's student Theodor Molien from Russia. – A. Krager (Karlsruhe) = Adolf Krazer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Freiburg ◻ W. Lorey, Leipzig ◻ G. Loria, Genua
A. E. H. Love, Oxford ◻ W. Ludwig, Dresden ◻ A. Markoff,
Petersburg ◻ E. Marx, Leipzig ◻ Mathematischer Verein der
Universität Göttingen ◻ Mathematisches Institut der
Technischen Hochschule München ◻ E. Mayer, Mühlheim
L. Mecking, Göttingen ◻ R. Mehmke, Stuttgart
H. Meldau, Bremen ◻ F. Mertens, Wien ◻ D. Meyer, Berlin
E. Meyer, Charlottenburg ◻ F. Meyer, Königsberg
G. A. Miller, Urbana ◻ G. Mittag-Leffler, Stockholm
J. Molk, Nancy ◻ E. H. Moore, Chicago ◻ O. Mügge,
Göttingen ◻ C. Müller, Hannover ◻ F. v. Müller, München
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J. O. Müller, Bonn ◻ R. Müller, Darmstadt ◻ K. Muthesius,
Weimar ◻ A. Nachtweh, Hannover ◻ L. Nelson, Göttingen
J. Neuberg, Lüttich ◻ F. Noether, Karlsruhe ◻ M. Noether,
Erlangen ◻ G. Noodt, Berlin ◻ v. Oechelhäuser, Niederwalluf
Olbricht, Chemnitz ◻ W. J. Osgood, Cambridge Mass.
E. Osterrath, Göttingen ◻ C. Ott, Bingen ◻ E. d'Ovidio, Turin
E. Papperitz, Freiberg ◻ R. P. Paranjape, Baamburda
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v. Petri, Nürnberg ◻ F. Pfeiffer, Danzig-Langfuhr ◻ E. Picard,
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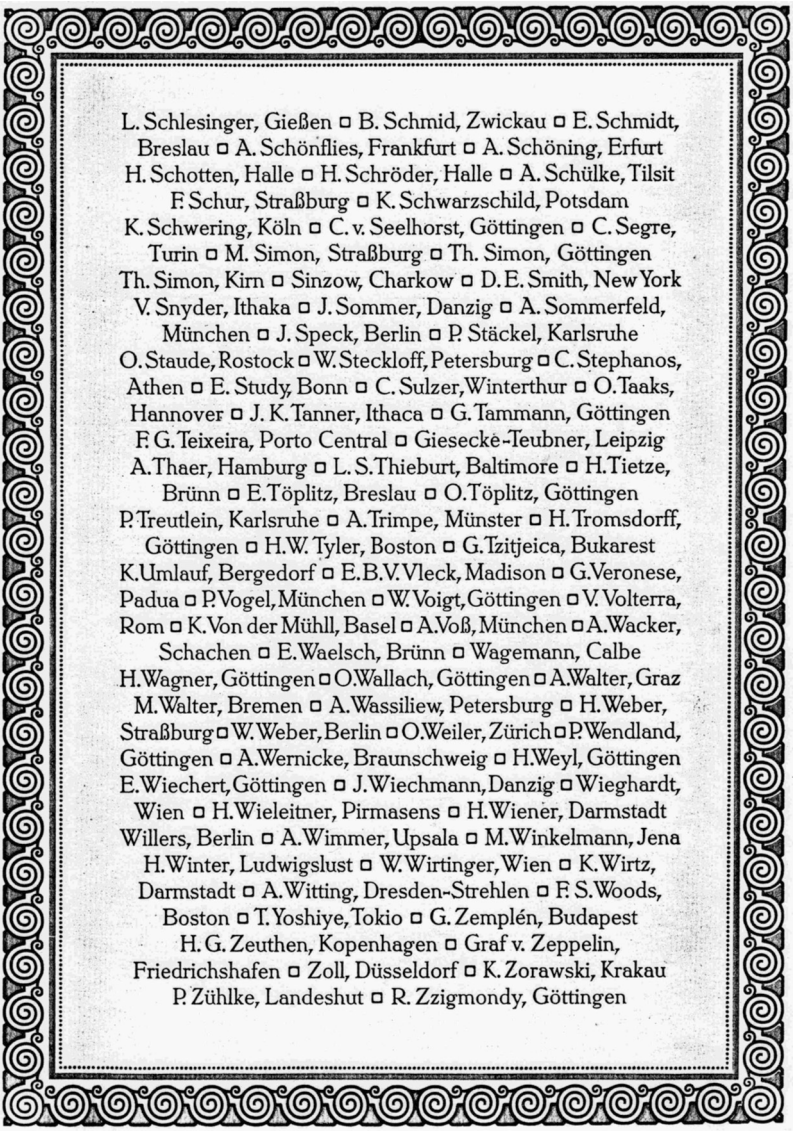
G. Rados, Budapest ◻ W. Reichardt, Dresden ◻ E. Reinstein,
Leipzig ◻ H. Reißner, Johannisthal ◻ E. Riecke, Göttingen
A. v. Rieppel, Nürnberg ◻ K. Rohn, Leipzig ◻ J. Rosanes,
Breslau ◻ A. Rosenthal, München ◻ F. Rudio, Zürich
R. Ruedenberg, Westend ◻ Ruer, Solln ◻ G. Rumelin, Göttingen
C. Runge, Göttingen ◻ P. Sack, Leipzig ◻ v. Sanden, Göttingen
A. Saß, Heide ◻ H. v. Schaper, Bremen ◻ W. Schiff, Petersburg
C. Schilling, Bremen ◻ Fr. Schilling, Danzig ◻ R. Schimmack,
Göttingen ◻ L. Schleiermacher, Aschaffenburg

Fig. 1.3 R.P. Paranjape = R.P. Paranjpye, Indian math. – R. Peter = Albert Peter, Prof. of Botany. – W. Schiff = Vera I. Schiff, (female) Prof. of Math. at the Women's Courses in St Petersburg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Hannover □ J. K. Tanner, Ithaca □ G. Tammann, Göttingen
F. G. Teixeira, Porto Central □ Gieseckè-Teubner, Leipzig
A. Thaer, Hamburg □ L. S. Thieburt, Baltimore □ H. Tietze,
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Straßburg □ W. Weber, Berlin □ O. Weiler, Zürich □ P. Wendland,
Göttingen □ A. Wernicke, Braunschweig □ H. Weyl, Göttingen
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Wien □ H. Wieleitner, Pirmasens □ H. Wiener, Darmstadt
Willers, Berlin □ A. Wimmer, Upsala □ M. Winkelmann, Jena
H. Winter, Ludwigslust □ W. Wirtinger, Wien □ K. Wirtz,
Darmstadt □ A. Witting, Dresden-Strehlen □ F. S. Woods,
Boston □ T. Yoshiye, Tokio □ G. Zemplén, Budapest
H. G. Zeuthen, Kopenhagen □ Graf v. Zeppelin,
Friedrichshafen □ Zoll, Düsseldorf □ K. Zorawski, Krakau
P. Zühlke, Landeshut □ R. Zzigmondy, Göttingen

