Mathematics Education in the Digital Era

Sergei Abramovich

Computational Experiment Approach to Advanced Secondary Mathematics Curriculum

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Preface

This book stems from the author's interest in and experience of using computing technology in K-12 mathematics teacher education. In particular, two decades of teaching prospective secondary mathematics teachers in technology-rich environments have led the author to believe that one of the most productive ways to teach mathematics in the digital era is through experimentation with mathematical concepts that takes advantage of computers' capability to plot sophisticated graphs, construct dynamic geometric shapes, generate interactive arrays of numbers, and perform symbolic computations. Whereas the notion of experiment in mathematics has several meanings (Baker 2008; Van Bendegem 1998), assigning the adjective *computational* to the word *experiment* implies that the meaning of the latter is narrowed down to the use of electronic computers as the means of mathematical experimentation.

The book utilizes a number of commonly available computer applications that allow for lucid presentation of advanced, though grade-appropriate, mathematical ideas. One application is the Graphing Calculator 4.0 produced by Pacific Tech (Avitzur 2001) that facilitates experimentation in algebra through the software's capability of constructing graphs from any two-variable equation, inequality, or a combination of those. Another application is an electronic spreadsheet used to support numerical experimentation, in particular, when carrying out probability simulations and modeling elementary number theory concepts. The book also takes advantage of Maple (Char et al. 1991) and Wolfram Alpha developed by Wolfram Research—software tools that allow for different types of experimentation with mathematical concepts, including the construction of graphs of functions and relations and carrying out complicated symbolic computations. Also, the book uses The Geometer's Sketchpad (GSP) created by Nicholas Jackiw in the late 1980s. Yet this dynamic geometry program is used more as a technical tool rather than as an experimental device. This lesser focus on experimentation with GSP is due to the book's stronger focus on algebra in comparison to geometry. Nevertheless, the idea of geometrization of algebraic concepts is one of the major mathematical ideas used in the book.

Throughout the book, a number of the modern day secondary mathematics education documents developed throughout the world are reviewed as appropriate.

These include Common Core State Standards (2010) for mathematical practice and the Conference Board of the Mathematical Sciences[1] (2001, 2012) recommendations for the preparation of mathematics teachers—the United States (the context in which the author prepares teachers); National mathematics curriculum (National Curriculum Board 2008)—Australia; Ontario mathematics curriculum (Ontario Ministry of Education 2005) and British Columbia mathematics curriculum (Western and Northern Canadian Protocol 2008)—Canada; Programs of mathematics study (Department for Education 2013a, b)—England; A secondary school teaching guide for the study of mathematics (Takahashi et al. 2006)—Japan; and Secondary mathematics syllabi (Ministry of Education, Singapore 2006)—Singapore.

The book consists of eight chapters. The first chapter provides theoretical underpinning of computational experiment approach to advanced secondary mathematics curriculum. In the focus are mathematics education research publications that started appearing in the second half of the twentieth century with the advent of computers as tools for the teaching of mathematics. The role of mathematics education reform in bringing computers first to the undergraduate level and gradually extending their use to include experimentation at the primary level is highlighted. Several theoretical frameworks leading to the development of the notion of technology-enabled mathematics pedagogy referred to as TEMP throughout the book are discussed. It is suggested that TEMP can become a major pillar of modern signature pedagogy of mathematics as it can focus on the unity of computational experiment and formal mathematical demonstration. The relationship between technology-enabled experiment and solution-enabled experiment is introduced as a structure that makes computational experiment a meaning making process. It is shown how visual imagery can support deductive reasoning leading to an error-free computational experiment.

One of the major differences between TEMP and a mathematics pedagogy (MP) that does not incorporate technology pertains to the interplay between mathematical content under study and the scope of student population to which this content can be made available. Whereas many problems discussed in the book under the umbrella of TEMP are fairly complex, using technology as a support system makes it possible to develop mathematical insight, facilitate conjecturing, and illuminate plausible problem-solving approaches to those problems. To a certain extent, the use of TEMP may be comparable to the use of computers in the modern day investigation of dynamical systems allowing one to carry out numeric/symbolic computations and graphical constructions not possible otherwise, yet being critical for understanding the behavior of those systems. In comparison with MP which, in particular, lacks empirical support for conjectures, using TEMP has great potential to engage a much broader student population in significant mathematical explorations. TEMP provides teachers with tools and ideas conducive to engaging students in the project-based, exploratory learning of mathematics by dividing a project in several stages—empirical, speculative, formal, and reflective. Even if TEMP helps

[1] The Conference Board of the Mathematical Sciences is an umbrella organization consisting of 16 professional societies in the United States.

a student to reach the level of conjecturing without being able to proceed to the next level, its use appears to be justified.

The second chapter is devoted to the use of computational experiment as support system for solving one-variable equations and inequalities. The importance of revealing the meaning of problem-solving techniques frequently considered as "tricks" by students and their teachers alike is emphasized. It is demonstrated how technology can be used as an agency for mathematical activities associated with formal demonstration of theoretical concepts that underpin commonly used algebraic techniques. The idea of parameterization of one-variable equations is explained through the lens of problem posing in the digital era, an approach conducive to the development of "mathematical reasoning and competence in solving increasingly sophisticated problems" (Department for Education 2013b, p. 1). Also, the chapter shows how through the integration of technology and historical perspectives the secondary mathematics content can be connected to its historical roots.

The third chapter is devoted to the study of quadratic equations and functions with parameters. Here, two methods of exploration made possible by computational experiment are discussed. One method deals with the possibility of transition from the traditional (x, y)-plane typically used to construct the graphs of functions to the (variable, parameter)-plane commonly referred to as the phase plane. Using the diagrams (loci) constructed in the phase plane, one can discern the most important information about a quadratic equation with a parameter, namely, the influence of the parameter on the solutions (roots) of the equation. Another method, in the case of equations with two parameters, deals with the qualitative study of solutions in the plane of parameters. It demonstrates how one can make a transition from representations in the plane of variables to representations in the plane of parameters when investigating the properties of quadratic functions and associated equations depending on parameters. In particular, qualitative methods for deciding the location of roots of quadratic equations with parameters about a point as well as about an interval are discussed. These methods make it possible to determine the location of the roots without finding their exact values. The need for such methods proved to be very useful in the context of the "S" and "E" components of STEM (science, technology, engineering, mathematics) where qualitative techniques are commonly used in exploring the corresponding mathematical models.