Fig. 1.4 J.K. Tanner = John Henry Tanner, Cornell University – A. Trimpe = Aloys Timpe, PhD with Klein in 1905. – O. Weiler = Adolf Weiler, PhD with Klein in 1873, Swiss math. – "A. Wimmer, Upsala" = Anders Wiman, Uppsala

Darboux:

> Here in Erlangen I was at first very isolated, especially scientifically; my only interactions were with beginners. Since then, two older people have come here from Göttingen who work independently on geometry. I hope very much that this number increases next semester and that, in Erlangen, I will gradually succeed in establishing a school of geometric productivity such as that which had gotten off to such a great start in Göttingen while Clebsch was still there.[1]

As a *Privatdozent* (after his habilitation in January 1871) at the University of Göttingen, Klein worked closely with Alfred Clebsch and his students. Clebsch established a new form of collaboration, which Klein quickly adopted. We agree with this statement: "Clebsch belonged to a new type of mathematician: he was characterised by a lively pedagogical activity, a lively community with younger mathematicians, in a word: the founding of a school. [...] A circle of enthusiastic co-workers and students gathered around Clebsch: Gordan, Brill, Lüroth, Zeuthen, and the most famous: [Max] Noether and Klein."[2]

Klein's working style – orientated towards Clebsch – corresponded to his own inner urge to influence others and help them to make progress. This is well illustrated by a statement in a letter from Klein to the Norwegian mathematician Sophus Lie, his collaborator in the early 1870s: "Until now, there has been no mathematics in Bavaria, and the southern German student seems to be a lazy thinker. That said, my rather strong social drive – if this is understood as the desire to affect other people – will presumably be satisfied."[3]

As a *Privatdozent*, Klein had already supervised his first doctoral student, the German Joseph Diekmann, and sent his dissertation to Lie with the remark: "I helped the author a great deal. [...]."[4] In his last semester there, Klein also guided his first *foreign* doctoral student, the Swiss Adolf Weiler, in the direction of his research. Weiler had previously studied with Wilhelm Fiedler in Zurich, who became famous for the German editions of the textbooks on analytic, projective and algebraic geometry by the Irishman George Salmon; Fiedler incorporated new findings by Clebsch, Klein and others directly into his expanded editions. Klein wrote to Fiedler in July 1872: "This summer I am giving a two-hour lecture course on line geometry with the express intention of encouraging others to work in this direction. Mr Weiler, to whom you probably suggested to come here, is in the audience. He has outstanding geometric talent [...]."[5]

Due to Klein's subsequent surprise appointment to Erlangen in October of 1872 and Clebsch's early death (7 November 1872), Weiler completed his doctorate under

[1] [PARIS] 64: Klein to Darboux, November 29, 1872, emphasis original.

[2] Shafarevich (1983, 139–140) – See in detail Tobies (2021, 47–53).

[3] [OSLO] A letter from Klein to Lie dated August 3, 1872. – For the discrepancies that later emerged between Klein and Lie, see the recent analysis by Kay (2023).

[4] [OSLO] A letter from Klein to Lie dated July 29, 1871. – Cf. on this Tobies (2021, 107).

[5] Confalonieri et al. (2019, 92–93) (Klein's letter to W. Fiedler, dated July 20, 1872).

Klein at the University of Erlangen in 1873. Overall, Klein's efforts to create a productive geometric school in Erlangen were successful despite the small number of students. In the course of just five semesters there, he inspired not only the *Habilitation* of Aurel Voss, but also the doctoral theses of six mathematicians, four of whom went on to university careers, including Adolf Weiler, who eventually became a professor in Zurich.

It was typical of Klein to publicise the good results of his students in letters to colleagues and also in the journal *Mathematische Annalen* (*Math.Ann.* for short), for whose edition Klein (with Adolph Mayer) took over the main responsibility from 1876 onwards. For example, Klein informed Lie: "Two dissertations will soon be completed here: the first [=Weiler's] concerns my classification of second-degree complexes [...]."[6] Ten years later, when the Italians Corrado Segre and Gino Loria further developed the classification system to all possible cases of harmonic complexes, even discovering errors in Weiler's thesis, Klein decided to disseminate the results. He published them in "his" journal,[7] and remained on very good terms with these (and other) Italian mathematicians throughout his life.[8] Both, like Weiler, were among the donors of Liebermann's Klein portrait (see Figs. 1.3 and 1.4).

The aforementioned Aurel Voss attested the young Klein "a remarkable ability to identify in the works of others, the point that related specifically to his own ideas" and "a talent of directing each of his students to the topic that best suited the latter's particular gifts and stage of development."[9] Klein continued to support (and challenge) young mathematicians; he developed a method of intensive mentoring, good preparation of the seminar lectures, and regular discussions with each individual. The Austrian mathematician Wilhelm Wirtinger (see Fig. 1.4) reported on his semester in Göttingen:

> In the summer term of 1889 I went to Göttingen to Felix Klein. He read about abelian functions and partial differential equations in physics and involved the advanced students. We were Burkhardt, Haskell, Osgood, White and I in the seminar at the time. With Klein, I immediately found the richest encouragement and inspiration. He was extremely adept at specifying and processing the burgeoning ideas and devoted a great deal of his time and energy to periodic meetings with each individual.[10]
>
> Klein never hesitated to communicate his results, and he also openly explained the paths he took. [...] He introduced his students not only to the subject matter, but also to the art of

[6] [OSLO] A letter from Klein to Lie dated June 28, 1873. – Weiler, A. (1874). "Ueber die verschiedenen Gattungen der Complexe zweiten Grades." *Math.Ann.* 7, pp. 145–207.

[7] See Rowe (2016), Tobies (2021, 42).

[8] As professors in Turin, Segre and his former student Gino Fano had an impressive collection of Klein's writings, books and elaborated lecture courses in their personal library. See Chap. 5 by Erika Luciano in this book.

[9] Voss (1919, 286).

[10] [ADW WIEN] Wirtinger 1939: 5. – On Wirtinger's relationship to Klein, see Peter Ullrich's contribution, Chap. 8.

> disposition and to scientific expression in general. Klein has allowed us to experience the manifold tensions and their solutions in mathematical problems.[11]

Felix Klein, who travelled to Paris, Italy, Great Britain and the United States several times and was friends with many foreign colleagues, had a reputation for guiding young mathematicians to their own creativity. Sophus Lie, who knew early on of Klein's desire to guide, work with and help young talents, wrote to his Norwegian student Elling Holst in 1874 recommending that he go to Klein: "Believe me, there is nothing for you in *Berlin* unless you were to be as lucky as I and meet another Klein there."[12]

At the time, Berlin was dominated by the mathematicians Ernst Eduard Kummer, Karl Weierstraß and Leopold Kronecker.[13] They had very different ideas about working with young people. Kummer is said to have likened the request for a dissertation topic to "a young man asking him to suggest a pretty young girl with whom he could fall in love".[14] In the last year of his life, Kronecker wrote in a letter to Georg Cantor, the creator of set theory and initiator of the *Deutsche Mathematiker-Vereinigung* (DMV, German Mathematical Society, founded in 1890):

> We neither want nor need a school, because in our absolutely clear science every new discovery can render the previous school knowledge worthless. [...] We can therefore expect nothing beneficial from such joint work. On the contrary, such work can only hinder the progress of mathematics.[15]

Internationally, however, there was talk of Klein's mathematical school. More and more foreign colleagues were sending their own students to study with him. For example, the Russian mathematicians Andrey A. Markov [Markoff] (see Fig. 1.3) and Nikolay Y. Sonin spoke of Felix Klein's mathematical school in their proposal to elect him as a corresponding member of the St Petersburg Academy.[16] On 26 December 1895, Charles Hermite congratulated Klein on his election to the St Petersburg Academy of Sciences and informed him (via his translator Léonce Laugel) that the Paris Academy intended to do the same at the earliest opportunity. Klein was elected on 17 May 1897, taking the place of the late

[11] Wirtinger (1919, 287–288). Original quotation: "Klein hat niemals mit der Mitteilung seiner Ergebnisse gekargt und auch die Wege auf denen er vordrang, hat er immer offen dargelegt. [...] Er hat seine Schüler nicht bloß in den Gegenstand, sondern auch in die Kunst der Disposition und die wissenschaftliche Ausdrucksweise überhaupt eingeführt."

[12] Quoted from Stubhaug (2002, 236).

[13] See in more detail, also on the Weierstraß school of function theory, Tobies (2023, pp. 1–5).

[14] Runge (1949, 40).

[15] Kronecker, Leopold (1892). "Auszug aus einem Briefe von L. Kronecker an Herrn Prof. G. Cantor, vom 18. September 1891." *Jahresbericht der Deutschen-Mathematiker-Vereinigung* [DMV] 1, pp. 23–25. See also Tobies (2023, pp. 2–3).

[16] [Archiv St. Petersburg] Fond 2, 810, Bl. 16–17 (Elected in December 1895). – See also Tobies (2018).

James J. Sylvester.[17] The Hungarian Julius [Gyula] König informed Klein that the Magyar Tudományos Akadémia in Budapest had elected him a foreign member in accordance with his application.[18] During his lifetime, Klein became a member of fifty-one academies and societies. In some cases, these memberships were initiated by younger mathematicians who had studied under him, such as the aforementioned Wilhelm Wirtinger in Vienna or Ernesto Pascal in Naples.[19]

The list of donors for the Felix Klein portrait painted by Max Liebermann (Fig. 1.1 to 1.4) could be a kind of signpost for Klein's extensive network. There is also evidence of links as far afield as India and Australia. This list has already been included in the appendix of Tobies (2021). In the meantime, however, some misspelled names in this printed greeting address have been identified, which has led to further discoveries. We are including the list here, with the errors corrected.

1.2 Felix Klein's Foreign Students

Early in his career Klein made a name for himself by demonstrating the non-contradictory nature of non-Euclidean geometry, and by using group theory to systematise the various geometric directions (his *Erlangen Programme* of 1872). His attempt to combine methods from different mathematical disciplines (geometry, algebra, function theory, number theory) became a special feature of his research and at the same time enabled him to supervise doctoral students in several disciplines, including applied fields.

Many of his works have been translated into English, French, Italian and other languages. The influence of his *Erlangen Programme*, which has also been translated into Polish, Russian and Hungarian since the 1890s, was and still is far-reaching. It was also distributed in Spain and Argentina (see the contribution by A. Ramirez) and translated into Japanese still in 1970 (see H. Kümmerle).[20] The Greek mathematician Constantin Carathéodory (see Fig. 1.1), who studied under Klein and was to become his successor in Göttingen in 1913, wrote enthusiastically about the scope of Klein's *Erlangen Programme* as a classification principle for mechanics and general relativity.[21] Eilenberg and Mac Lane saw their category theory in the footsteps of the programme: "This may be regarded as a continuation

[17] "Nous sommes heureux d'avoir à vous annoncer que, dans la Séance de se jour, l'Académie vous a nommé à la place de Correspondant, devenue Vacante dans la Section de Géométrie, par suite du décès de M. Sylvester." ([UBG] Cod. MS. F. Klein 114: 25). – See in more detail in Tobies (2016).

[18] [UBG] Cod. MS. F. Klein 10: 531 (G. König's letter to Klein, dated 8 May 1899).

[19] Pascal (see Fig. 1.3) attended Klein's cours on Abelian functions in the winter semester 1888/89 and in the summer semester 1889, while Wirtinger only enrolled for one semester. [UBG] Cod. MS. F. Klein 7 E, pp. 161v, 162v). – For Pascal see Maria Giulia Lugaresi's section in this book.

[20] These results are missing in Rowe (2025), who presents a new translation of the *Erlangen programme* into English, combined with a selected insight into the history of its reception.