The fourth chapter is devoted to the systematic study of algebraic equations with parameters (including simultaneous equations) using the computationally supported locus approach when explorations take place in the (variable, parameter)-plane. Here, the computational experiment approach is applied to "make use of structure" (Common Core State Standards 2010) of a complicated mathematical situation and to develop its deep understanding by using locus as a thinking device. In doing so, one can come across various extensions of the situation to include new concepts, representations, and lines of reasoning that connect different grade appropriate mathematical ideas. The notion of collateral learning in the spirit of Dewey (1938) is highlighted. In the case of two-variable simultaneous equations with parameters it is shown how the parameter can be given a proper geometric interpretation enabling the computational experiment to be carried out in a two-dimensional context, that is, in the plane of variables.

Inequalities are the major means of investigation in mathematics, both pure and applied. The fifth chapter is devoted to the systematic study of inequalities, including systems of inequalities, with parameters. In this context, it is demonstrated how computational experiment approach facilitates solving traditionally difficult problems not typically considered in the secondary school mathematics curriculum. The idea of extending an original exploration of an inequality with a parameter to allow for a deep inquiry into closely related ideas is discussed. One of the aspects associated with the dependence of solution to an inequality on a parameter is a possibility of using locus of an inequality with a parameter as a tool for posing one-variable inequalities with no parameter. In the digital era, such pedagogical perspective makes it possible to present problem solving and problem posing as two sides of the same coin. Also, the chapter focuses on the so-called technology-enabled/technology-immune tasks in the sense that whereas technology may be used in support of problem solving, its direct application is not sufficient for achieving the end result. It is shown that such tasks can be developed in the context of inequalities with parameters. A point is made that the computational experiment approach may be inconclusive, thereby requiring an analytic clarification of the experiment to make sense of the structure of a situation. A two-dimensional sign-chart method that can be used for solving inequalities with parameters is presented. Finally, the applied character of problem-solving techniques developed for solving inequalities with parameters is illustrated through their application to quadratic equations when the location of roots about a given point can be determined without solving an equation.

Trigonometry is known as a subject matter of great importance for the study of engineering disciplines. The sixth chapter shows how concepts in trigonometry can be approached from a computational perspective. Here, a single trigonometric equation with parameters is used as a springboard into several geometric ideas, thereby, demonstrating a closed connection of the two contexts. Technology such as Wolfram Alpha with its own unique algorithm of solving trigonometric equations and inequalities is presented as an agent of rather sophisticated mathematical activities stemming from the need to justify the equivalence of different forms of solution expressed through inverse circular functions. Whereas in the presence of technology (including just a calculator) such equivalence can be easily established numerically, the appropriate use of technology should motivate learners of mathematics to appreciate rigor and to enable the development of formal reasoning skills. Having experience with proving the equivalence of two solutions obtained through different methods can be construed as support system for research-like experience that prospective secondary mathematics teachers need for the successful teaching of the subject matter. This further provides experience with STEM-related techniques, something that can contribute to the efforts of introducing secondary mathematics teacher candidates and their future students alike to the ideas that develop the foundation of engineering profession.

The seventh chapter deals with geometric probabilities. It shows how the computational experiment approach can work in calculating probabilities of events associated with the behavior of solutions of algebraic equations with parameters. In particular, both theoretical and experimental probabilities are computed in the space of parameters and compared within a spreadsheet. Here, most of the explorations extend the ideas considered in the previous chapters and developed in the context of making a transition from the plane of variables to the plane of parameters. The material of this chapter is aimed at providing teacher candidates with experience important for applying mathematics to science and engineering the models of which typically depend on parameters. Whereas the construction of spreadsheet-based computational environments for computing geometric probabilities experimentally does not require any mathematical or technological sophistication, the skills in using such environments are important for understanding how to do explorations of mathematical models in engineering and science.

The last chapter illustrates how the computational experiment approach can make concepts of number theory more accessible to prospective teachers and their students alike. It provides a number of illustrations of using modern technology tools in exploring classic topics in elementary theory of numbers through a computational experiment. Here, one can learn how technology can be used to develop theoretical knowledge on the basis of a simple experiment so that, in turn, the knowledge so developed can inform and facilitate similar yet more complicated experiments. The chapter highlights the duality of computational experiment and formal demonstration in the sense that whereas one needs theory to validate experimental results, once can benefit from computing when discovering and correcting unexpected flaws that theory may sometimes comprise. The chapter demonstrates how TEMP that encourages collateral learning can be brought to bear by emphasizing geometrization of algebraic concepts and the appropriate use of digital tools. The deficiency of reasoning by induction in the context of basic summation formulas that can result in overgeneralization is discussed. The topic of Pythagorean triples is explored in depth using jointly a spreadsheet and Wolfram Alpha. Within this topic, it is demonstrated how computational experiment approach can motivate mathematical insight and encourage natural curiosity of the learners of mathematics.

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Contents

Chapter 1
Theoretical Foundations of Computational Experiment Approach to Secondary Mathematics

1.1 Introduction

This chapter provides theoretical underpinning of computational experiment approach to pre-college mathematics curriculum. It reviews mathematics education research publications and (available in English) educational reform documents from Australia, Canada, England, Japan, Singapore and the United States related to the use of computers as tools for experimenting with mathematical ideas. The chapter links pioneering ideas by Euler about experimentation with mathematical ideas to the use of the word experiment in the modern context of pre-college mathematics curricula. It emphasizes the role of mathematics education reform in bringing computers first to the undergraduate level and gradually extending their use to include experimentation at the primary level. Several theoretical frameworks including signature pedagogy, Type I/Type II technology applications, parallel structures of teaching and learning, agent-consumer-amplifier framework, and collateral learning in the digital era are highlighted leading to the development of the notion of technology-enabled mathematics pedagogy (TEMP). One of the major characteristics of TEMP is its focus on the idea with ancient roots—the unity of computational experiment and formal mathematical demonstration. The relationship between technology-enabled experiment and solution-enabled experiment is introduced as a structure that makes computational experiment a meaning making process. It will be demonstrated how visual imagery can support deductive reasoning leading to an error-free computational experiment.

1.2 Experiment in Mathematics Education

In this book, the word experiment is connected to the use of electronic computers in the context of advanced secondary mathematics curriculum and, in particular, mathematics teacher education. These modern tools when used in mathematics instruction create and enhance conditions for one's inquiry into mathematical structures, which may include interactive graphs, dynamic geometric shapes, and electronically generated and controlled arrays of numbers. That is, the modern experiment in

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Fig. 1.1 One out of 12 Latin squares of order 3

1	2	3
2	3	1
3	1	2

mathematics can be a computational one. In the context of mathematical education in general, a computational experiment approach to mathematics makes use of such computer-enabled experiments designed by a teacher and carried out by students, or jointly by teacher and students. Hereafter, the words teacher and student are understood broadly: the former is one who teaches and the latter is one who is taught. Because through such interaction, both parties can learn, the word learner will be applied to any individual engaged in the learning of mathematics.

Whereas the notion of experiment in the context of education has multiple meanings, learning as the goal of experiment is what all the meanings have in common. In a seminal book on experiment in education, McCall (1923) recognized the power of experiment as a milieu where "teachers join their pupils [i.e., students] in becoming question askers" (p. 3). Similarly, about a century later, Hiebert et al. (2003) emphasized the importance for teachers to treat lessons as experiments towards the end "of making some aspects of teachers' routine, natural activity more systematic and intensive" (p. 207). In other words, by treating lessons as experiments, teachers, "by focusing attention on, and making more explicit, the process of forming and testing hypotheses" (ibid, p. 207), are expected to learn both about and from teaching. Mathematics is especially conducive to the development of an environment in which reflective inquiry—a problem-solving method that blurs the distinction between knowing and doing by integrating knowledge with experience (Dewey 1933)—is the major learning strategy.

There is an interesting connection between the notions of experiment in mathematics and experiment in education. This connection can be revealed through the concept of Latin square. The latter is a square matrix each row and column of which contains any element one and only one time (Fig. 1.1). Latin squares have been commonly utilized in the design of educational experiments (Fisher 1935; Campbell and Stanley 1963) in different disciplines where it is required to construct a matrix under specific conditions on the location of its entries. For example, in the study by Gall et al. (1978) involving 12 teachers from 12 classrooms (experimental units), three recitation treatments (Probing and Redirection, No Probing and Redirection, Filler Activity) and one control treatment (Art Activity) were arranged in three 4×4 Latin squares (with each treatment, randomly assigned to the experimental units, appearing only once in a row and once in a column), provided that each teacher taught all four treatments. Another major application of Latin squares is in agriculture (Lakić 2001). Here, a field can be divided into sections and different seeds sown or treatments applied are recorded in the form of a Latin square with the goal to diminish the influence of other factors. In sum, Latin squares are great tools within which data can be conveniently stored, meaningfully observed, and appropriately analyzed.

In mathematics Latin squares were used by Euler, the great Swiss mathematician of the eighteenth century considered the father of all modern mathematics. In particular, Euler's name is associated with the so-called *Officers Problem* of placing 36 officers of six ranks and six regiments in a Latin square so that no officer of the same rank or of the same regiment would be in the same row or in the same column (MacNeish 1922). In his insightful approaches to mathematics, especially to number theory, Euler emphasized the importance of observations and so-called quasi-experiments or thought processes (experiments) that stem from observations: "the properties of the numbers known today have been mostly discovered by observation, and discovered long before their truth has been confirmed by rigid demonstration" (Pólya 1954, p. 3; according to Lakatos (1976), Pólya, who made the translation, "mistakenly attributes the quotation to Euler" (p. 9) instead of crediting it to the Editor of Euler's work). So, quite unexpectedly, a mathematical tool used by a pioneer of experimentation with mathematics, nowadays is utilized for the rigorous description of educational experiments.

The ideas about making numeric quasi-experiments as part of pre-college mathematics curricula with an emphasis on discovery learning, mathematical investigations, and drawing conclusions informed by inductive reasoning have begun gaining popularity around the world in the second part of the twentieth century. This is evidenced by a number of publications on and standards for mathematics teaching and learning (Cambridge Conference on School Mathematics 1963; Fletcher 1964; National Council of Teachers of Mathematics 1970, 1989; Peterson 1973; Wheeler 1967). According to Mason (2001), in England, this approach to mathematics can be traced back to the writings of Wallis[1] (1685) who used the word investigation to refer to 'my method of investigation' which, however, when supported by (empirical) induction alone can lead to erroneous conjectures (see Chap. 8 for examples). Therefore, it has been cautioned, "we should take great care not to accept as true such properties of the numbers which we have discovered by observation and… should use such a discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them; in both cases we may learn something useful" (Pólya 1954, p. 3). In that, the importance of theory that augments mathematical experimentation by appropriate demonstration and formal justification was equally emphasized. Therefore, as an experiment provides basis for insight, one can conclude that observation is at the core of any experiment. By the same token, experiment leads to the development of theory, which, in turn, can inform experiment as its conditions grow in complexity. All this is true for a modern day computational experiment.

1.3 Computational Experiment and its Validation

In education, any experiment can be associated with two types of validity: internal and external (Campbell and Stanley 1963). Internal validity of experiment is characterized by the basic set of skills and abilities without which any experiment

[1] John Wallis (1616–1703)—an English mathematician whose work, in particular, provided foundation for the development of integral calculus.

Fig. 1.2 Computing partial sums of consecutive odd numbers

	A	B
1	1	1
2	3	4
3	5	9
4	7	16
5	9	25
6	11	36
7	13	49
8	15	64
9	17	81
10	19	100

cannot be interpreted in some meaningful way. Meaningful interpretation creates conditions for external validity or, to put it differently, generalization of the experiment. In the case of a computational experiment, one has to possess some basic (grade appropriate) mathematical knowledge and skills in order to be able to interpret information generated through the experiment and, therefore, develop a more general perspective on this interpretation.