[21] Carathéodory (1919, 300) – In this book Christine Phili explores Klein's relations with Greek mathematicians, including C. Stéphanos' early attempt to publish Klein's programme in French translation.

of the Klein *Erlanger Programm*, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings."[22] It has also influenced philosophical systems. Schiemer (2020) described Klein's programme as an important origin of structuralism in the philosophy of mathematics. Ernst Cassirer, see Ihmig (1998); and Alfred North Whitehead are particularly worth mentioning.[23]

Let us stick to the historical order of some of the links with younger foreign students who studied with Klein. Even before the mentioned Elling Holst the Swede Victor Bäcklund, who already held a doctorate, came to Erlangen in 1874 on a six-month travelling scholarship. Klein published Bäcklund's work, and he managed to get Holst a Norwegian travelling scholarship to continue his studies with him at the Polytechnic in Munich.[24] Holst taught and researched mainly in the field of geometry and eventually got a position (docent) at the University of Oslo in 1894. Later, during Klein's time in Göttingen, other Scandinavians studied with Klein, for example the Norwegian Alf Guldberg in 1888–1889, the Dane Poul Heegaard in 1894, and Elizabeth Stephansen in 1902/03, the first Norwegian woman to obtain a doctorate in mathematics.[25]

Klein's basic courses in Munich were attended by many (engineering) students, including numerous from abroad, especially from Austria-Hungary. Apart from Holst, only four other foreigners, the German-speaking Bohemian Anton Puchta and the Czech Ludvík Kraus (see M. Bečvářová's section) and two Italians, attended Klein's advanced courses in Munich.

When Klein moved to the *University of Leipzig* in October 1880, he was joined for the first time by a Frenchman, Georges Brunel, a Briton, Arthur Buchheim,[26] and mathematicians from the USA, Russia, and other European countries. An overview of their involvement in Klein's courses and his inspiration can be found in Tobies (2021). Some aspects are explored in more detail in this book.

[22] Eilenberg and Mac Lane (1945, 237).

[23] Thanks to Michael Rahnfeld for analysing the Whitehead literature, see esp. Kraus (2018, 145).

[24] See Tobies (2021). Holst, E. (1877). "Ein paar allgemeine metrische Sätze für algebraische Curven." *Math.Ann.* 11, pp. 341–46. – For his further publications see Poggendorff (1898–1926), vol. III, p. 652; vol. IV.1, p. 660.

[25] Elizabeth Stephansen studied on a scholarship at Zurich Polytechnic (with Klein's student Hurwitz). After completing her studies, she got a job as a teacher in Norway and worked on a dissertation, with which she was able to complete her doctorate in absentia under Heinrich Burkhardt at the University of Zurich in 1902. Burkhardt had previously habilitated with Klein in Göttingen. See also Siegmund-Schultze (2024), and Tobies (2020), which lists all the women who attended Klein's courses.

[26] June Barrow-Green and Brigitte Stenhouse will analyse Klein's influence in Britain in this book.

Brunel's studies with Klein are of particular importance in connection with the correspondence between Felix Klein and Henri Poincaré[27] It should be especially emphasized that Darboux had recommended Brunel to Klein, who led him to his first publication, and that Brunel contacted Poincaré on Klein's behalf. Brunel was to help Poincaré understand Klein's results. However, in his letters to Poincaré, whom he did not know personally, Brunel made strong French nationalist statements.[28] This may have influenced Poincaré to be reluctant to share his own concrete methods with Klein. The collaboration Klein wanted (as he had them with many other mathematicians) turned into a kind of competition.

Parshall and Rowe (1994) provides an excellent study of Klein's impact in the United States. Nevertheless, there are still new aspects that have been discovered. For example, the list (Figs. 1.1 to 1.4) contains 14 US-Americans, including some not mentioned by Parshall & Rowe, such as Arthur Bowes Frizell and Peter Field, both of whom studied in Göttingen and completed their doctorates in the USA.[29] In Chap. 6 of this book, H. Heller gained new insights into the life and work of the American M.W. Haskell, who had already spent a semester with Klein in Leipzig (without enrolling in a course) before moving to the *University of Göttingen* with his teacher to complete his doctorate.

In Göttingen, Klein also established probability theory, actuarial mathematics and statistics as fields of teaching and research. In this book (Chap. 15), H. Fischer analyses the seminars Klein held to promote these fields.

We should also mention another dimension of Klein's "influence", namely the promotion of a collegial and scientific culture at foreign universities, modeled on what Klein's former students had previously experienced in Göttingen. Klein's model was important in various places in the world, e.g. in Italy (Turin, Naples

[27] See the most recent annotated edition (Nabonnand et al. 2024). Poincaré died on 17 July 1912 and is therefore not contained in the list of donors, but he belonged to the seventy-one people, who signed an appeal for donations for funds for the portrait in early 1912, see Tobies (2021, 613).

[28] See the analysis of the Darboux – Klein correspondence (Tobies 2016; Tobies 2021, 244–245), and the letters written by Brunel to Poincaré. "il ne faut pas croire pour cela que je me plaise beaucoup au milieu de ce peuple; plus j'apprends à le connaître et plus je le déteste." […] "Je dois dire cependant que relativement j'ai toujours trouvé que Mr Klein était le plus aimable et le plus obligeant. Enfin, quels qu'ils soient, il faut savoir profiter de ce qu'ils ont de bon; c'est toujours là ce que je me suis dit et c'est même pour cela que je suis en Allemagne. […] Je me mets tout entier à votre service. Français, notre devoir est de combattre les Allemands par tous les moyens possibles, mais loyalement." Brunel's letter to Poincaré, dated June 22, 1881. Poincaré, H., "La correspondance d'Henri Poincaré avec des mathématiciens de A à H." (1986), pp. 91–92. – Klein later managed to get Brunel to write an article for the Encyklopädie, "Bestimmte Integrale" [Definite integrals] (1899), in vol. 2.1.1, pp. 135–188.