As an illustration, consider the case of computing partial sums of consecutive odd numbers within a spreadsheet (Fig. 1.2). This computational tool, used in mathematics education for more than three decades (Baker and Sugden 2003), was conceptualized by its inventor as “an electronic blackboard and electronic chalk in a classroom” (Power 2000). In particular, spreadsheets were recommended to teacher candidates for exploring ideas in number theory (Conference Board of the Mathematical Sciences 2001). In order to grasp the intended meaning and appreciate the mathematical significance (in educational sense) of this computational experiment (it will be referred to throughout this chapter as appropriate), one has to be able to recognize the nature of numbers appearing in column B of the spreadsheet. Otherwise, that is, if the partial sums are not recognized as square numbers, one is unable to proceed from experiment to theory. This transition is needed to establish the external validity of experiment that deals with the issue of generalizability. In the case of the partial sums of odd numbers, the theoretical justification of the experiment consists in formal demonstration of the emerging conjecture, namely, that the sum of the first n odd numbers is the square of n. This conjecture, in fact, is a proposition well known for almost two thousand years due to a Greek mathematician Theon (Smith 1958). Therefore, if internal validity of experiment is not established, one is not prepared for the discussion of its external validity, or, at the very least, one’s efforts in establishing the external validity of experiment may go astray.

A possible incompleteness of the required knowledge and skills on the part of the learners of mathematics calls for the introduction of additional techniques that enable one’s success at the stage of internal validation of experiment. In the case when one has to connect partial sums of odd numbers to squares, a diagrammatic

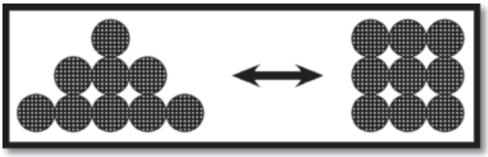

Fig. 1.3 Visual proof of the relation $1+3+5=3\times 3$

representation through concrete materials can be brought to bear. It is important to keep in mind that, in general, one's intuitive recognition of geometric figures develops at an early age (Freudenthal 1978, p. 100) and that geometric knowledge and skills are often superior to their numeric equivalent (Clements 1999). In particular, young children are capable of noticing, "that only some numbers could be formed into squares" (Ball 1993, p. 384). In general, this follows an ancient tradition originated in Chinese mathematics about tenth century B.C. (Beiler 1964) of ascribing geometric characteristics to numbers from where such terms as triangular, square, pentagonal, etc. numbers are derived. Such alternative geometric representation of the spreadsheet-based experiment is shown in Fig. 1.3. It can be facilitated through the task of rearranging counters forming a triangle with one counter at the vertex and with each row having two more counters than the previous one into a square.

Considering "every obstacle an opportunity for the exercise of ingenuity instead of an insuperable barrier" (McCall 1923, p. 7), one can precede a somewhat abstract computational experiment with a more concrete hands-on activity involving basic geometric shapes. This is in agreement with van Bendegen's (1998) dual categorization of an experiment in mathematics that involves computation and physical manipulation. As noted by Freudenthal (1978), "It is independency of new experiments that enhances credibility… [for] repeating does not create new evidence, which in fact is successfully aspired to by independent experiments" (pp. 193–194). Having different representations of an experimentally developed concept makes it easier to establish internal validity of experiment. In particular, it is not the amount of square numbers appearing in column B of the spreadsheet (Fig. 1.2) or, in Baker's (2008) terms, the number of calculated instances of the emerging hypothesis about the partial sums of odd numbers that facilitates the interpretation of the experiment. Rather, it is just one activity of rearranging counters from a triangle into a square shape (or vice versa) that provides experimental evidence to the analytically formulated mathematical proposition and is conducive for the development of new insight.

1.4 Defining Computational Experiment in the Pre-College Context

There are several ways the notion of experimental mathematics—the bedrock of any computational experiment—is currently defined. For example, Borwein and Bayley (2004) define experimental mathematics as the use of advanced computing technology in mathematics research. However, according to Epstein et al. (1992), "many mathematical experiments these days are carried out on computers, but others are still the result of paper-and-pencil work" (p. 1). The latter definition includes the use of both computational and non-computational tools in a mathematical experiment. Comparing the two definitions suggests that any kind of computing technology, if used appropriately, can support learners in their experimentation with mathematics. Characterizing experimental mathematics from a philosophical perspective, Baker (2008) did not emphasize the use of computers as essential tools for experimentation with mathematical ideas and, instead, argued that it "involves calculating instances of some general hypothesis" (p. 331) thus leaving it open as to how calculations are carried out. However, all the definitions draw on mathematical research and do not reflect on computational experiments in the context of mathematics education whether they are carried out at the tertiary, secondary, or primary level.

Boas (1971) was probably the first to introduce the notion of computational experiment in tertiary mathematical education at the meeting of the American Association of the Advancement of Science by arguing "calculus should be presented to the student in the same spirit as the experimental sciences" (p. 664). McKenna (1972) went even further in his argument in favor of computational experiment extending it from calculus to the whole mathematics: "the computer provides mathematicians with an unparalleled opportunity to motivate students towards experimentation with mathematics" (p. 295). In particular, referring to the use of computer in linear algebra, Hethcote and Schaeffer (1972) reported how "one student found experimentally that the limit of the ratio of terms in a Fibonacci sequence is independent of the two given initial values" (p. 293). But the same computational experiment can be already demonstrated in the secondary mathematics classroom using, for example, a spreadsheet (Arganbright 1984). It can be further extended from Fibonacci-like sequences such as Lucas numbers, as shown in Posamentier and Lehmann (2007), to Fibonacci-like polynomials leading to the discovery of new knowledge about the behavior of the so-called generalized golden ratios (Abramovich and Leonov 2011). Therefore, integrating computational experiment approach in the secondary mathematics curriculum has the potential for motivated students to experience mathematics in the making.

Understanding the idea of computational experiment in mathematics education as a way of learning mathematics with technology brings about the term "experimental mathematics" into the modern pre-college classroom (Sutherland 1994). It is interesting to note that the notion of mathematical experiment as a teaching method can be found in the Common Core, the modern day educational document in the United States: "A spreadsheet, or a computer algebra system (CAS) can be used to *experiment* with algebraic expressions, perform complicated algebraic manipulations, and understand how algebraic manipulations behave" (Common Core State Standards 2010, p. 62, italics added). Likewise, in Japan, the use of computing technology has been recommended to "be taken into account for the instructional content related to numerical calculations, as well as in teaching through activities like observation, manipulation and *experimentation*" (Takahashi et al. 2006, p. 257, italics added).

One can see how the advent of computers made it possible to extend the notion of experimental mathematics to include the idea of advanced problem solving in the context of computationally supported mathematical education environments. Through such an extension, both the complexity of mathematics to be involved and the sophistication of technology to be used can vary on the spectrum from precollege level to that of mathematical research. Although the use of paper and pencil allows for some basic computations also, the current emphasis on the use of technology in the teaching of mathematics in Australia (National Curriculum Board 2008), Canada (Expert Panel of Student Success in Ontario 2004; Ontario Ministry of Education 2005), Japan (Takahashi et al. 2006), Singapore (Ministry of Education, Singapore 2006), England (Advisory Committee on Mathematics Education 2011; Department for Education 2013b), the United States (Conference Board of the Mathematical Sciences 2001, 2012; International Society for Technology in Education 2007, 2008; National Council of Teachers of Mathematics 2000, 2011; President's Council of Advisors on Science and Technology 2010), and other countries as a way of making its learning more accessible is the main reason for the author to emphasize the use of the tools available in the digital era.

So, in the context of this book, the term computational experiment means an approach to mathematics teaching and learning made possible by the use of various commonly available and user friendly computational tools. Furthermore, methodology of the experiment remains the same regardless at which grade level it is used and what tools it employs. Namely, this methodology draws on the power of computers to perform a variety of numeric computations and geometric/graphic constructions in the case when ideas and objects under study are too complex to be approached by using mathematical machinery and mental computation alone. Of course, this complexity is relative to a grade level and learner's mathematical background knowledge.

1.5 Computational Experiment as a Signature Pedagogy of Mathematics

The notion of signature pedagogy was introduced by Shulman (2005a) in the context of professional education towards the goal of developing in students habits of mind of professionals working in the field they are preparing to join. Emphasizing teaching as a scholarly endeavor, this notion was explored for a variety of disciplines, including mathematics, in the book edited by Guring et al. (2009). In the specific context of mathematics, Cuoco et al. (1996) underscored the importance of drawing on the habits of mind of a mathematician as an organizing principle for school mathematics curriculum. Common characteristics for all signature pedagogies varying across disciplines comprise three entities called by Shulman (2005a) the structures of signature pedagogy, namely, surface structure of teaching, deep structure of teaching, and implicit structure of teaching. When one possesses only basic subject matter knowledge, his or her pedagogical skills and abilities lie within the surface structure of teaching. In mathematics, teaching at the surface structure level fails to appreciate such pillars of knowledge development as problem solving, reflective inquiry and quest for meanings. Instead, someone holding to the surface structure of mathematics teaching emphasizes memorization of rules without understanding their meaning and avoids using them as problem-solving tools. Computational experiment approach cannot be used in the classroom unless a teacher is prepared psychologically and pedagogically to move away from the surface structure of teaching.

Pedagogy that characterizes the deep structure of teaching is the result of one's strong possession of content he or she teaches, understanding how different areas and concepts of a discipline can be connected, and how students and teachers may become 'partners in advancement' (Bruner 1985) by exploring jointly generated questions. In mathematics, deep structure of teaching implies the need for a teacher to understand mathematics, its ideas and concepts in a profound way, to know how to demonstrate and interpret connections among the concepts, and have a rich repertoire of motivational techniques for the introduction of such concepts. In other words, a teacher must have a strong command of pedagogical content knowledge (Shulman 1986, 1987), something that is currently considered as the basis for students' progress in learning mathematics (Baumert et al. 2010). As was mentioned earlier, motivational techniques may include experimentation with mathematics (McKenna 1972). Therefore, with the advent of technology, deep structure of mathematics pedagogy can also be characterized by a computational experiment that stems from one's understanding of how mathematics and technology interact and becomes an element of implicit structure of teaching. Indeed, teachers' beliefs about the use of computers in the classroom formed by their experiences with using technology in teaching, are the major components of the implicit structure of their profession.

1.6 Nurturing Mathematical Mindset in Students

Shulman (2005b) described signature pedagogy using the following three descriptors: uncertainty, engagement, and formation. In general, the current signature pedagogy of mathematics both at the pre-college (National Council of Teachers of Mathematics 2000; Common Core State Standards 2010) and tertiary (Committee on the Undergraduate Program in Mathematics 2004; Conference Board of the Mathematical Sciences 2001, 2012) levels provides students with the opportunity of "doing mathematics rather that hearing about mathematics" (Ernie et al. 2009, p. 265). This focus on *doing* mathematics through problem solving is the distinctive signature of a professional mathematician, whether one uses technology or does not use it.

The need for nurturing mathematical mindset in students was introduced in North American secondary education half a century ago and the following two quotes represent the most prominent sources: "To know mathematics means to be able to do mathematics" (Alfors 1962, p. 189), and "For efficient learning, an exploratory phase should precede the phase of verbalization and concept formation" (Pólya 1963, p. 609). The former source is a memorandum signed by 75 professional mathematicians from the United States and Canada. It was written in response to the need to design a new mathematics curriculum of secondary school to improve and advance mathematical preparation of students at the pre-college level.

A call for the revision of school mathematics curriculum can also be found in the writings of Freudenthal (1973) who ascertained that mathematics pedagogy lacks connecting everyday problems to mathematical method available because, in general, "people never experience mathematics as an activity of solving problems, except according to fixed rules" (p. 95). As a remedy to this state of mathematics pedagogy and teacher training, Freudenthal (1978) emphasized the need for "a shift of stress from pedagogics to subject matter" (p. 68), echoing an earlier remark about "the dominance of education by professional educators who may have stressed pedagogy at the expense of content" (Ahlfors 1962, p. 189). Teaching prospective teachers to solve mathematical problems through a manifold of approaches, including computational experiment approach, develops their strength in understanding the subject matter of mathematics. However, according to Wittmann (2005), despite all correct recommendations, Freudenthal's call for the emphasis on subject matter preparation, mathematics education research "had lost the connection with mathematics [focusing, instead, on] artificial 'applications' and… lists of 'competences'. Step by step mathematical substance was pushed into the background and lost" (p. 296). As a secondary mathematics teacher candidate expressed (through the student opinion of faculty instruction) her belief about the role of a mathematics education course for secondary teachers: "The course has to prepare us to teach mathematics without knowing mathematics." This comment, stemming from the teacher's implicit structure of signature pedagogy, naively, yet mistakenly, defies the role of content in teaching mathematics, something that, on the contrary, as Ball et al. (2008) put it, "stands on its own as a domain of understanding disposition and skill needed by teachers for their work" (p. 398).