[29] Both are listed in the *math genealogy project*. Frizell joined the German Mathematical Society in 1906 (see Toepell 1991, 115); after receiving his doctorate from the University of Kansas in 1910, he became a professor at the Nautical School in Kansas. – Peter Field was one of Klein's 'scientific grandsons'; his doctoral thesis (*On the Form of Unicursal Quintic Curves*, Cornell University, 1902) was supervised by Klein's student Virgil Snyder. In the years 1908–09 Field studied in Göttingen, where Klein inspired him to further results, published in *Math.Ann.*, vol. 67 and 69, and in *Zeitschrift f. Mathe. & Physik*, vol. 61, see Poggendorff, vol. V, p. 364.

esp.), in Russia (Moscow, Tomsk), Japan (Tohoku/Sendai), Budapest, Prague, in the United States, in India (Calcutta), Spain and Argentina, and also at Polish universities, especially after Poland gaines its independence. In Smyrna (then Greek), Carathéodory orientated himself on Klein's university model, which also provided for the promotion of East European / oriental languages.[30]

1.2.1 The Italians: Ricci-Curbastro, Bianchi, Veronese, and Others

Felix Klein had already established very good contacts with a number of important mathematicians since his trips to Italy in 1874 and 1878.[31] After Klein had visited Enrico Betti, Ulisse Dini and others in Pisa in April 1878, Gregorio Ricci-Curbastro came to Munich in the autumn on a scholarship. Ricci attended Klein's courses on number theory (1878/1879) and algebraic equations (1879), and participated in the seminars. There he gave two lectures: on "Certaines équations du degré 2^n", and on a paper by Henry Stephen Smith, which had been published in French in an Italian journal: "Les courbes modulaires: Rapport sur un mémoire de M. Stephen Smith communiqué à la Académie Royale des Lyncées, 1877".[32] Klein analysed the latest literature in the seminars and gave several talks on his own results. This increased Ricci's knowledge of the results of Riemann, Lipschitz and others. P. Speziali wrote: "Ricci greatly admired Klein, and his esteem was soon reciprocated [...]".[33]

When Ricci later developed tensor analysis with his student Tullio Levi-Civita, Klein requested that these results be published in *Math.Ann.*[34] Georg Pick would later refer Albert Einstein to the article by Ricci and Levi-Civita, when they were both professors in Prague (Pick had studied with Klein for two semesters and continued to work closely with him).[35] Although Einstein still needed help to grasp its significance and understand it mathematically, the work of Ricci and Levi-Civita provided the crucial foundation for his general theory of relativity.[36] These results would form the basis of Klein's own work on this topic, to which he also

[30] For Italy Chap. 5 by E. Luciano and Chap. 7 by M.G. Lugaresi, for Japan H. Kümmerle (Chap. 12), for Prague M. Bečvářová (Chap. 3) , for Greece Ch. Phili (Chap. 9), A. Ramírez Ogando for Spain and Argentina (Chap. 14); E. Mühlhausen on W.H. Young's attempt at the University of Calcutta (India, Chap. 10). Nicolas Michel looked at the conditions at the University of Chicago (see Report 2023, 2733–2736).

[31] See Tobies (2021, 153–158).

[32] [PROTOKOLLE] vol. 1.2, pp. 75–78 (Lecture, dated 28 July 1879).

[33] Speziali, Pierre. "Ricci-Curbastro, Gregorio." Dictionary of Scientific Biography.

[34] Ricci-Curbastro, Gregorio; Levi-Civita, Tullio (1901). "Méthodes de Calcul différentiel absolu et leurs applications." *Math.Ann.* 54, pp. 125–201. – For Ricci see also Poggendorff, Vol. III, p. 1118.

[35] Tobies (2023, 34).

[36] Klein (1927, 189–195, 205).

introduced Emmy Noether.[37] In 1899 Ricci had become a member of the DMV, which published a speech by him in German in its 1902 annual report, in which he drew a historical arc from Euclid to non-Euclidean geometry.[38]

Luigi Bianchi arrived in the summer term of 1879 and stayed for three semesters. Unlike Ricci at the time, he had an excellent command of German and enjoyed an intensive collaboration with Klein, who first asked him to talk about his doctoral thesis, and he invited him to write an article about it for the *Math.Ann.*[39] Klein included Bianchi in his field of research and recommended topics for five further lectures. Thus, Bianchi made contributions to Klein's level theory [*Stufentheorie*] and spoke about the tetrahedral irrationality (on 29 May and 5 June 1880) and about the icosahedral irrationality (on 13 June, 4 and 11 July 1880).[40] In Klein's work "Über unendlich viele Normalformen des elliptischen Integrals erster Gattung" [On Infinitely Many Normal Forms of the Elliptic Integral of the First Kind], which was first presented on July 3, 1880 at the Academy of Sciences in Munich and was an effort to expand his level theory with the help of doubly periodic functions, Klein acknowledged Bianchi's results as follows: "At my request, Mr. Bianchi recently investigated the fifth level, and what I will communicate below are essentially results discovered by him."[41] Bianchi expanded the proofs in an additional article, of whose beginning he thanked Klein "for his many suggestions and support in my work."[42] About the relationship between Bianchi's work and his own, Klein later explained:

> By treating, in the summer of 1880 (*Math.Ann.* 17), the elliptic curves that I would later call elliptic normal curves of the 3rd and 5th order by means of the σ function, he overcame my reservations about using this resource and that of theta series. He [Bianchi] thus did his best to build a bridge from my work to the developments of Weierstraß's school, particularly to my friend [Ludwig] Kiepert's studies, which were written around the same time."[43]

Conversely, Bianchi wrote to Klein: "I will never forget what you have done for me. And if I can be useful to you, remember that I am entirely at your disposal."[44]

Bianchi soon obtained a professorship in Pisa, where Ricci became his colleague; both – it is well known – developed methods (*Bianchi identities* for the Riemann tensor), which play an important role in the general theory of relativity. Bianchi

[37] See in detail in particular (Tollmien 2023, 273–283).

[38] Ricci, Gregorio (1902). "Anfänge und Entwicklung der neueren Auffassungen der Grundlagen der Geometrie" (held in Padua on 5 November 1901). *Jahresbericht der DMV* 11, pp. 382–403.

[39] Bianchi, L. (1880). "Ueber die Flächen mit constanter negativer Krümmung." *Math.Ann.* 16 (1880), pp. 577–82.

[40] Bianchi's seven lectures are recorded in [PROTOKOLLE] vol. 1.2, pp. 94–97; and vol. 2, pp. 6–10, 12–27.

[41] Klein (1921–1923), Vol. III, pp. 179–185, at p. 183.

[42] Luigi Bianchi, "Ueber die Normalformen dritter und fünfter Stufe des elliptischen Integrals erster Gattung," *Math.Ann.* 17 (1880), pp. 234–62, quotation p. 234.

[43] Klein (1921–1923), Vol. III, p. 6 – Ludwig Kiepert, a student of Weierstraß and a friend of Klein.