Interestingly, teaching students to do mathematics is inherently linked to Shulman's (2005b) descriptors of signature pedagogy—uncertainty, engagement, and formation. Indeed, if a teacher promotes reflective inquiry pedagogy, thereby acting at the deep structure of teaching, a student, by asking an unexpected question, can, in fact, pose a problem. This problem might be too difficult to solve even in an experimental fashion. The realization of this fact points to the uncertainty of mathematics pedagogy. Next, doing something presupposes engagement; so the problem-solving focus of current mathematics pedagogy does require students' engagement. Finally, regardless of the outcome of this engagement, one develops a kind of professional disposition towards the discipline of mathematics and even more so if a problem is not solved from the first attempt. In this case, it is very important to provide a qualified assistance in order to prevent students' interest towards mathematics to bifurcate into the state of unstable equilibrium. As noted by de Lange (1993), the students' excitement with problem solving and the teacher's growing confidence with mathematical content have great potential to overcome the issue of uncertainty in the classroom. In the digital era, at any grade level, problem solving and mathematical exploration can be greatly assisted through the use of computers, which bring elements of uncertainty, engagement, and formation to the mathematics classroom. Therefore, these ever changing teaching tools, in Shulman's (2005a) words, "create an opportunity for reexamining the fundamental signatures we have so long taken for granted" (p. 59). Several ideas of revisiting signature pedagogy of mathematics in the digital era through the lens of the computational experiment approach will be presented below.

1.7 Type I/Type II Technology Applications

Using computers at the surface structure level is what Maddux (1984) has termed Type I application of technology in education, seeing it quite different from Type II application. He referred to the latter type as "new and *better* ways of teaching" (p. 38, italics in the original) in the technological paradigm. In the context of mathematics teacher preparation it was argued, "technology used in a superficial way, without connection to mathematical reasoning, can take up precious course time without advancing learning" (Conference Board of the Mathematical Sciences 2012, p. 57). Thus, one can say that Type I application of technology to mathematics teaching stands in the way of advancing mathematical learning.

The concept of Type I/Type II application of technology turned out to be a very powerful theoretical shield against sometimes rather strong critique and persistent skepticism regarding the worth and purpose of using computers in the schools. More recently, Maddux and Johnson (2005) argued that "the boring and mundane uses to which computers were often being applied [at the infancy of their educational applications] had set the stage for a major backlash against bringing computers into schools" (p. 2). At the same time, just as the boring and ineffective uses of mathematics can motivate, at least in educational sense, the development of its concepts (e.g., repeated addition motivates multiplication which, in turn, motivates the use of logarithms), one's unsatisfactory experience with Type I application of technology can set the stage for the development of new ideas of teaching with computers, thereby, bringing about Type II technology applications. Among Type II applications is the computational experiment approach. However, this approach cannot be viewed as an educationally successful tool unless a teacher is capable of dealing with uncertainty when supporting students' engagement in computationally enabled problem solving and possesses skills necessary to foster mathematical mindset of the students. One can see how the concepts of Type II technology application, signature pedagogy, and computational experiment become connected.

Encouraging reflection and supporting analysis of the action by a student implies that one acts at the deep structure level of teaching. This kind of professional behavior requires broad pedagogical knowledge of what a specific computer environment affords, intellectual courage to motivate students to reflect on their actions, readiness to answer unexpected questions, and willingness to support students' natural curiosity by helping them learn in the "zone of proximal development" (Vygotsky 1987). This zone, described as a dynamic characteristic of cognition, measures the distance between two levels of the one's cognitive development as determined by independent and assisted performances in a problem-solving situation. When a student's performance becomes assisted within the zone, the teaching occurs at the deep structure level. Teaching at that level also requires knowledge of national standards of the subject matter taught, understanding connections that exist between concepts that belong to different grade levels, and, in the digital era, skills of using computers to support concept learning. When a computer is used at the deep structure level of teaching, Type II application of technology occurs.

1.8 Parallel Structures of Teaching and Learning

The theoretical construct of signature pedagogy can be augmented to include students as the beneficiaries of the pedagogy. This augmentation is consistent with the underlying principles of educational scholarship, which sees the concept of signature pedagogy as an application of the theory of learning to the practice of teaching (Shulman 2005a). In particular, in the context of mathematical education this concept serves as a link between teaching and learning (Ernie et al. 2009). However, in general, the proposed extension is neither grade nor content specific and it may be applied to any discipline.

By extending the notion of signature pedagogy to include students, two separate but interdependent universes can be considered: teacher's universe and student's universe. Each universe comprises three levels echoing Shulman's classic structures of signature pedagogy, which can be then considered as a part of the whole teaching and learning process. In this process, teaching affects learning and vice versa; that is, the way students learn (or aspire to learn) can affect the way teachers teach. Due to such reciprocity of teaching and learning, the same three structures—surface, deep, and implicit—can be considered in the student's universe. As the concept map in Fig. 1.4 shows, the teacher's universe includes surface structure of teaching (SST), deep structure of teaching (DST), and implicit structure of teaching (IST). Likewise, the student's universe consists of surface structure of learning (SSL), deep structure of learning (DSL), and implicit structure of learning (ISL).

Moreover, the concept map indicates that in the teacher's universe, IST influences SST and DST (the same size of arrows does not imply the same level of influence). At the same time, IST is a dynamic characteristic and its current state depends on teacher's experience with SST and DST. For example, successful experience with DST changes an IST state and the latter, in turn, sends a signal directly to SST to reduce its weight in the craft of teaching. In the student's universe, the solid arrow from SSL to DSL means the inborn desire of learning. The dashed arrow shows that one's presence in DSL typically depends on support from a 'more knowledgeable other'. And like in the teacher's universe, the components of student's ISL depend on his/her learning experience and, in addition, affect teacher's IST.

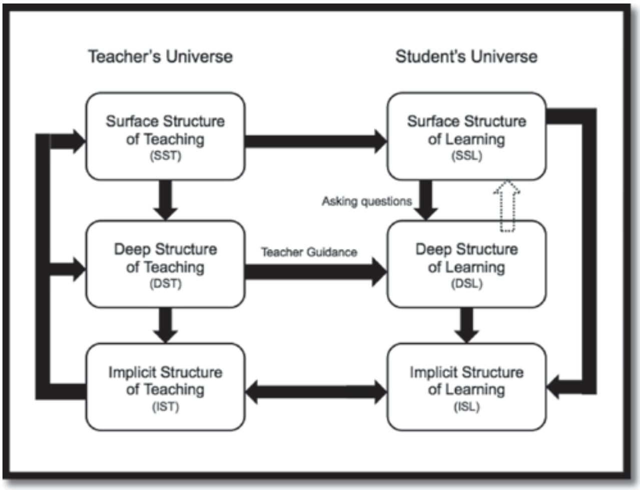

Fig. 1.4 Parallel structures of teaching and learning: a concept map. (Source: Abramovich et al. 2012)

The teacher's and the student's universes not only comprise matching strands but constantly affect each other as both parties make their way through the levels of teaching and learning. In the digital era, one can talk about computer-enhanced didactics of mathematics. The SST, perhaps inadvertently, keeps a student at the SSL level so that he or she would interact with a computer for the purpose of enjoyment only. However, the teacher cannot have full control of the student's use of technology. For example, the student might recognize patterns that the computer generates and then ask the teacher various questions about those patterns. In that way, the computer becomes a thinking device, thereby, bringing the student to the DSL level. However, the student's immersion into the DSL may be rather unstable and the extent of its instability depends on the teacher's willingness, in turn, to enter the DST; in other words, it depends on what style of assistance a teacher is prepared to offer (Fig. 1.5).

Similarly to the two types of technology integration, one can talk about two styles of assistance that teachers can offer to their students. Style I assistance is typified by the surface structure of teaching and it is limited by one's teaching philosophy which does not view teaching mainly as assisted performance (Tharp and Gallimore 1988). Style II assistance is typified by the deep structure of teaching and it is open to promoting reflective inquiry and taking an intellectual risk by going into an uncharted territory brought to light through an open-ended classroom discourse. Likewise, a cursory knowledge of technology by a teacher offers students Style I assistance only. By the same token, Style II assistance in the students' design and/or utilization of a computational learning environment requires a high level of technological literacy on the part of a teacher. These two styles of assistance, observed within the general instructional setting, underlie one's implicit structure of signature pedagogy and determine the composition of two other structures.

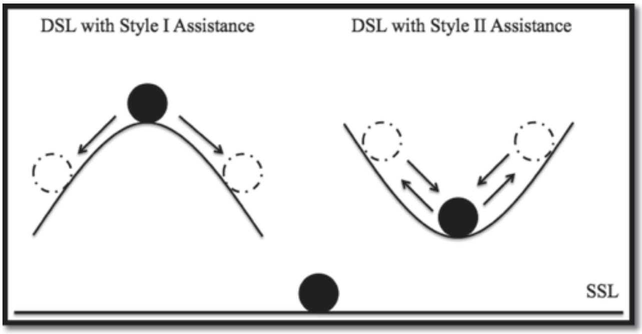

Fig. 1.5 Unstable (*left*) and stable (*right*) equilibriums of DSL. (Source: Abramovich et al. 2012)

If the student enters the DSL, but does not receive Style II assistance from the teacher, it is quite likely that he or she would exit it back to the SSL. Furthermore, receiving no support for natural curiosity—defined by James[2] (1983) as "the *impulse towards better cognition* in its full extent" (p. 37, italics in the original)—affects one's cognitive disposition towards the continuation of having 'a good time' at the SSL level. This kind of a student's functioning within his or her universe is consistent with the dynamism of cognition expressed through the theoretical construct of the zone of proximal development. The longer both the teacher and the student function at the deep structure of their universes, in other words, the longer Style II assistance in the context of Type II application of technology is provided, the more concept learning can result from the use of technology-enabled mathematics pedagogy (hereafter referred to as TEMP).

By examining the computational experiment approach through the combined lens of teaching and learning, one can recognize significant merits of TEMP and its potential for achieving substantial learning outcomes. A student's entrance into DSL may be motivated by a sudden recognition of a mathematical concept that a computer supports, be it by a teacher's design (manifesting DST) or not. In the student's universe, the ISL includes previous learning experiences and beliefs about what it means to learn and do mathematics (Ernie et al. 2009). Just as in the case of IST, the ISL affects both SSL and DSL. An example of this relationship is a student's belief that any mathematical model, be it symbolic or iconic, serves only a single problem rather than multiple problems. Even if the same model emerges in different contexts, this belief prevents one from recognizing the sameness, affects one's desire to move from SSL to DSL and, thereby, hinders conceptual understanding of mathematics. However, if teachers function at the DST level, they can guide students to understanding that just as different problem-solving strategies can be applied to a single problem, different problems may be resolved through a single approach.

[2] William James (1842–1910), a Harvard-based philosopher and psychologist known for being the first to offer a psychology course for teachers in the United States.

1.9 Collateral Learning in the Technological Paradigm

John Dewey, one of the most influential philosophers behind the reform of American educational system in the first part of the twentieth century, argued that a pedagogy based on encouraging continuous reflection on the material studied with the assistance of 'the more knowledgeable other' creates conditions for what he called *collateral learning*, something that does not result from the immediate objective of the curriculum under study but rather stems from its hidden domain. The following tenet emphasizes the educational significance of that type of learning: "Perhaps the greatest of all pedagogical fallacies is the notion that a person learns only the particular thing he is studying at the time" (Dewey 1938, p. 49). The appropriate use of technology requires that teachers have deep knowledge of mathematics and understanding of how to integrate mathematics and technology in order to guide students in their journey through often hidden opportunities for collateral learning. For example, collateral learning may result from what Kantorovitch (1998) referred to as "unintentional discovery" (p. 33), an intellectual phenomenon which stems from a pedagogy encouraging the freedom of exploration so that one can solve several (not necessarily appearing related) problems at a time, and, in addition, continuously utilizes known concepts as the building blocks of concepts to be discovered by serendipity. Computational experiment approach is conducive to this kind of discovery, be it a new knowledge for a student, teacher, or professional mathematician. For a curious mind, the use of a computer at a tool for experimentation with mathematics is conducive to the discovery of facts that were not expected to come about at the outset of a computational experiment. That is, mathematical experimentation in the technological paradigm is an essential mechanism of collateral learning.