[44] [UBG] Cod. MS. F. Klein 8: 91 (Bianchi's letter to Klein dated August 14, 1880).

is famous for his work on differential geometry, about which he also wrote a comprehensive textbook *Lezioni di geometria differenziale* (Pisa 1894), which went through two German editions (1899, 1910) and was emphasised by Felix Klein.[45] Later, in February of 1924, Klein successfully arranged for Bianchi to be made a corresponding member of the Göttingen Academy of Sciences.[46]

The next Italian to join Klein was Giuseppe Veronese. As the son of poor parents, Veronese studied at Zurich *Polytechnic* from 1872 to 1876, where he was influenced by the above-mentioned Wilhelm Fiedler. Veronese then became Luigi Cremona's assistant in Rome. From there, he wrote to Klein in Munich on 1 May 1880, enclosing two notes of his own "Ueber einige bemerkenswerte Gruppen von Punkten, Geraden u. Ebenen, Kegelschnitten u. Flächen 2^{ten} Grades" [On some remarkable groups of points, straight lines and planes, conic sections and surfaces of the second degree], which were related to Klein and Lie's work on W-curves.[47] Veronese wished to study with Klein, who invited him to Leipzig, where he just had received the first professorship at a German university dedicated solely to geometry. Klein encouraged Veronese in his geometric interests, even if he himself was now more broadly orientated – as he informed Fiedler:

> [...] I have become somewhat alienated from geometry itself. I may always have been a geometer at the bottom of my mathematical thinking. One has been explicitly entrusted with the representation of geometry in Leipzig, as you probably know. For this reason I do not think I can change my present affairs, which, out of a certain inner necessity, are gradually embracing wider areas of abstract mathematics. But I will always be forced to give lectures on purely geometrical topics.[48]

Klein first asked Veronese for an overview of his previous work on projective configurations, which he handwrote in the protocol book under the title „Ueber einige merkwürdige Configurationen" [On some unusual configurations].[49] In the following summer semester of 1881, Veronese gave a presentation entitled "Ueber die Darstellende Geometrie im Raume von 4 Dimensionen" [On descriptive geometry in the space of 4 dimensions] (25 April 1881).[50] Klein, who quickly thought his way into other areas, wrote to Fiedler: "You may know that G. Veronese has been here since the beginning of winter. I am currently busy discussing a paper with him that he has delivered on the synthetic geometry of n-dimensional

[45] See Klein (1927, 148) – Bianchi also refers to Klein's results in his textbook.

[46] See Tobies (2021, 585).

[47] [UBG] Cod. MS. F. Klein 12: 42 (Veronese's letter to Klein, dated May 1, 1880). – For the works on W-curves see Tobies (2021, 78–80).

[48] Confalonieri et al. (2019, 111) (Klein's letter to W. Fiedler, dated July 30, 1880).

[49] [PROTOKOLLE] vol. 2, pp. 80–92 (Veronese's lecture on 3 January 1881).

[50] [PROTOKOLLE] vol. 3, pp. 1–13 (Lecture by Veronese on 25 April 1881 – Felix Klein's 32nd birthday).

space."[51] This work was published in the *Math.Ann.*, in which Veronese expressly thanked "Prof. Klein for his many suggestions and support during my mathematical studies in Leipzig".[52] After his studies in Leipzig, Veronese was appointed professor of descriptive geometry in Padua. He became known for his non-Archimedean geometry (1890), the consistency of which was doubted by Giuseppe Peano but recognised by David Hilbert in his *Grundlagen der Geometrie* (1899). Klein also appreciated Veronese's theory, but raised the question of whether it was appropriate [*zweckmäßig*] to deal with it.[53]

Two other Italian mathematicians studied with Klein in Leipzig; Francesco Gerbaldi came for the summer semester of 1883, and Giacinto Morera for two semesters from autumn 1883. They took part in Klein's research seminars, which at the time were dedicated to the theory of elliptic functions. Klein promoted publications by both of them.[54] Gerbaldi became a member of the German Mathematical Society (DMV) in 1897.

Klein's relations with Italian mathematicians, especially with representatives of algebraic geometry in Turin, were close and are generally well analysed.[55] Erika Luciano and C.S. Roero stressed, that Klein, in particular, was a "reference interlocutor" for Segre, Gino Fano, Federigo Enriques and Guido Castelnuovo.[56] Klein exerted a notable influence on mathematical studies, on the teaching of mathematics, on publishing activity, and on the organization of cultural and academic life in Turin. It was there that he received his first honorary doctorate in 1880 (out of a total of ten). Klein recruited the aforementioned Italians as authors for his major project, the *Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen* (Leipzig: B.G. Teubner, 1898–1935), Encyklopädie for short; a total of 16 contributions were written by eleven Italian scientists, most of them for Vol. 3 (Geometry). Ten Italians honoured Klein by supporting the portrait in 1912 (Fig. 1.1, 1.3, and 1.4).

In Göttingen, Alfonso Sella, later professor of physics in Rome, attended Klein's lectures on potential theory (in 1888) and partial differential equations in physics

[51] Confalonieri et al. (2019, 120) (Klein's letter to W. Fiedler, dated June 10, 1881).

[52] Veronese, G. (1881). "Die Anzahl der unabhängigen Gleichungen, die zwischen den allgemeinen Charakteren einer Curve im Raume von *n* Dimensionen stattfinden." *Math.Ann.* 18, p. 448; and Veronese, G. (1882). "Behandlung der projectivischen Verhältnisse der Räume von verschiedenen Dimensionen durch das Princip des Projicirens und Schneidens." *Math.Ann.* 19, pp. 161–234; acknowledgement, p. 162.

[53] Klein (1928, 156–157) English ed. (2016, p. 170). – Veronese's non-Archimedean geometry was included in his book *Grundzüge der Geometrie von mehreren Dimensionen und mehreren Arten gradliniger Einheiten in elementarer Form entwickelt* (Leipzig: B.G. Teubner, 1894; Italian Original: Padova 1891).

[54] The main aspects of their relationship with Klein are presented in Tobies (2021, 247–248) – Both otherwise only published in Italian journals, see Poggendorff, Vol. III, IV, and V.

[55] See in particular (Coen 2012; Casnati et al. 2016), and Erika Luciano's section in this book.

[56] Luciano, E.; Roero, C.S. (2016). "Corrado Segre and His Disciples: the construction of an International Identidy for the Italian School of Algebraic Geometry," in Casnati et al., 93–241.

(1888/1889 and summer 1889), as well as lectures by the theoretical physicist Woldemar Voigt. The above-mentioned Pascal was enrolled for the Abel functions for two terms (see M.G. Lugaresi's section). Gino Fano, who had translated Klein's *Erlangen Programme* (inspired by Segre), arrived in autumn 1893 for two semesters.[57] His seminar talk shows Klein's further turn to approximation theory and applications in physics.[58] Fano became even a serious candidate for Klein, whom he proposed for a professorship in Göttingen.[59]

1.2.2 Hungarian Mathematicians with Klein: Rados and Beke

The five Hungarian mathematicians listed in Fig. 1.1, 1.2, 1.3, and 1.4 all held positions in Budapest. Among them is the above mentioned Julius [Gyula] König, who was the same age as Felix Klein. As early as Easter 1873, he and Moritz [Mór] Réthy had attended the mathematicians' meeting in Göttingen, which was initiated by Alfred Clebsch and organised by Klein.[60] J. König was introduced to mathematics by Leo Königsberger in Heidelberg. In 1874, König was appointed full professor at the Royal Hungarian Technical (Joseph's) University in Budapest, which is famous for its leading role in mathematics. This Technical University was later known as the "Göttingen of the [Austro-Hungarian] Monarchy". Réthy, who had also studied in Germany (Göttingen, Heidelberg), became the first Hungarian professor of theoretical physics (in Klausenburg). In 1886 he moved to Budapest as professor of geometry and theoretical physics at the same university as Julius König. Réthy and J. König represented important fields (elliptic functions, non-Euclidean geometry – Bolyai – , hydrodynamics etc.), which had many connections to Klein's work, and both joined the DMV in 1893.[61] No wonder that a number of their students and other younger Hungarians found their way to Leipzig and Göttingen. In the order of their study under Klein, these include: Gusztáv Rados (Raussnitz) in 1884/1585 and 1885; Emanuel [Manó] Beke in 1893/1894; Ludwig [Lajos] Steiner in 1895; Karl [Károly] Goldziher in 1900/1901 and 1901; Leopold (Lipót)

[57] Klein, F. (1890). "Considerazioni comparative intorno a ricerche geometriche recenti." *Annali di matematica pura ed applicata* 17, pp. 307–43.

[58] [PROTOKOLLE] vol. 12, pp. 33–38 (G. Fano: "Allgemeine Bemerkungen über Fourier'sche Reihen (Vgl. Klein. Phys. Diffgl. II)". Fano gave this lecture on 13 and 20 June 1894. – In 1898/99, Adolfo Viterbi spoke in Klein's seminar; he was to become a professor of theoretical geodesy at the University of Pavia.

[59] Tobies (2021, 447).

[60] See ibid, pp. 117–121.

[61] Klein felt responsible for the DMV from the outset, was president three times and also invited many foreign colleagues and former doctoral students to become members (see in more detail (Tobies 2021, 367–372)).

Fejér in 1902/1903; Friedrich [Frigyes] Riesz in 1903/1904;[62] Győző Zemplén in 1904/1905; Alfréd Haar 1905/1906, 1906, 1907; 1909; Theodore von Kármán in 1906/1907, 1907/1908, 1908/1909; 1909; Dénes [Dionys] Kőnig – a son of Julius [Gyula] König – from 1904/1905 to 1907; Otto Szász and Aladár Visnya both in 1907/1908 and 1908.[63]

For the time being, we will only refer to Rados and Beke. The others worked mainly in Klein's interdisciplinary seminars (see point 3).

In the winter term of 1884/1885, Gusztáv Rados (Raussnitz until 1884/1885) attended Klein's courses without giving a lecture. In the summer of 1885, Klein treated the theory of algebraic functions of one variable in his seminar. He began by explaining that Riemann's theory can essentially be assumed to be known. His aim was to compare recent work with a geometric treatment of function theory with work based on an arithmetic approach.[64] With an already strong background in number theory from his studies with Julius König in Budapest, Rados spoke here on the work of Richard Dedekind and Heinrich Weber "Theorie der algebraischen Functionen einer Veränderlichen" [Theory of Algebraic Functions of One Variable][65] and on the content and methods of Kronecker's Festschrift, which was titled *Über den Zahlbegriff* [On the Concept of Number].[66] Rados analysed the latter book especially thoroughly, thereby demonstrating that his attitude toward research was closely aligned to Klein's own. He criticized Kronecker for limiting his focus to "the development of the characteristics [*Eigenschaften*] of rational whole numbers and functions without using methods from other areas of mathematics." Very much in the spirit of Klein, Rados promised, that he would use "all existing tools available to us." In his presentation, he outlined the fundamental concepts of Kronecker's work and praised the "algebraic materialisation [*Materialisirung*] of Kummer's ideal numbers or Dedekind's ideals," but also discussed other approaches to the topic.[67]

At the same time, Klein corresponded intensively with the aforementioned Georg Pick, whom he had persuaded to prepare Klein's work on elliptic module functions for the planned monograph.[68] There were still many unresolved issues on the topic.

[62] F. Riesz enrolled in the seminar in 1903/1904 ([UBG] Cod. MS. F. Klein 7 E, 219v); he did not give a lecture here, but he talked about his own new findings at the Göttingen Mathematical Society.

[63] See *Amtliche Verzeichnisse des Personals und der Studierenden der (Königlichen) Georg-August-Universität zu Göttingen*; [UBG] Cod. MS. F. Klein 7 E; and [PROTOKOLLE].

[64] [PROTOKOLLE] vol. 7, pp. 1–2 (Klein's introductory lecture, dated April 27 1885).

[65] Published in *Journal für die reine und angewandte Mathematik* 92 (1882), pp. 181–290.

[66] Kronecker's contribution to a *Festschrift* (*Philosophische Aufsätze*. Leipzig: Fues, 1887) for the philosopher Eduard Zeller must have been available to the seminar before publication. An extended version appeared in *Journal für die reine und angewandte Mathematik* 101 (1887), pp. 337–55.

[67] [PROTOKOLLE] vol. 7, pp. 27–50, 125–50 (lectures given on 1 June, 20 July, 27 July, and 3 August 3 1885). See also Tobies (2021, 249–250).

[68] The final product was Felix Klein and Robert Fricke: *Vorlesungen über die Elliptischen Modulfunctionen*. 2 vols. Teubner, Leipzig. Engl. trans. Arthur M. Dupre, *Lectures on the Theory of Elliptic Functions*. Beijing: Higher Education Press, 2017.