In a more general context, both collateral learning and unintentional discovery concepts bring to mind another educational construct known as hidden curriculum—"those nonacademic but educationally significant consequences of schooling that occur systematically" (Martin 1983, p. 124). This kind of learning experience is taking place within a context that is much broader than a topic of any given lesson and, through reflection, enables students to become aware of rules and guidelines typically associated with social relations and control of individual actions. The notion of hidden curriculum can be extended to include collateral learning and unintentional discovery that may take place within a pure academic domain when one is expected and even encouraged to make connections among seemingly disconnected ideas and concepts related to a specific subject matter. Thus, one can talk about hidden mathematics curriculum (Abramovich and Brouwer 2006)—a didactic approach to the teaching of mathematics that motivates learning in a larger context that one "is studying at the time." Computing technology provides a learning environment to support this approach through which hidden messages of mathematics can be revealed to students by teachers as 'more knowledgeable others'. By the very design of a computational experiment and the nature of the current signature pedagogy of mathematics, students are provided with ample opportunities for collateral learning and unintentional discovery as they develop mathematical habits of mind through continuous reflection on the results of the experiment.

In a classroom setting supported by TEMP, the computational experiment approach requires teachers' ability to teach at the deep structure level of teaching. By the same token, students are expected to learn at the deep structure of learning. Here, signature pedagogy based on the notion of computational experiment comes into play as the complex nature of extended explorations calls for the development of computational environments through which hidden mathematical messages can be revealed. In the context of TEMP, the activities of doing mathematics and using technology are not in the relation of dichotomy as some learners believe (Niess 2005), but rather are framed by their unity.

As will be shown throughout the book, mathematics teachers (or teacher candidates for that matter) may consider TEMP both as a teaching and learning tool. By turning an inherently open-ended problem into a small technology-supported mathematical project for a student, a teacher can learn to appreciate the didactic value of dividing the project into four stages—empirical, speculative, formal, and reflective—each of which depends on the previous stage(s). Such an appreciation develops gradually as students move at their own pace from one stage to another in rather challenging contexts. Helping students with this move or mediating their struggle with a peculiarity of each stage requires Style II assistance, something that belongs to the deep structure of teaching. The self-awareness of even a slight teaching success happening at each stage and the emerging recognition of collateral learning opportunities that stem from an open-ended context ultimately build into background of experience for the teachers in using TEMP.

1.10 Computational Experiment as a Meaning-Making Process

In the context of the theory of semiotic (i.e., sign-based) mediation the word text refers to any meaningful verbal and non-verbal semiotic structure (Lotman 1988). Thus, mathematical text may be as short as a single character π which describes the ratio of the circumference of any circle to its diameter and as complex as a proof, due to Lindemann (1882), that π may not be a root of any polynomial with rational coefficients, i.e., π is a transcendental number. The most important issue associated with text, in general, and mathematical text, in particular, is to what extent it functions as a generator of new meanings and serves the goal to enhance the development of new knowledge. In other words, uncovering hidden messages in a text brings about the collateral learning phenomenon.

Whereas the principle assumptions of TEMP dwell on the notions of reflective inquiry, dialogic discourse and conceptual development, many uses of technology, especially those associated with Type I applications, support authoritative discourse the pedagogy of which promotes automatism alone and does not encourage insight, reflection and dialogue. As noted by Freudenthal (1983), "sources of insight can be clogged by automatisms" (p. 469). In the context of Type I application of technology, a possible static meaning structure of computer-generated text is not conducive to be taken as a thinking device by a student. In that way, one can observe a didactic contradiction when the use of TEMP is limited to Type I technology application. A question to be answered is: How can computational experiment approach bring about a non-authoritative text of TEMP? A variety of interpretations of modeling data emerging from a computational experiment can help to resolve a seemingly contradictory computer-entrusted didactic situation. This section, using the sociocultural approach to mediated action (Wertsch 1991), will highlight computational experiment through the lens of modern mathematics signature pedagogy with its non-authoritative texts.

Proceeding from neo-Vygotskian tradition "to treat human learning and cognitive development as a process which is culturally-based, not just culturally influenced" (Mercer 1994, p. 92), this approach views humans as coming into contact with the learning environment through the action in which they engage. In turn, the action employs different tools and signs called mediational means. Mediated by tools, the action can generate signs which, in turn, can be used to create new tools. Sometimes it is useful to distinguish between different types of mediation and to single out a semiotic mediation as being relevant to computational experiment that draws on the power of technology in the creation of arrays of numbers, geometric images and graphs.

Conceptualization of computational experiment as a meaning-making process can be tied to the ideas about two major functions of text—transmission of constant information and dialogic interanimation. The sociocultural perspective makes a clear distinction between authoritative and internally persuasive discourse in a problem-solving context; that is, it discriminates between static and dynamic meaning structures of associated texts. Consider two types of texts—Type I and Type II applications of technology to the teaching of mathematics. The former type can be advanced to the latter type if, in Lotman's (1988) terms, one can extract a text from a state of semiotic equilibrium, activate its self-development, and make it function as a generator of new meanings.

Type I application is aimed at the correct answer only and, typically, it is framed by authoritative discourse. In the context of the spreadsheet pictured in Fig. 1.2, the goal of computation could be to find the sum of the first ten odd numbers. Type II application, however, provides means to consider computational experiment as a thinking device so that one can shift the attention from a particular inquiry (finding the number 100) to a whole class of similar explorations for which technology application proves to be equally effective. As noted by Kline (1985), "A farmer who seeks the rectangle of maximum area with given perimeter might, after finding the answer to his question, turn to gardening, but a mathematician who obtains such a neat result would not stop there" (p. 133). This note characterizes the crux of the current signature pedagogy of mathematics. Exploring problems in open way, seeking for extensions and generalizations, reflecting meaningfully on effective strategies in search for new results constitute the essence of an internally persuasive discourse of TEMP. For example, within a Type II application of the spreadsheet pictured in Fig. 1.2 one can see the sequence in column A as an arithmetic sequence the difference of which can be varied. Through this variation, the numbers in column B would vary as well so that the arithmetic sequence with difference $d \geq 1$ (and the first term equal to one) yields polygonal numbers of side $d+2$ as its partial sums.