It is therefore not surprising that Klein inspired Rados and Pick to tackle a problem almost simultaneously.[69] Rados published his findings in Budapest in 1885.[70] From 1885 Rados held positions (as a full professor in 1893) at the same technical university as Julius König and Réthy. Klein and Rados remained in contact. From 1892 Rados was in charge of the Hungarian journal *Mathematikai és Physikai Lapok*. His first paper in the *Math.Ann.* appeared 1897.[71] When the Hungarian Academy of Sciences established the János Bolyai Prize (10,000 *Kronen*), Felix Klein and Gaston Darboux were the only two external members who were appointed to the prize committee in 1905, which also included Julius [Gulya] König and Gusztáv Rados.[72]

After his studies in Budapest, Emanuel [Manó] Beke became a mathematics teacher; he did his doctorate in 1884. He came to Göttingen on a one-semester scholarship in 1893/1894. Beke also joined the Göttingen Mathematical Society, which Klein had founded in 1892. In Klein's seminar on the theory of linear differential equations, Beke presented new results on homogeneous linear differential equations, which Klein immediately accepted for the *Math.Ann.*[73] In the same semester, women were first allowed to attend Klein's courses,[74] and Beke was enthusiastic about Klein's commitment to promoting women mathematicians and training secondary school teachers.[75] Back in Budapest, he was appointed professor at the College of Education (*Pädagogische Hochschule*) in 1895 and also taught at the first girls' secondary school there. In 1900, he became professor at the University of Budapest.

Beke and Klein later worked closely together as part of the International Commission for Mathematical Instruction (IMUK, today ICMI) founded in Rome in 1908. Klein was unable to travel to Rome, but was elected President of the IMUK in his absence. Beke said in his presentation in Rome: "By and large, we are all influenced by the movement that was inaugurated by Prof. Klein and further promoted by the German Commission."[76]

[69] See Tobies (2023, pp. 94–96), (Georg Pick's letters to Klein, dated 30 April, and 3 May 1885).

[70] Rados, G. (1885). "Az algebrai függvények elméletéhez" [On the Theory of algebraic functions]. *Matematikai és Természettudományi Értesíto* 3, pp. 185–87.

[71] Rados, G. (1897). "Theorie der adjungirten Substitutionen". *Math.Ann.* 48, pp. 417–24. See also Poggendorff, Vol. IV, p. 1206.

[72] See on this also Tobies (2016), and Tobies (2021, 250).

[73] [PROTOKOLLE] vol. 11, pp. 302–16 (Beke's lectures on January 10, 17 and 24, 1894). – Beke, E. (1894). "Die Irreducibilität der homogenen linearen Differentialgleichungen". *Math.Ann.* 45, pp. 278–94; and „Die symmetrischen Functionen bei den linearen homogenen Differentialgleichungen." Ibid., 295–300, elaborated in January and March 1894 in Göttingen.

[74] In 1893/94, the American Mary F. Winston and the Englishwoman Grace E. Chisholm, who completed their doctorates under Klein, began their work.

[75] [UBG] Cod. MS. F. Klein 8: 76A, 76B/1 (Beke's letters to Klein, dated March 8 and August 21, 1895).

[76] Quoted in Tobies (2021, 493) – For the history of the ICMI, see also (Furinghetti and Giacardi 2022).

1.2.3 Students and Elderly Mathematicians from Russia

Seven of the nine mathematicians from the Tsarist Russia[77] on the donators list (Figs. 1.1 to 1.4) are named with the location "Petersburg", including the two women. This is of course an expression of Klein's special relationships there, but it does not characterise the diversity of his contacts.

Klein's first student from Tsarist Russia, Theodor Molien, arrived in Leipzig in the autumn of 1883 after studying at the then German-speaking University of Dorpat (now Tartu, Estonia).[78] He stayed for three semesters, was twice a speaker at the seminars and was led to publishable results in Klein's research field at the time, the theory of elliptic functions.[79] With these results Molien was able to progress to the next academic level (Master's) at Dorpat, as he wrote to Klein.[80] Molien's most important paper "Ueber Systeme höherer komplexer Zahlen" [On systems of higher complex numbers] appeared in *Math.Ann.* [41 (1893), pp. 83–156], and earned him a doctorate from Dorpat. In 1900, he was appointed professor at the Technological Institute in Tomsk (from 1917 the Faculty of Physics and Mathematics of the University). Molien was the first professor of mathematics in Siberia and organised teaching, research and a library following the example of Klein.[81]

In addition to Molien, three older mathematicians from what was then Russia visited Klein in Leipzig, including Kiev-born Matvey A. Tichomandrickij and Kazan-born Alexander V. Vasilev [Wassiliew], both of whom had studied in St Petersburg and whose efforts were to contribute significantly to the dissemination of Klein's findings.

Tichomandrickij had already submitted a note (On the Converse Problem of Elliptic Integrals) for publication in *Math.Ann.* – before arriving in Leipzig. After giving a lecture in Klein's seminar (21 July 1884), a second note was published, which is still cited in modern scholarship.[82] In 1883, Tichomandrickij had just been appointed as a *Dozent* at the University Kharkov [Charkow] in 1883. He had a broad knowledge of Eastern European mathematical institutions. At Klein's request, Tichomandrickij sent him a multi-page list of researchers, institutions, main

[77] Until 1917, Tsarist Russia included Finland and parts of Poland, from where students also came. The Finn Alexander Leonard Hjelmman, for example, spoke at Klein's seminar in 1897/98 and became a professor in Helsinki. D. Ciesielska (Chap. 2) and J. Zwierzyńksa (Chap. 11) examine the links between Polish mathematicians and Klein in our book.

[78] See Siilivask (1985).

[79] Molien, Theodor (1885). "Ueber gewisse, in der Theorie der elliptischen Functionen auftretende Einheitswurzeln." *Berichte der Kgl. Sächsischen Gesellschaft der Wissenschaften zu Leipzig* 37, pp. 25–38 (submitted by Klein, 12 January 1885). – Ueber die lineare Transformation der elliptischen Functionen [On the linear transformation of elliptic functions] 24p., Dorpat 1885 (Master thesis).

[80] [UBG] Cod. MS. F. Klein 10: 1283 (Molien's letter to Klein, 14/26 December 1885).

[81] Molien's name is misspelled in Fig. 1.2, and I identified him only late. See Kanunow, N.F. (1983). *Fjodor Eduardowitsch Molin 1861–1941*. Moskwa: Isdatelstwo Nauka (Russ.).

[82] See Tobies (2021, 250–251), [PROTOKOLLE] vol. 6, 121–26; *Math.Ann.* 22 (1883), pp. 450–54; *Math.Ann.* 25 (1885), pp. 197–202.